Generals 2011 Part II Section 3A

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a) Linearize the cold electron fluid equations:

$$\frac{\partial \tilde{n}}{\partial t} + \nabla \cdot (n_0 \tilde{v}) = 0 \tag{1}$$

$$m_e n_0 \frac{\partial \tilde{v}}{\partial t} = -e\vec{E} \tag{2}$$

$$\nabla \cdot \vec{E} = -\frac{e\tilde{n}}{\epsilon_0} \tag{3}$$

Linearize using $\partial_t \to -i\omega$, $\nabla \to ik$, eliminate $E, \tilde{n}, \tilde{v}$ to get $\omega^2 = \frac{n_0 e}{\epsilon_0 m_e} = \omega_{pe}^2$.

b)Using Poisson's equation from above:

$$\nabla \cdot \vec{E}(t=0) = -E_0 k sin(kx) = -\frac{e\tilde{n}}{\epsilon_0} \tag{4}$$

from which $\tilde{n}(t=0)$ and $\tilde{v}(t=0) = \frac{\omega}{kn_0}\tilde{n}(t=0)$ follow directly.

c) If the wave is not completely extinguished, it has completely flattened the distribution function for some width Δv around $v = \omega/k$. See the very well-graph graph below. d) Electrons with velocities between $\frac{\omega}{k} - \Delta v$ are accelerated, and those with velocities $\frac{\omega}{k} - \Delta v$ are decelerated. We can approximate the number of accelerated particles $N_a \approx \Delta v f_0(\frac{\omega}{k} - \Delta v)$ and similarly the number of decelerated particles $N_d \approx \Delta v f_0(\frac{\omega}{k} + \Delta v)$. Since we are out on the tail of the distribution function, we can Taylor expand:

$$f_0\left(\frac{\omega}{k} \pm \Delta v\right) \approx f_0\left(\frac{\omega}{k}\right) \mp \Delta v \left|\frac{\partial f_0}{\partial v}\right|_{\omega/k} \tag{5}$$

So the net number of accelerated electrons is:

$$N = N_a - N_d \approx (\Delta v)^2 \left|\frac{\partial f_0}{\partial v}\right|_{\omega/k} \tag{6}$$

The average energy gained by one of these electrons:

$$E_{gained} \approx \frac{1}{2} m_e \left[\left(\frac{\omega}{k}\right)^2 - \left(\frac{\omega}{k} - \Delta v\right)^2)\right] \approx m_e \Delta v \frac{\omega}{k} \tag{7}$$

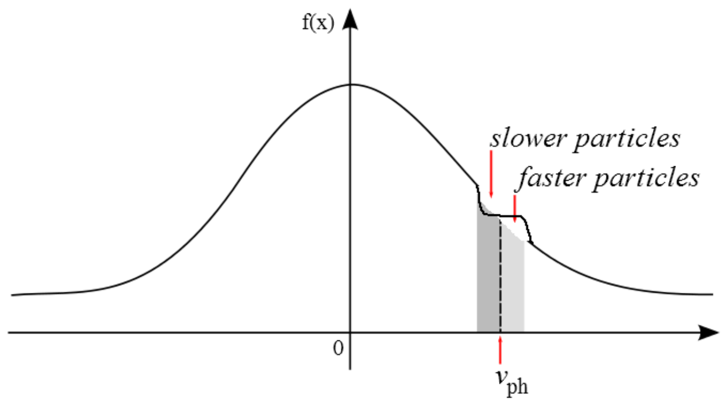

Figure 1: Note that the flattening should really be farther out on the tail in this case, but the idea is the same.

Now we look at the (linearized approximation of) the maximum energy the $\vec{E}$ field could transfer to an electron:

$$W = \int \vec{F} \cdot dl \approx \frac{eE_0}{k} \tag{8}$$

Within a factor of unity, E_{gained} and W are equal, so we equate them to get a value for $\Delta v = \sqrt{eE_0/mk} = v_{tr}$. Thus, v_{tr} corresponds to the maximum velocity of an electron that could remained trapped in the potential well of this standing wave.

For $E(x, t \to \infty)$ to remain finite despite Landau damping, we require that the total energy of the wave is greater than the energy spent flattening the distribution function:

$$\epsilon_0 |E_0|^2 > N \times E_{gained} = m_e (\Delta v)^3 \frac{\omega_p}{k} \left| \frac{\partial f_0}{\partial v} \right|_{\omega/k} \tag{9}$$

where we noted that $\omega \approx \omega_p$. Rearranging and using $\Delta v = v_{tr}$ we obtain

$$v_{tr} > \frac{\omega_p^3}{k^3} \left| \frac{\partial f_0}{\partial v} \right|_{\omega/k} \tag{10}$$

as desired.