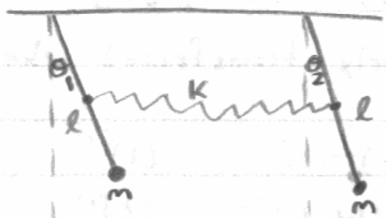
Seth Dorfman Prelims January 4, 2006

May 2004 CM

2) a.



$\Delta X_{sping} = \frac{\ell}{2}\sin\theta_2 - \frac{\ell}{2}\sin\theta_1$

$I_{pendula} = m\ell^2$

$T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2$

$T = \frac{1}{2}m\ell^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)$

$U = -mg\ell\cos\theta_1 - mg\ell\cos\theta_2 + \frac{1}{2}K\left(\frac{\ell}{2}\right)^2(\sin\theta_2 - \sin\theta_1)^2$

$L = T - U$

$L = \frac{1}{2}m\ell^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) + mg\ell(\cos\theta_1 + \cos\theta_2) - \frac{1}{8}K\ell^2(\sin\theta_2 - \sin\theta_1)^2$

b. $\frac{\partial L}{\partial \theta_1} = -mg\ell\sin\theta_1 + \frac{1}{4}K\ell^2(\sin\theta_2 - \sin\theta_1)\cos\theta_1$

$\frac{\partial L}{\partial \theta_2} = -mg\ell\sin\theta_2 - \frac{1}{4}K\ell^2(\sin\theta_2 - \sin\theta_1)\cos\theta_2$

$\frac{\partial L}{\partial \dot{\theta}_2} = m\ell^2\dot{\theta}_2 \quad \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\theta}_2}\right] = m\ell^2\ddot{\theta}_2$

$\Rightarrow m\ell\ddot{\theta}_1 = -mg\sin\theta_1 + \frac{1}{4}K\ell(\sin\theta_2 - \sin\theta_1)\cos\theta_1$

$m\ell\ddot{\theta}_2 = -mg\sin\theta_2 - \frac{1}{4}K\ell(\sin\theta_2 - \sin\theta_1)\cos\theta_2$

For small oscillations $\sin(\theta) \approx \theta$, $\cos(\theta) \approx 1$

$\Rightarrow m\ell\ddot{\theta}_1 = -mg\theta_1 + \frac{1}{4}K\ell(\theta_2 - \theta_1)$

$m\ell\ddot{\theta}_2 = -mg\theta_2 - \frac{1}{4}K\ell(\theta_2 - \theta_1)$

$m\ell(\ddot{\theta}_1 + \ddot{\theta}_2) = -mg(\theta_1 + \theta_2)$

$(\ddot{\theta}_1 + \ddot{\theta}_2) + \frac{g}{\ell}(\theta_1 + \theta_2) = 0$

$\theta_1 + \theta_2 \sim \sin\left(\sqrt{\frac{g}{\ell}}\,t\right) \qquad w_1 = \sqrt{\frac{g}{\ell}}$

$m\ell(\ddot{\theta}_2 - \ddot{\theta}_1) = -\frac{1}{2}K\ell(\theta_2 - \theta_1) - mg(\theta_2 - \theta_1)$

$(\ddot{\theta}_2 - \ddot{\theta}_1) + \left(\frac{K}{2m} + \frac{g}{\ell}\right)(\theta_2 - \theta_1)$

$\theta_2 - \theta_1 \sim \sin\left(\sqrt{\frac{K}{2m} + \frac{g}{\ell}}\,t\right) \qquad w_2 = \sqrt{\frac{K}{2m} + \frac{g}{\ell}}$

Normal Modes:

(1) $\theta_1 = \theta_2$ frequency: $\sqrt{\frac{g}{\ell}}$

(2) $\theta_1 = -\theta_2$ frequency: $\sqrt{\frac{K}{2m} + \frac{g}{\ell}}$

c. $\theta_1(t) = A\cos(w_1 t) + B\sin(w_1 t) + C\cos(w_2 t) + D\sin(w_2 t)$

$\theta_2(t) = A\cos(w_1 t) + B\sin(w_1 t) - C\cos(w_2 t) - D\sin(w_2 t)$

$\theta_2(0) = 0 \Rightarrow A - C = 0 \qquad \dot{\theta}_2(0) = 0 \quad Bw_1 - Dw_2 = 0$

$\theta_1(0) = \theta_0 \Rightarrow A + C = \theta_0 \qquad \dot{\theta}_1(0) = 0 \quad Bw_1 + Dw_2 = 0$

$\therefore A = C = \frac{\theta_0}{2} \qquad \therefore B = D = 0$

$\therefore \; \theta_1(t) = \frac{\theta_0}{2}\left[\cos(w_1 t) + \cos(w_2 t)\right]$

$\quad\;\; \theta_2(t) = \frac{\theta_0}{2}\left[\cos(w_1 t) - \cos(w_2 t)\right]$

c. The energy is completely transferred when

$\theta_1(t) = \dot{\theta}_1(t) = 0$

$\therefore \; \cos(w_1 t) + \cos(w_2 t) = 0 \qquad (1)$

$w_1 \sin(w_1 t) + w_2 \sin(w_2 t) = 0 \qquad (2)$

$\cos^2(w_2 t) + \sin^2(w_2 t) = \cos^2(w_1 t) + \left(\frac{w_1}{w_2}\right)^2 \sin^2(w_1 t)$

$1 = 1 - \sin^2(w_1 t) + \left(\frac{w_1}{w_2}\right)\sin^2(w_1 t)$

$0 = \sin^2(w_1 t)\left[\left(\frac{w_1}{w_2}\right)^2 - 1\right]$

if $w_1 \neq w_2$ $\quad \sin(w_1 t) = \sin(w_2 t) = 0$

to satisfy (2)

From (1) $\quad w_1 t = a\pi \quad w_2 t = b\pi \quad$, $a - b$ odd

$$t = \frac{a\pi}{w_1} = \frac{b\pi}{w_2}$$

$$\frac{a}{b} = \frac{w_1}{w_2} = \frac{1}{\sqrt{Kl/2mg + 1}}$$

Let $\frac{a}{b}$ be the fraction $\frac{1}{\sqrt{\frac{Kl}{2mg} + 1}}$ in reduced form so that a and b are integers,

Then the energy is transferred completely at

$$t = a\pi\sqrt{\frac{l}{g}} = b\pi\sqrt{\frac{2ml}{Kl + 2mg}}$$