2007 Part 1 Q2 Nonneutrals

a. $\vec{V}(\vec{x}) = \dfrac{\int d^3\vec{p}\ \vec{v} f}{\int d^3\vec{p}\ f}$ $\qquad f = \dfrac{\hat{n}}{(2\pi m T)^{3/2}} \exp\left(\dfrac{-(H - \Omega P_\theta)}{T}\right)$

$H = \frac{1}{2m}(p_r^2 + p_\theta^2 - p_z^2) - e\phi(r) \qquad P_\theta = r(p_\theta - \frac{1}{2} m r \omega_c) \qquad \vec{p} = m\vec{v}$

$H - \Omega P_\theta = \frac{1}{2m}\left(p_r^2 + p_\theta^2 - 2mrp_\theta\Omega + p_z^2\right) - e\phi(r) + \frac{1}{2} m r^2 \omega_c \Omega$

complete the square: $p_\theta^2 - 2mrp_\theta\Omega = p_\theta^2 - 2mrp_\theta\Omega + m^2r^2\Omega^2 - m^2r^2\Omega^2$
$= (p_\theta - mr\Omega)^2 - m^2r^2\Omega^2$

$H - \Omega P_\theta = \dfrac{1}{2m}\left(p_r^2 + (p_\theta - mr\Omega)^2 + p_z^2\right) - e\phi(r) + \frac{1}{2} m r^2 \omega_c \Omega - \frac{1}{2} m r^2 \Omega^2$

$\frac{1}{m}\int d^3\vec{p}\ p_r f = 0$ because f is even in p_r. Ditto p_z: $V_z = 0$, $V_r = 0$

$\frac{1}{m}\int d^3\vec{p}\ p_\theta f = \frac{1}{m}\int d^3\vec{p}\left[(p_\theta - mr\Omega) + mr\Omega\right] f$ First term vanishes because f is even in $p_\theta - mr\Omega$

$= \frac{1}{m} \cdot mr\Omega \int d^3\vec{p}\ f$

$V_\theta = r\Omega \dfrac{\int d^3\vec{p} f}{\int d^3\vec{p} f} = r\Omega$ $\quad \Omega$ is the constant mean angular velocity of the column

b. We have already shown $H - \Omega P_\theta = \dfrac{1}{2m}\left(p_r^2 + (p_\theta - mr\Omega)^2 + p_z^2\right) + \psi(r)$

where $\psi(r) = \frac{1}{2} m r^2 (\Omega\omega_c - \Omega^2) - e\phi(r)$

$n(r) = \int d^3\vec{p}\ f$. momentum integrals cancel out denominator in f

$n(r) = \hat{n} \exp\left(-\dfrac{\psi(r)}{T}\right)$

c. $\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 4\pi e\, r\, n(r)$

Near $r=0$, $\psi = 0$, and $n(r) \approx \hat{n}$

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) \approx 4\pi e\hat{n}\, r$$

$$\Rightarrow r\frac{d\phi}{dr} = 2\pi e\hat{n} r^2$$

$$\frac{d\phi}{dr} = 2\pi\hat{n}er \quad \Rightarrow \quad \phi(r) = \pi\hat{n}er^2 \qquad \text{with } \phi(0)=0$$

$$\Rightarrow \phi(r) \approx \frac{1}{4}\frac{m}{e}\omega_p^2 r^2, \qquad \text{with } \omega_p^2 \equiv \frac{4\pi\hat{n}e^2}{m}$$

d. For $n(r) = \hat{n}\exp\left(-\frac{\psi(r)}{T}\right)$ to be monotonically decreasing near $r=0$, $\psi(r)$ should be monotonically increasing.

$$\psi(r) \cong \tfrac{1}{2}mr^2(\Omega\omega_c - \Omega^2) - \tfrac{1}{4}m\omega_p^2 r^2 \qquad \text{near } r=0$$

$$\psi(r) \cong \tfrac{1}{2}mr^2(\Omega\omega_c - \Omega^2 - \tfrac{1}{2}\omega_p^2)$$

$\rightarrow$ For $\psi(r)$ increasing, $\Omega\omega_c - \Omega^2 - \frac{1}{2}\omega_p^2 > 0$

e. Boundaries of confinement: $\Omega = \dfrac{\omega_c \pm \sqrt{\omega_c^2 - 2\omega_p^2}}{2}$ max + min

largest value of $\dfrac{\omega_p^2}{\omega_c^2} = \dfrac{1}{2}$, at which $\Omega = \dfrac{\omega_c}{2}$