2008 Part II, Question 3, General Plasma Phenomena

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1 Part a.

An alternate and easy way to determine the kinematic equations for the relativistic particle is to use the relativistic Lagrangian. The relativistic Lagrangian for a classical particle in an electromagnetic field is:

$$\mathcal{L} = \frac{-mc^2}{\gamma} - q\phi(\mathbf{x}, t) + \frac{q}{c}\mathbf{v} \cdot \mathbf{A}(\mathbf{x}, t), \tag{1}$$

where the EM field is represented by the scalar and vector potentials, q is the particle electronic charge, and $\gamma = \sqrt{1 + \frac{\mathbf{p}^2}{m^2c^2}} = 1/\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}$ is the standard relativistic mass factor. The equations of motion are calculated using exactly the same Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}, \tag{2}$$

where the q_i are the generalized coordinates for the particle.

From the given form of $\mathbf{B}$ and the vector potential equation $\nabla \times \mathbf{A} = \mathbf{B}$, we can see that

$$\begin{aligned}
\mathbf{B} &= \hat{\mathbf{x}}\left(-B_w \cos k_w z\right) + \hat{\mathbf{y}}\left(-B_w \sin k_w z\right) \\
&= \hat{\mathbf{x}}\left(-\frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z}\right) \\
\Rightarrow \mathbf{A} &= \frac{B_w}{k_w}\left(\hat{\mathbf{x}} \cos k_w z + \hat{\mathbf{y}} \sin k_w z\right).
\end{aligned} \tag{3}$$

Now that we have the vector potential for the wiggler field, we can immediately write the single particle relativistic Lagrangian using Eqs. 1 and 3:

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{v_x^2 + v_y^2 + v_z^2}{c^2}} - \frac{eB_w}{ck_w}\left(v_x \cos k_w z + v_y \sin k_w z\right) \tag{4}$$

Determining the equations for v_x',v_y', and v_z' is now trivial. For instance, notice that both x and y are cyclic coordinates, i.e. they are not present in the Lagrangian $\mathcal{L}$. Thus $\partial\mathcal{L}/\partial x = \partial\mathcal{L}/\partial y = 0$, and there are two conserved quantities that will immediately yield v_x' and v_y':

$$\frac{\partial\mathcal{L}}{\partial v_x} = \text{constant} = \gamma m v_x - \frac{eB_w}{ck_w}\cos k_w z,$$
$$\frac{\partial\mathcal{L}}{\partial v_y} = \text{constant} = \gamma m v_y - \frac{eB_w}{ck_w}\sin k_w z.$$

Our condition that $v_x' = 0 = v_y'$ if $B_w = 0$ means that we should choose both constants to be zero, and thus we find:

$$v_x' = \frac{a_w c}{\gamma}\cos k_w z', \tag{5}$$

$$v_y' = \frac{a_w c}{\gamma}\cos k_w z', \tag{6}$$

with

$$a_w \equiv \frac{eB_w}{mc^2 k_w}. \tag{7}$$

Finally, the full Euler-Lagrange equation for v_z gives

$$\gamma m \dot{v}_z = -\frac{eB_w}{c}\left(-v_x \sin k_w z + v_y \cos k_w z\right) = 0,$$

where the use of equations 5 and 6 led to the result that $\dot{v}_z = 0$, so v_z' is a constant, implying $z'(t') = z + v_z(t' - t)$. We now have equations for all of the relevant velocities in this problem. Inspection of Eqs. 5 and 6 leads to the conclusion that energy, and thus γ', is conserved, but a more formal proof involves taking the time derivative of γ' and using the Lorentz force equation:

$$\frac{\partial\gamma'}{\partial t'} = \frac{d}{dt'}\left(\sqrt{1 + \frac{\mathbf{p}'^2}{m^2c^2}}\right) = \frac{1}{\gamma'}\frac{\mathbf{p}'\cdot\frac{d\mathbf{p}'}{dt'}}{m^2c^2} = \frac{\mathbf{v}'\cdot\left(\frac{e}{c}\mathbf{v}'\times\mathbf{B}\right)}{mc^2} = 0. \tag{8}$$