The effect of damping on autoresonant (nonstationary) excitation

J. Fajans[a] and E. Gilson
Department of Physics, University of California, Berkeley, Berkeley, California 94720-7300

L. Friedland
Racah Institute of Physics, Hebrew University of Jerusalem, 91904 Jerusalem, Israel

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When a nonlinear oscillator with an amplitude dependent frequency is driven by a swept frequency drive, the oscillator's amplitude will, in some circumstances, automatically adjust itself so that the oscillator's nonlinear frequency closely matches the drive frequency. This phenomenon is called autoresonance, and allows the amplitude of the oscillator to be controlled simply by sweeping the drive frequency. Previous studies of autoresonance were in undamped systems; the effect of damping on autoresonance is considered here. In particular, the question of a threshold for entering autoresonance in a dissipative system is investigated. The resulting theory accurately describes the behavior of experiments on the diocotron mode in pure-electron plasmas. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1338539]

I. INTRODUCTION

When a nonlinear oscillator with an amplitude dependent frequency is driven by a swept frequency drive, the oscillator's amplitude will, in some circumstances, automatically adjust itself so that the oscillator's nonlinear frequency closely matches the drive frequency. This phenomenon is called autoresonance, and allows the amplitude of the oscillator to be controlled simply by sweeping the drive frequency. Autoresonant effects were first observed in particle accelerators,[1] and have since been noted in atomic physics,[2,3] fluid dynamics,[4] plasmas,[5,6] and nonlinear waves.[7,8]

Experimental evidence[5] and theoretical analysis[6] both show that autoresonance occurs only when the normalized drive amplitude, ϵ, exceeds the critical value

$$\epsilon_c = \frac{1}{\sqrt{\beta\omega_0}}\left(\frac{\alpha}{3\mu}\right)^{3/4}, \tag{1}$$

where α is the sweep rate, ω_0 is the linear frequency of the oscillator, β is a nonlinearity parameter discussed later,[5,6] and μ is close to unity and describes a correction due to inertia. However, this scaling was found for systems without damping ($Q=\infty$) and the inertial corrections were ignored. In this paper we show that autoresonance will still occur with damping if the damping is not too great. When the drive ϵ is not much greater than the critical drive ϵ_c, the damping cannot be large, but when the drive is significantly greater than ϵ_c, the system can tolerate quite substantial damping. In Sec. II we show how autoresonance theory is modified to include the effects of damping, and in Sec. III we present experimental measurements confirming the theoretical results.

II. THEORY

We will use the action-angle description of our system. This canonical representation is advantageous in studying resonantly driven oscillators since, in the absence of driving and dissipation, the oscillator action I is constant, while the angle rotates uniformly in time, $\theta=\theta_0+\Omega(I)t$, $\Omega(I)$ being oscillator's frequency. In weakly driven and damped systems the action becomes a *slow* variable and the equations describing the evolution of the system are[6]

$$\frac{dI}{dt} = -\gamma I - 2\epsilon I^{1/2}\sin\Phi, \tag{2}$$

$$\frac{d\Phi}{dt} = \Omega(I) - \omega_0 - \alpha t - \epsilon I^{-1/2}\cos\Phi, \tag{3}$$

where $\Phi=\theta-\int\omega(t)dt$ is the phase mismatch between the oscillator and the drive and $\omega(t)$ is the driving frequency. We use normalized (dimensionless) action I and oscillator strength ϵ, and γ is the damping rate. We start with the oscillator at rest ($I=0$) and assume a linear frequency chirp, $\omega(t)=\omega_0+\alpha t$, such that the driving frequency passes the linear oscillator's frequency, $\omega_0=\Omega(0)$ at $t=0$. Different $\Omega(I)$ describe different systems. For example, $\Omega(I)=\omega_0/(1-\beta I)$ describes the diocotron mode used in the experiments discussed below, $\Omega(I)=\omega_0(1-I/8+\mathcal{O}(I^2))$ describes a pendulum, and $\Omega(I)=\omega_0(1+\beta I)$ describes a Duffing oscillator. We will show below that the influence of $\Omega(I)$ is well described by its linear simplification; thus, to determine the existence of autoresonance, all these systems reduce to the Duffing oscillator. With this simplification, Eq. (3) becomes

$$\frac{d\Phi}{dt} = \beta\omega_0 I - \alpha t - \epsilon I^{-1/2}\cos\Phi. \tag{4}$$

The undamped, $\gamma=0$ realization of this system was described in Ref. 6 and we will analyze the general case similarly. We begin by differentiating Eq. (4), yielding

$$\frac{d^2\Phi}{dt^2} = -\alpha + S\frac{dI}{dt} + \epsilon I^{-1/2}\sin\Phi\frac{d\Phi}{dt}, \tag{5}$$

[a]Electronic mail: joel@physics.berkeley.edu

where $S=\beta\omega_0+\epsilon/(2I^{3/2})\cos\Phi$. As in Ref. 6, we assume that the driven system enters the phase locked state $\Phi\approx 0$ at some $t<0$, and stays in this state beyond the linear resonance for some appreciable time. Substituting Eq. (2) into Eq. (5) yields

$$\frac{d^2\Phi}{dt^2}=-\alpha-S(\gamma I+2\epsilon I^{1/2}\sin\Phi)-\frac{1}{2}\left(\frac{1}{I}\frac{dI}{dt}+\gamma\right)\frac{d\Phi}{dt}, \tag{6}$$

where we have used the approximation $S\approx\beta\omega_0+\epsilon/(2I^{3/2})\equiv S(I)$.

Since γ and ϵ in Eq. (6) are both assumed to be small, the right-hand side of the equation can be expanded around the instantaneous equilibrium action I_0 given by setting the time derivative in Eq. (4) to zero, namely

$$-\alpha t+\beta\omega_0 I_0-\epsilon I_0^{-1/2}\approx 0. \tag{7}$$

This equilibrium action is simply the action plotted in the classical driven nonlinear oscillator response curves given in many references.[9–11] Note that as time increases, the equilibrium action will increase correspondingly, always keeping the oscillator frequency near the drive frequency. Using the equilibrium action, Eq. (6) reduces to

$$\frac{d^2\Phi}{dt^2}=-\frac{\partial V_{\text{pseudo}}}{\partial\Phi}-\gamma_{\text{eff}}\frac{d\Phi}{dt}, \tag{8}$$

where the pseudopotential is defined by

$$V_{\text{pseudo}}=[\alpha+\gamma I_0 S(I_0)]\Phi-2\epsilon I_0^{1/2}S(I_0)\cos\Phi, \tag{9}$$

and the effective dissipation rate is

$$\gamma_{\text{eff}}=\frac{1}{2}\left(\frac{1}{I_0}\frac{dI_0}{dt}+\gamma\right). \tag{10}$$

Note that the time dependence in the pseudopotential and the effective dissipation rate comes only through the time dependence of I_0. Thus, the system behaves like a pseudoparticle moving in a slowly varying pseudopotential in the presence of a small effective friction force.

If we temporarily neglect the dissipative terms, the analysis of Eq. (8) is equivalent to that given in Ref. 6. The pseudopotential consists of a series of tilted wells when the amplitude of the cosine term exceeds the slope of the linear term, namely when

$$2\epsilon I_0^{1/2}S(I_0)>\alpha. \tag{11}$$

If this condition is not met, the cosine term is overwhelmed by the pseudopotential tilt, and no wells exist.

Assuming that condition (11) is met, and the system is initially trapped in one of the pseudopotential wells, the system will remain trapped in that well so long as the pseudopotential changes slowly. Then the phase mismatch Φ will stay small, the oscillator and drive frequency will phaselock, I will be forced to stay near I_0, and the oscillator amplitude will increase appropriately. The condition (11) is most difficult to meet when $I_0^{1/2}S(I_0)$ is small; this quantity has a minimum at the critical action

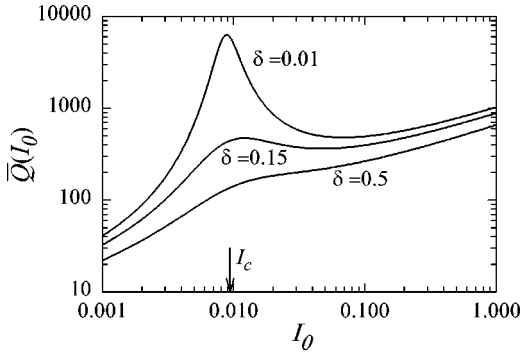

FIG. 1. The function $\overline{Q}(I_0)$ defined by Eq. (15). The system's Q must exceed this function for the pseudopotential wells to exist and for the system to stay in autoresonance. $\overline{Q}(I_0)$ is plotted as a function of I_0 for three values of drive strength $\epsilon=(1+\delta)\epsilon_c$. The arrow marks the critical action for $\delta=0$, and $\beta=0.6$.

$$I_c=\left(\frac{\epsilon}{\beta\omega_0}\right)^{2/3}. \tag{12}$$

When this formula is evaluated in the limit $\epsilon/\beta\omega_0\ll 1$ common to many realistic systems, I_c will be rather small. This is the origin of the $\Omega(I)$ invariance of this system: as only the linear correction to $\Omega(I)$ is needed, all $\Omega(I)$ can be reduced to the Duffing oscillator $\Omega(I)$.

We have found that we can slightly relax condition (11), i.e., we can write it as $2\mu\epsilon I_0^{1/2}S(I_0)>\alpha$, where $\mu\gtrsim 1$. The factor μ reflects the effect of inertia and damping: if the system briefly violates condition (11), the pseudoparticle may not have enough time to escape before the pseudopotential well is established again. Substitution of I_c into the relaxed condition (11) leads directly to the autoresonant drive amplitude threshold, Eq. (1). Numerical simulations show that $\mu\approx 1.15\equiv\mu_c$ for the undamped case. Thus the threshold [Eq. (1)] is reduced by $\mu_c^{-3/4}\approx 0.9$ by these inertial effects.

Now we return to the case where the damping is nonzero. In an analogy with condition (11), the relaxed condition for the existence of the pseudopotential wells becomes

$$2\epsilon\mu I_0^{1/2}S(I_0)>\alpha+\gamma I_0 S(I_0). \tag{13}$$

Clearly, this condition is stricter than in the case of no damping; at the minimum of $I_0^{1/2}S(I_0)$, $\epsilon>\epsilon_c$. For a given ϵ we can find the minimum $Q\equiv\omega_0/\gamma$ that will allow the system to be in autoresonance[12]

$$Q>\frac{\omega_0 I_0 S(I_0)}{2\epsilon\mu I_0^{1/2}S(I_0)-\alpha} \tag{14}$$

$$=\frac{\omega_0 I_0^{1/2}}{2\epsilon\mu-\alpha/[I_0^{1/2}S(I_0)]}\equiv\overline{Q}(I_0). \tag{15}$$

The function $\overline{Q}(I_0)$ always asymptotes to a line proportional to $I_0^{1/2}$ at large action I_0 (see Fig. 1). For drives $\epsilon=\epsilon_c(1+\delta)$, $\delta\ll 1$, not much greater than ϵ_c, $\overline{Q}(I_0)$ has a local maximum at small I_0 very near the critical action I_c. Since the system generally starts with the action near zero, it will encounter this local maximum first when the maximum ex-

ists. Assuming that the first order correction to μ near ϵ_c can be written as $\mu=\mu_c(1+r\delta)$, then, to order δ, the maximum value of $\overline{Q}(I_0)$ is

$$Q_c \approx \frac{3\omega_0 I_c^{1/2}}{2(4+3r)\delta\epsilon_c}, \tag{16}$$

where the expansion coefficient r will be found later. Substituting for I_c and ϵ_c gives

$$Q_c \approx \frac{3\omega_0}{2(4+3r)\delta(\alpha/3)^{1/2}}. \tag{17}$$

Comparison of this threshold formula with the results of numerical simulations yields excellent agreement for $r\approx -0.66$, i.e., $Q_c \approx (3\omega_0/4\delta)(\alpha/3)^{-1/2}$. If the system Q is below Q_c, autoresonance will terminate at a value of the action less than I_c; the damping has a strong effect on the system.

If the system Q is greater than Q_c, the action will successfully pass through the critical region and autoresonantly continue to a maximum action I_{0m} set by $Q=\overline{Q}(I_{0m})$. When the drive amplitude ϵ is not too much greater than the critical drive ϵ_c, this action will be significantly greater than the critical action I_c, and Eq. (15) will simplify to

$$\overline{Q}(I_0) \approx \frac{\omega_0 I_0^{1/2}}{2\epsilon}. \tag{18}$$

Inverting gives the maximum attainable action, i.e., the action at which the pseudopotential wells finally disappear

$$I_{0m} = \left(\frac{2\epsilon Q}{\omega_0}\right)^2, \tag{19}$$

as a function of the system Q. Since $I_{0m} \gg I_c$ here, the system will appear to have a sharp Q threshold: for system Q's below Q_c, the final action will be small, for Q's above Q_c the final action will be large. Indeed, I_{0m} may be so large that the growth is stopped by some other process. In this case, the system will appear to be unaffected by damping.

As shown in Fig. 1, the local maximum near I_c is small when the drive amplitude ϵ is significantly greater than the critical drive ϵ_c. If the actual system Q is not much larger than Q_c, the action may not grow much larger than I_c. In this case the Q_c threshold will not be significant. For sufficiently large ϵ it will disappear entirely.

The limit described by Eq. (18) or Eq. (19) has a physical explanation. Referring back to Eq. (2), the value of I_{0m} is simply the action at which the drive is no longer sufficient to overcome the damping losses at the most favorable value of Φ, namely $-\pi/2$. This is the conventional maximum action described in many references.

III. EXPERIMENT

The effects described above have been confirmed with experiments using the $l=1$ diocotron mode of a pure-electron plasma confined in a Malmberg–Penning trap (Fig. 2). This mode has been used to confirm many aspects of autoresonance theory.[5,6,13,14] The traps consist of a series of collimated conducting cylinders immersed in a strong, axial magnetic-field **B**. The plasma forms a cylindrical column inside a center cylinder, and appropriately biased end cylinders provide longitudinal confinement. The axial magnetic field provides radial confinement. The $\mathbf{E}\times\mathbf{B}$ drifts that result from the plasma's self-electric field cause the plasma to rotate around itself (see Fig. 3). If the plasma is moved off center, it undergoes an additional $\mathbf{E}\times\mathbf{B}$ drift from the electric field of its image. As this drift always points azimuthally, the plasma orbits around the trap center. This motion, at frequency ω_D, is called the diocotron mode and is very stable when undamped, lasting for hundreds of thousands of rotations.

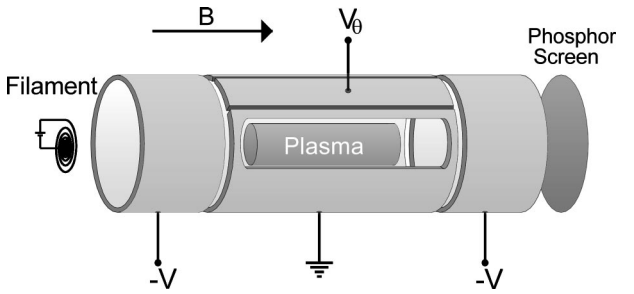

FIG. 2. Basic trap geometry. The plasma is emitted from the filament, and loaded into the trap by temporarily grounding the left cylinder. Details of the trap operation can be found in Ref. 18.

Assuming that the plasma column's charge per unit length is λ, then the electric field of its image, E, is approximately radial and constant across the plasma, $E\approx 2\lambda D/(R^2-D^2)$ (cgs-Gaussian units). Here R is the wall radius, and D is the offset of the plasma column from the center, i.e., the mode amplitude. The diocotron mode frequency ω_D follows by equating $\omega_D D$ to $c\mathbf{E}\times\mathbf{B}/B^2$, giving

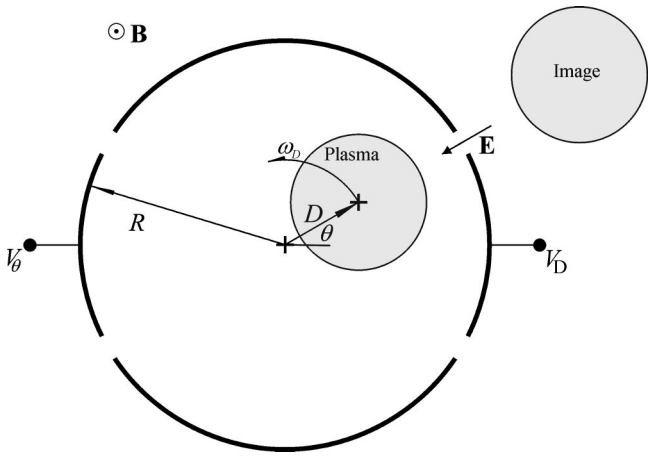

FIG. 3. Endview of the trap showing the confining wall at radius R, the plasma at angle θ and distance D from the trap center, the plasma image, the image electric field **E**, and the diocotron drift at frequency $\omega/2\pi$. For our experiments, $\omega_0/2\pi=26.5$ kHz. The mode is detected by monitoring the image charge on the pickup sector V_θ, and driven by applying a voltage to the drive sector V_D. Damping is introduced in the system by feeding back the appropriately phase-shifted signal detected by the pickup sector to the drive sector. The Q is set by the amplitude of the feedback. Further details are given in Refs. 6 and 18.

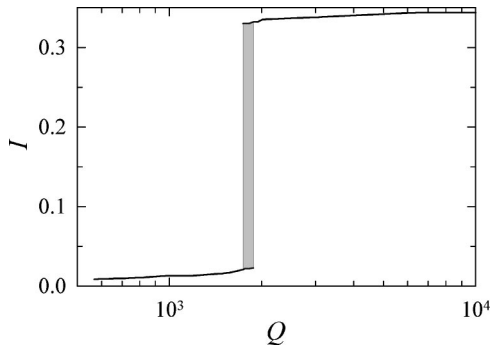

FIG. 4. The maximum action attained during an autoresonance sweep as a function of the Q. Below the threshold near $Q \approx 1800$ the action is very low, while above the threshold the action takes a big jump and quick rises to $I \approx 0.4$. (At $I \approx 0.4$ the plasma column hits the trap wall and the autoresonant interaction ends.) For Q's in the shaded region, the maximum action depends on very small differences in the initial conditions, and oscillates between the high and low values for different runs. For this data, $\delta = 0.03$, and $\alpha = 1.131 \times 10^5$ rad/s^2.

$$\omega_D = \omega_0 \left(\frac{1}{1 - I^2} \right), \tag{20}$$

where $\omega_0 = 2c\lambda/BR^2$ is the linear resonant frequency and $I = (D/R)^2$ is the action. Note that the mode frequency increases with the action.[15,16] We can determine the mode amplitude by measuring the image charge at a particular angle on the trap wall as a function of time. More precisely, we measure the time dependence of the surface charge on an azimuthal sector like the one labeled V_θ in Fig. 3. The received signal is calibrated to the displacement D by imaging the plasma on a phosphor screen at the end of the trap. We drive the diocotron mode by applying a signal to a driving sector V_D.[17] This signal creates electric fields which induce the additional drifts responsible for driving the mode.

The Q of the system can be controlled by using negative feedback.[18] This technique was developed by Warren White at U.C. San Diego, but has not been published in a refereed journal. It is used extensively by experimentalists. In brief, the signal detected on the pickup sector, V_θ, appropriately phased and amplified, is applied to the drive sector. Since the detected signal is proportional to the mode amplitude, and the mode's response to a drive is proportional to the drive strength, the mode is thus driven by a signal proportional to its amplitude. This results in exponential mode growth or damping, as selected by the phase of the feedback signal. We select the correct phase to give damping, and control the damping rate by adjusting the pickup signal amplification. The resulting damping is exponential over at least three orders of magnitude, and can be varied from $Q \approx 20$ to $Q > 5 \times 10^5$ with an accuracy of about 5%.

The experiments reported here were done at $B = 1485$ G in a trap with wall radius $R = 1.905$ cm. The plasma density was $\sim 2 \times 10^7$ cm^{-3}, temperature $T = 1$ eV, and plasma radius 0.6 cm. The measured linear diocotron frequency[19] was ~ 26.5 kHz. The plasma was confined within negatively biased cylinders separated by 10.25 cm. Finite length and plasma radius effects, discussed in Ref. 20, increase the linear frequency from that given by Eq. (20) by $\sim 40\%$. These effects also change the frequency's nonlinear dependence. We find $\beta = 0.6$ in the formula $\Omega(I) = \omega_0/(1 - \beta I)$ discussed in the theory section.

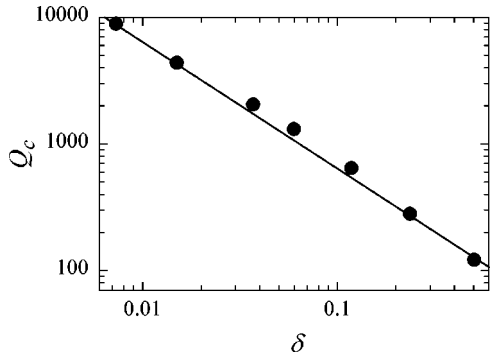

FIG. 5. The measured minimum autoresonant Q_c as a function of the drive amplitude $\epsilon = (1 + \delta)\epsilon_c$. The results (circles) are compared to the prediction (line) of Eq. (17); there are no adjustable parameters. Note that $Q_c \propto 1/\delta$. The sweep rate was $\alpha = 1.131 \times 10^7$ rad/s^2. The parameter δ is relative to the measured ϵ_c, which can be determined to better than 0.2%. The Q threshold value can be determined to about 5%.

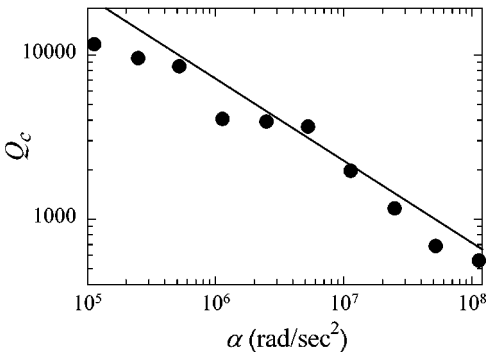

FIG. 6. The measured minimum autoresonant Q_c as a function of the sweep rate α, for $\delta = 0.03$. The results (circles) are compared to the prediction (line) of Eq. (17); there are no adjustable parameters. Note that $Q_c \propto 1/\sqrt{\alpha}$. The parameter δ is relative to the measured ϵ_c.

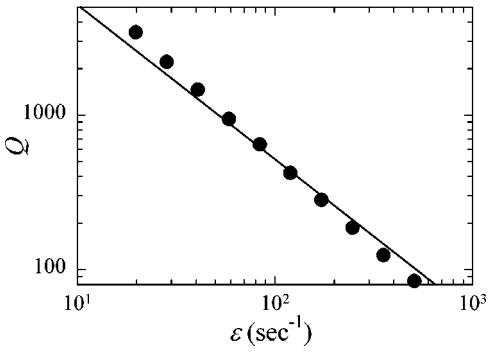

FIG. 7. The measured minimum autoresonant Q as a function of the drive strength ϵ, that allows the mode to autoresonantly grow to at least $I = 0.39$. The results (circles) are compared to the prediction (line) of Eq. (18); there are no adjustable parameters, but the drive strength ϵ must be related to the unnormalized drive amplitude used in the experiments. Previously (see Ref. 6) we showed that $\epsilon = cA_1 V_{pp}/BR_w^2$, where A_1 comes from the geometry, and V_{pp} is the voltage applied to the drive sector V_D. From the geometry and the other system parameters, $\epsilon = 460 V_{pp}$. The sweep rate was $\alpha = 1.131 \times 10^7$ rad/s^2. Note that $Q \propto 1/\epsilon$.

The results of measuring the maximum action attained at a given Q are shown in Fig. 4. As expected, there is a large threshold since ϵ is near ϵ_c. Below the threshold, the mode amplitude never makes it past the critical action. Above the threshold, the mode grows until it hits the trap wall.

Figure 5 compares the measured minimum Q that yields autoresonance to the value predicted by Eq. (17) as δ varies. The data follows the predicted $1/\delta$ scaling. Figure 6 compares the minimum autoresonant Q to the predicted value as the sweep rate α varies. As expected, the Q scales as $1/\sqrt{\alpha}$.

We have also measured (Fig. 7) the minimum Q that permits the mode to grow to at least $I=0.39$ as a function of the drive strength ϵ. This data falls in the conventional regime in which the energy lost to damping overcomes the energy supplied by the drive. As expected, the results are close to that predicted by Eq. (18).

IV. CONCLUSION

Autoresonance is a very general phenomenon found in many nonlinear oscillators. Previous papers have described autoresonance with drives at the fundamental,[5,6] subharmonics[13,21] and superharmonics,[14] but studied undamped systems. Here we found that the presence of damping does not necessarily inhibit autoresonance. As in the undamped systems, the drive strength must exceed a critical value [Eq. (1)], but if the drive is significantly stronger than this critical value, autoresonance will occur up to a value of the oscillator action set by the rate at which the drive can pump energy into the oscillator. This is the steady-state limit, familiar from many references on nonlinear oscillators. Close to the critical drive strength, however, there is a new, significantly more stringent limit on the damping, which had not been previously identified.

ACKNOWLEDGMENTS

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