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Kinetic-ballooning-mode theory in general geometry

KINETIC-BALLOONING-MODE THEORY IN GENERAL GEOMETRY

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ABSTRACT. A systematic procedure for studying the influence of kinetic effects on the stability of MHD ballooning modes is presented. The ballooning mode formalism, which is particularly effective for analysing high-mode-number perturbations of a plasma in toroidal systems, is used to solve the Vlasov-Maxwell equations for modes with perpendicular wavelengths on the scale of the ion gyroradius. The local stability on each flux surface is determined by the solution of three coupled integro-differential equations which include effects due to finite gyroradius, trapped particles, and wave-particle resonances. More tractable forms of these equations are then obtained in the low- ($\omega < \omega_{bi}, \omega_{ti}$) and intermediate- ($\omega_{bi}, \omega_{ti} < \omega < \omega_{be}, \omega_{te}$) frequency regimes with ω_{bj} and ω_{tj} being the average bounce and transit frequencies of each species. After further simplifying approximations, the kinetic results here are shown to be reducible to the MHD-ballooning-mode equations in the analogous limits, $\omega \lesssim \omega_s$ where $\omega_s = c_s/L_c$, with c_s being the acoustic speed and L_c the connection length.

1. INTRODUCTION

One of the most important practical problems in the area of magnetic-confinement research has been to calculate properly the limiting beta (ratio of plasma to magnetic pressure) for stability in toroidal systems. These theoretical estimates are generally obtained from ideal magnetohydrodynamic (MHD) calculations determining the stability of the plasma against high-mode-number perturbations called ballooning modes [1, 2]. Extensions of the MHD analysis to include resistivity have led to results indicating that these critical beta values could be lower [3]. On the other hand, recent experimental results from the ISX-B tokamak have given no evidence of a beta limit caused by ballooning instabilities [4]. It is, therefore, of some interest to examine whether kinetic effects, absent in the MHD-analysis, could significantly modify the beta criteria. The starting point of such an investigation involves obtaining an appropriate set of kinetic equations governing ballooning modes. In this paper, the primary aim is to present these equations and to establish their relationship to the familiar ideal-MHD ballooning-mode equations [3, 5].

It is now generally agreed that the most effective method for treating high-mode-number perturbations in toroidal systems is to adopt the so-called ballooning representation [3, 5 – 7]. Since these perturbations are characterized by short wavelengths perpendicular to the magnetic field and long parallel wavelengths (on the scale of the equilibrium variations, L), the eikonal representation is a natural choice. To resolve the resultant complications associated with the periodicity constraints for a torus with a sheared magnetic field, the ballooning transformation maps the familiar poloidal angle-like variable onto an infinite domain where periodicity is not required. This approach has proven to be quite successful in dealing with the MHD ballooning modes and has also been found useful in the kinetic treatment of electrostatic drift-type instabilities in toroidal systems when the characteristic perpendicular wavelengths are comparable to the ion gyroradius, ρ_i [8 – 10]. The extension to a general formulation encompassing both electrostatic and electromagnetic modes, thus requiring the solution of the Vlasov-Maxwell equations, has been described in several earlier papers [10 – 12]. Collisional effects associated, for example, with trapped-particle scattering, have usually been estimated by employing Krook-model operators [10]. For the highly collisional regimes where trapping effects can be ignored, more sophisticated collision operators have been used in particular cases [7, 12].

To investigate kinetic modifications to the MHD ballooning modes, attention is focused in the present paper on the collision-free Vlasov-Maxwell set of equations which govern both drift and Alfvén waves. Although the ideal-MHD ballooning-mode equation is

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derived from an energy principle strictly valid only in highly collisional regimes, it is well known that the form of this energy principle is closely related to the kinetic-energy principle derived for a collisionless plasma merely modified by trapped-particle effects [13 – 16]. By comparing the kinetic equations governing the Alfvén modes with the ideal-MHD ballooning-mode equation, it should be possible to assess the importance of additional kinetic effects due to finite gyroradius and wave-particle resonances. The kinetic effects noted here can also give rise to a finite parallel electric field which is absent in the usual MHD analysis.

The remainder of this paper is organized as follows. In Section 2, the ballooning representation is applied to the Vlasov equation governing an arbitrary-beta plasma in general geometry. The usual small-gyroradius ($\rho_i/L \ll 1$), low-frequency ($\omega/\Omega_i \ll 1$, with Ω_i being the ion gyrofrequency) ordering is adopted, and solutions are obtained for fully electromagnetic perturbations (including effects due to field compression). Using this result, with the quasi-neutrality condition and two components of Ampère's law, then generates a system of three coupled one-dimensional integral equations determining Φ (the perturbed electrostatic potential), $A_{\parallel}$ (the perturbed parallel magnetic vector potential), and $\delta B_{\parallel}$ (the perturbed parallel magnetic field).

The MHD equations with which the kinetic results are compared are presented in Section 3.1. To facilitate this comparison, it is necessary to reduce the set of kinetic equations derived in Section 2 to more tractable forms by considering particular frequency ranges. Specifically, in Section 3.2, the mode frequency is taken to fall below the particle bounce frequency (for both electrons and ions) along the magnetic field, i.e. $\omega < \omega_{bi}$, with ω_{bi} being the average ion bounce frequency. In this low-frequency regime, the trapped-particle responses of both species are taken into account. As expected, this system of equations is closely related to those derived in earlier kinetic studies [17]. By taking beta to be a small expansion parameter (a reasonable approximation for tokamaks), these reduce to two equations, and, if further simplifying approximations are made, it is possible to recover the usual ideal-MHD ballooning-mode equations [6].

For higher-mode number perturbations it becomes necessary to consider an intermediate frequency range where the frequency of the wave lies above the bounce and transit frequencies of the ions but below those for the electrons, i.e. $\omega_{bi}, \omega_{ti} < \omega < \omega_{be}, \omega_{te}$. In this regime, trapping effects are retained only for the electrons, and the appropriate system of three coupled equations is presented in Section 3.3. The reduction to two equations for $\beta < 1$, and the further simplification to a single governing equation, obtainable from an analogous limit of the ideal-MHD equations [3], are also demonstrated in this section.

Finally, in Section 4, the main results of this paper are briefly summarized, and the implications of the results discussed.

2. GENERAL KINETIC THEORY

In theoretical studies of axisymmetric toroidal systems, it is often convenient to adopt the ψ, χ, ζ co-ordinate system, where ψ is the poloidal flux within a magnetic surface and acts as the radial variable, χ is the poloidal angle-like co-ordinate, and ζ is the toroidal angle. The spatial gradient operator and volume element in these co-ordinates are, respectively,

$$\nabla = \vec{e}_\psi \, RB_\chi \frac{\partial}{\partial\psi} + \vec{e}_\chi \frac{1}{JB_\chi}\frac{\partial}{\partial\chi} + \vec{e}_\zeta \frac{1}{R}\frac{\partial}{\partial\zeta} \tag{2.1}$$

and

$$d\tau = J \, d\psi \, d\zeta \, d\chi \tag{2.2}$$

where $\vec{e}_\psi$, $\vec{e}_\chi$, and $\vec{e}_\zeta$ are unit vectors, R is the major radius, B_χ is the poloidal magnetic field, and J is the Jacobian defined by

$$J \equiv (\nabla\psi \cdot \nabla\zeta \times \nabla\chi)^{-1} \tag{2.3}$$

In general, the equilibrium magnetic field can be expressed in the form

$$\vec{B} = -\nabla\psi \times \nabla\zeta + I(\psi,\chi)\nabla\zeta \tag{2.4}$$

with I being a prescribed function. The safety factor then becomes

$$q = \frac{1}{2\pi} \oint d\chi IJ/R^2 \tag{2.5}$$

For tokamak plasmas it is usually appropriate to consider a local Maxwellian equilibrium distribution,

$$F^{(0)}(E,\psi) = F_M$$

with E being the kinetic energy per unit mass, implying a corresponding isotropic pressure, p. In such cases the quantity I in Eq.(2.4) becomes a function of ψ only, and the explicit form of the Jacobian is determined by

$$\frac{1}{J}\frac{\partial}{\partial\psi}(JB_\chi^2) = -\frac{d}{d\psi}p - \frac{I}{R^2}\frac{d}{d\psi}I \tag{2.6}$$

The superscript in $F^{(0)}$ refers to the fact that this is the lowest order (in ρ_i/L) of the equilibrium distribution function. Regarding the basic ordering, ϵ is taken as the fundamental smallness parameter with

$$\varepsilon \sim \rho_i/L \sim \omega/\Omega_i$$

$k_\perp\rho_i$ is taken to be $O(\epsilon^0)$ while $k_\parallel\rho_i$ is taken to be $O(\epsilon)$.

To solve the well known problem of satisfying the periodicity constraint in χ when using an eikonal form for high-mode-number perturbations, the ballooning representation is introduced for all perturbed quantities, i.e.

$$\Phi(\psi,\chi,\zeta) = \sum_{p=-\infty}^{\infty} \bar{\Phi}(\psi,\chi-2\pi p,\zeta) \tag{2.7}$$

with a similar form for f, the perturbed particle distribution function, and for $\vec{A}$, the perturbed magnetic vector potential. Note here that although $\bar{\phi}$, which extends from $-\infty$ to ∞ in χ, is not itself a periodic function, the infinite sum is necessarily periodic. Also, to ensure convergence of this sum, $\bar{\phi}$ must vanish sufficiently fast as $\chi\to\pm\infty$. As emphasized in earlier work [9], the linear operator, $\mathscr{L}$, acting on the perturbed quantities and determining the eigenfrequency, ω, is periodic in χ. This implies that if $\mathscr{L}\bar{\phi}(\chi)=0$, then $\mathscr{L}\phi(\chi)=0$ is automatically satisfied. In short, the problem now reduces to one of solving for the perturbed quantities over an infinite range in χ with no periodicity constraint.

As noted in Section 1, the perturbations of interest are characterized by short perpendicular and long parallel wavelengths, i.e. $k_\perp\rho_i\sim 1$ and $k_\parallel\rho_i\sim\epsilon$. Hence, it is appropriate to adopt the eikonal representation along with the ballooning transformation of Eq.(2.7), i.e.

$$\begin{pmatrix}\bar{f}\\ \bar{\Phi}\\ \vec{A}\end{pmatrix} = \begin{pmatrix}\hat{f}\\ \hat{\Phi}\\ \vec{\hat{A}}\end{pmatrix}\exp(iS/\varepsilon) \tag{2.8}$$

with S being the eikonal accounting for the rapid cross-field variations and $\hat{f}$, $\hat{\psi}$, and $\hat{A}$ accounting for the slow variations along the field line. The specific form for S is determined by the requirement $k_\parallel\rho_i\sim\epsilon$, which implies

$$\vec{n}\cdot\nabla S = 0 \tag{2.9}$$

with $\vec{n}\equiv\vec{B}/B$. Since all perturbations can be Fourier-decomposed in the ignorable co-ordinate ζ, Eq.(2.9) yields

$$\frac{1}{BJ}\left(\frac{\partial}{\partial\chi} + in\frac{IJ}{R^2}\right)S = 0 \tag{2.10}$$

with n being the toroidal mode number, so that the most general form for $S/\epsilon\equiv nS$ must be

$$S = \left[\zeta - \int_0^\chi d\chi'\, IJ/R^2 + \int^\psi k(\psi)\,d\psi\right] \tag{2.11}$$

with $k(\psi)$ being a function to be determined by a higher-order radially non-local analysis [6, 10]. In terms of S, the perpendicular wavenumber can be expressed as

$$\vec{k}_\perp = k_\psi\vec{e}_\psi + k_b\vec{e}_b = n\nabla S \tag{2.12}$$

with $\vec{e}_b\equiv\vec{n}\times\vec{e}_\psi$. Using Eq.(2.11) then gives the effective radial and binormal components of $\vec{k}_\perp$ as

$$k_\psi = -n\,RB_\chi\left[\int_0^\chi d\chi'\frac{\partial}{\partial\psi}(IJ/R^2) - k(\psi)\right] \tag{2.13}$$

and

$$k_b = nB/RB_\chi \tag{2.14}$$

To take into account compressional electromagnetic effects, it is necessary to consider $\vec{A}_\perp$. Adopting the Coulomb gauge,

$$\nabla\cdot\vec{A} = 0$$

then gives

$$A_b = -(k_\psi/k_b)A_\psi \tag{2.15}$$

with

$$\vec{A}_\perp = A_\psi\vec{e}_\psi + A_b\vec{e}_b$$

It is convenient here to introduce the perturbed parallel magnetic field, $\delta B_\parallel$, as the third scalar dependent variable along with Φ and $A_\parallel$. Since

$$\delta B_\parallel = \vec{n}.(\nabla \times \vec{A}) = \vec{e}_\psi \cdot \nabla A_b - \vec{e}_b \cdot \nabla A_\psi$$

it follows that

$$A_\psi = (ik_b / k_\perp^2)\, \delta B_\parallel \tag{2.16}$$

Employing the ballooning-eikonal representation of Eqs (2.8) and (2.11) for the perturbed quantities and a local Maxwellian equilibrium distribution of the usual form,

$$F^{(o)} = F_M = n_o \left(\frac{m}{2\pi T}\right)^{3/2} \exp(-\, mE/T) \tag{2.17}$$

with T being the temperature, the governing linearized gyrokinetic equation [10, 11] in the collisionless limit becomes

$$\frac{v_\parallel}{JB} \frac{\partial}{\partial \chi} \hat{h} - i\hat{h}\,(\omega - \vec{k}_\perp \cdot \vec{v}_D)$$

$$= -\frac{ie}{T} F_m \quad (\omega - \omega_*^T) \left[J_o(\alpha)\left(\hat{\phi} - \frac{v_\parallel}{c}\hat{A}_\parallel\right)\right.$$

$$\left. + J_1(\alpha)\frac{v_\perp}{k_\perp}\frac{\delta \hat{B}_\parallel}{c}\right] \tag{2.18}$$

with

$$\hat{f} = -\frac{e\hat{\phi}}{T} F_m + \hat{h} \exp\,(iL) \tag{2.19}$$

$$\vec{v}_D = \frac{1}{\Omega}\vec{n} \times (\mu \nabla B + v_\parallel{}^2\, \vec{n}\cdot\nabla n) \tag{2.20}$$

where

$$\mu \equiv v_\perp{}^2/2B$$

and equilibrium electric fields are ignored,

$$\omega_*{}^T = \omega_* \left[1 + \eta \left(\frac{mE}{T} - \frac{3}{2}\right) \right] \tag{2.21}$$

with

$$\eta \equiv d\ell n\, T/d\ell n\, n_o$$

The diamagnetic drift frequency ω_* is defined by

$$\omega_* = \frac{ncT}{e} \frac{d}{d\psi} \ell n(n_o) \tag{2.22}$$

$$\alpha = k_\perp v_\perp / \Omega$$

and

$$\vec{v}_\perp = v_\perp\, (\cos\phi\, \vec{e}_\psi + \sin\phi\, \vec{e}_b)$$

with ϕ being the gyrophase.

The last term in Eq.(2.18) accounts for compressional effects and is obtained from the appropriate gyrophase average of

$$\vec{A}_\perp \cdot \vec{v}_\perp$$

Specifically, using Eqs (2.15) and (2.16),

$$\left\langle \vec{A}_\perp \cdot \vec{v}_\perp \right\rangle_\phi = i\delta B_\parallel \frac{v_\perp}{k_\perp} \frac{1}{2\pi} \int_o^{2\pi} d\phi\, \cos(\phi + \gamma)\, \exp\,(-iL) \tag{2.23}$$

with

$$k_\perp = n|\nabla S|$$

J_1 being the Bessel function of the first kind, and

$$L \equiv \frac{v_\perp}{\Omega}\left(k_\psi \sin\phi - k_b \cos\phi\right) = -\frac{k_\perp v_\perp}{\Omega} \cos\,(\phi + \gamma) \tag{2.24}$$

where

$$\gamma \equiv \tan^{-1}(k_\psi/k_b) \tag{2.25}$$

The appropriate boundary conditions for Eq.(2.18) are that

$$\hat{h} \to 0 \quad \text{as} \quad |\chi| \to \infty$$

for circulating particles and that the forward and backward streams match at the turning points for the trapped particles. Following the usual procedure of introducing an integrating factor of the form

$$I_a^b = \int_a^b d\chi'\, (\omega - \omega_D)\, JB/|v_\parallel| \tag{2.26}$$

with

$$\omega_D \equiv \vec{k}_\perp \cdot \vec{v}_D$$

and assuming that ω has a positive imaginary part corresponding to a growing eigenmode to ensure convergence as $|\chi| \to \infty$, the general solution for circulating particles is

$$\hat{h}_{\pm}(\chi) = \mp \frac{ie}{T} F_m(\omega - \omega_*^T) \int_{\mp\infty}^{\chi} d\chi' (J\bar{B}/|v_\parallel|) \exp\left(\mp i I_{\chi}^{\chi'}\right) \times \left[J_o\left(\hat{\phi} \mp \frac{|v_\parallel|}{c} \hat{A}_\parallel\right) + J_1 \frac{v_\perp}{k_\perp} \frac{\delta\hat{B}_\parallel}{c}\right] \quad (2.27)$$

with $\pm$ referring to the sign of $v_\parallel$ for the particles in question.

In the case of trapped particles, a similar procedure is followed with the specific boundary conditions at the turning points being

$$\hat{h}_+(\chi_1) = \hat{h}_-(\chi_1) \quad \text{and} \quad \hat{h}_+(\chi_2) = \hat{h}_-(\chi_2) \quad (2.28)$$

Here, χ_1 and χ_2 refer to the nearest reflection points surrounding χ. After some straightforward algebraic steps, the results for the trapped-particle perturbed distribution function yield:

$$\frac{1}{2}\left(\hat{h}_+ + \hat{h}_-\right) = \frac{e}{T} F_m(\omega - \omega_*^T) \left\{ \int_{\chi}^{\chi_2} d\chi' \frac{JB}{|v_\parallel|} \left[\left(J_o\hat{\phi} + J_1 \frac{v_\perp}{k_\perp} \frac{\delta\hat{B}_\parallel}{c}\right) \times \cos I_{\chi_2}^{\chi'} \cos I_{\chi_1}^{\chi} + iJ_o \frac{\hat{A}_\parallel |v_\parallel|}{c} \sin I_{\chi_2}^{\chi'} \sin I_{\chi_1}^{\chi} \right] + \int_{\chi_1}^{\chi} d\chi' \frac{JB}{|v_\parallel|} \left[\left(J_o\hat{\phi} + J_1 \frac{v_\perp}{k_\perp} \frac{\delta\hat{B}_\parallel}{c}\right) \cos I_{\chi_1}^{\chi'} \cos I_{\chi_2}^{\chi} + iJ_o \frac{\hat{A}_\parallel |v_\parallel|}{c} \sin I_{\chi_1}^{\chi'} \cos I_{\chi_2}^{\chi} \right] \right\} \left(\sin I_{\chi_1}^{\chi_2}\right)^{-1} \quad (2.29)$$

and

$$\frac{1}{2}\left(\hat{h}_+ - \hat{h}_-\right) = i\frac{e}{T} F_m(\omega - \omega_*^T) \left\{ \int_{\chi}^{\chi_2} d\chi' \frac{JB}{|v_\parallel|} \left[\left(J_o\hat{\phi} + J_1 \frac{v_\perp}{k_\perp} \frac{\delta\hat{B}_\parallel}{c}\right) \times \cos I_{\chi_2}^{\chi'} \sin I_{\chi_1}^{\chi} + iJ_o \frac{\hat{A}_\parallel |v_\parallel|}{c} \sin I_{\chi_2}^{\chi'} \sin I_{\chi_1}^{\chi} \right] + \int_{\chi_1}^{\chi} d\chi' \frac{JB}{|v_\parallel|} \left[\left(J_o\hat{\phi} + J_1 \frac{v_\perp}{k_\perp} \frac{\delta\hat{B}_\parallel}{c}\right) \sin I_{\chi_1}^{\chi} \cos I_{\chi_1}^{\chi'} + iJ_o \frac{\hat{A}_\parallel |v_\parallel|}{c} \sin I_{\chi_2}^{\chi} \sin I_{\chi_1}^{\chi'} \right] \right\} \left(\sin I_{\chi_1}^{\chi_2}\right)^{-1} \quad (2.30)$$

The basic set of three coupled one-dimensional integral equations governing the eigenmodes of the system can now be written in terms of the general solutions for the perturbed distribution functions given by Eqs (2.27), (2.29), and (2.30). These are:

(i) the quasi-neutrality condition,

$$0 = -\sum \frac{n_o e^2}{T} \hat{\phi} + \sum 2\pi e \int dE d\mu \frac{B}{|v_\parallel|} (\hat{h}_+ + \hat{h}_-) J_o \quad (2.31)$$

with the summation being over particle species,

(ii) the parallel current equation (component of Ampère's law along $\vec{B}$),

$$k_\perp^2 \hat{A}_\parallel = (k_b^2 + k_\psi^2) \hat{A}_\parallel = \frac{4\pi}{c} \hat{j}_\parallel \quad (2.32)$$

with k_b and k_ψ given by Eqs (2.12) and (2.13) and

$$\hat{j}_\parallel = \sum 2\pi e \int dE d\mu B \, (\hat{h}_+ - \hat{h}_-) \, J_o \quad (2.33)$$

(iii) the radial current equation (one component of Ampère's law perpendicular to $\vec{B}$),

$$ik_b \delta\hat{B}_\parallel = \frac{4\pi}{c} \hat{j}_\psi = \frac{4\pi}{c} \sum 2\pi e \int dE d\mu \frac{B}{|v_\parallel|} v_\perp (\hat{h}_+ + \hat{h}_-) \left\langle \cos\phi \exp(iL) \right\rangle_\phi$$

or

$$\delta\hat{B}_\parallel = -\frac{1}{k_\perp} \frac{4\pi}{c} \sum 2\pi e \int dE d\mu \frac{B}{|v_\parallel|} v_\perp (\hat{h}_+ + \hat{h}_-) J_1 \quad (2.34)$$

Instead of constructing $\hat{j}_\parallel$ as indicated in Eq.(2.33), it is often convenient [18] to obtain an equation governing $\hat{j}_\parallel$ by taking the moment

$$\sum e \int d^3v \, \exp(iL)$$

of the gyrokinetic equation, Eq.(2.18), The result of this operation is

$$\frac{i}{\omega}\frac{1}{J}\frac{\partial}{\partial\chi}\left(\frac{\hat{j}_{\parallel}}{B}\right) = -\sum\frac{e^2}{T}\int d^3v F_m \left\{\left[1-\left(1-\frac{\omega_*^T}{\omega}\right)J_o^2\right]\hat{\phi}\right.$$

$$\left. -\left(1-\frac{\omega_*^T}{\omega}\right)J_oJ_1\frac{v_\perp}{k_\perp}\frac{\delta\hat{B}_\parallel}{c}\right\} + \sum e\int d^3v\ (\hat{h}_+ + \hat{h}_-)\frac{\omega_D}{\omega}J_o$$

$$+\frac{i}{\omega}\sum e\int d^3v\ (\hat{h}_+ - \hat{h}_-)|v_\parallel|\frac{1}{JB}\frac{\partial}{\partial\chi}J_o \qquad (2.35)$$

This form, which automatically accounts for much cancellation implicit in Eq.(2.33) ensuing from the quasi-neutrality condition given by Eq.(2.31), will be used in Sections 3.B and 3.C.

3.1. MHD ballooning-mode equations

As noted in Section 1, it is necessary to consider limits where the mode frequency is either large or small compared to the average transit and bounce frequencies of the particles in order to reduce Eqs (2.31), (2.34) and (2.35) to more tractable forms. Before presenting the results of these low-and-intermediate-frequency-regime calculations in the next two sub-sections, it is appropriate to first recall the ideal-MHD equations with which the kinetic results will be compared. As shown in Ref.[3], the ideal-MHD ballooning modes are governed by two coupled ordinary differential equations. In terms of the variables used in the present paper these are

$$\frac{1}{J}\frac{\partial}{\partial\chi}\left(\frac{1}{JB^2}|\nabla S|^2\frac{\partial}{\partial\chi}\hat{\psi}\right) + \frac{n_oM_i\omega^2}{B^2}|\nabla S|^2\hat{\psi}$$

$$+ 2\kappa_w\left[\frac{1}{B^2}(\nabla P\times\vec{B}\cdot\nabla S)\hat{\psi} + \frac{B^2\gamma P}{B^2+\gamma P}\left(\frac{1}{J}\frac{\partial}{\partial\chi}\hat{\xi} - 2\kappa_w\hat{\psi}\right)\right] = 0 \qquad (3.1)$$

and

$$\frac{1}{J}\frac{\partial}{\partial\chi}\left[\frac{B^2\gamma P}{B^2+\gamma P}\left(\frac{1}{J}\frac{\partial}{\partial\chi}\hat{\xi} - 2\kappa_w\hat{\psi}\right)\right] + n_oM_i\omega^2B^2\hat{\xi} = 0 \qquad (3.2)$$

where

$$\kappa_w \equiv \frac{1}{B^2}(\vec{n}\cdot\nabla\vec{n})\times\vec{B}\cdot\nabla S \qquad (3.3)$$

$\hat{\psi}$ is the stream function for the cross-field displacement, $\hat{\xi}$ is the displacement along the field line, γ is the ratio of specific heats, and ∇S is specified by Eqs (2.12) to (2.14). Noting that the ratio of the second to the first term in Eq.(3.2) is of order $(\omega/\omega_s)^2$ where

$$\omega_s \sim (\gamma P/n_oM_i)^{1/2}/L_c$$

is the sound transit frequency, with L_c being the connection length, it is clear that in the limit

$$(\omega/\omega_s)^2 \ll 1$$

this equation implies that

$$\frac{1}{J}\frac{\partial}{\partial\chi}\hat{\xi} - 2\kappa_w\hat{\psi} \simeq 0 \qquad (3.4)$$

Hence, Eq.(3.1) in this low-frequency limit reduces to the familiar form of the single second-order differential equation treated in Ref.[6], i.e.

$$\frac{1}{J}\frac{\partial}{\partial\chi}\left(\frac{1}{JB^2}|\nabla S|^2\frac{\partial}{\partial\chi}\hat{\psi}\right) + \frac{n_oM_i\omega^2}{B^2}|\nabla S|^2\hat{\psi}$$

$$+ 2\kappa_w\frac{1}{B^2}\left(\nabla P\times\vec{B}\cdot\nabla S\right)\hat{\psi} = 0 \qquad (3.5)$$

It is also of interest to note that in the opposite limit,

$$(\omega/\omega_s)^2 \gg 1$$

Eqs (3.1) and (3.2) again reduce to a single second-order differential equation. Specifically, for Alfvén modes,

$$\omega \sim \omega_A \sim \omega_s\beta^{-1/2}$$

so that for

$$\beta \ll 1,\ \omega_A \gg \omega_s$$

In this case, Eq.(3.2) implies that $\hat{\xi} \cong 0$. Noting also that

$$\beta \sim \gamma P/B^2 \ll 1$$

Eq.(3.1) reduces to

$$\frac{1}{J}\frac{\partial}{\partial\chi}\left(\frac{1}{JB^2}|\nabla S|^2\frac{\partial}{\partial\chi}\hat{\psi}\right) + \frac{n_oM_i\omega^2}{B^2}|\nabla S|^2\hat{\psi}$$

$$+ 2\kappa_w\frac{1}{B^2}\left(\nabla P\times B\cdot\nabla S\right)\hat{\psi} - 4\gamma P(\kappa_w)^2\hat{\psi} = 0 \qquad (3.6)$$

Equations (3.5) and (3.6) will be compared with the analogous kinetic equations obtained, respectively, in the low ($\omega < \omega_{bi}, \omega_{ti}$) and intermediate ($\omega > \omega_{bi}, \omega_{ti}$) frequency regimes. Since ω_{bi}, ω_{ti} are roughly equivalent in magnitude to ω_s, it is expected that the results obtained in Sections 3.2 and 3.3 should correspond closely to Eqs (3.5) and (3.6) when the kinetic effects (due to finite gyroradius, trapped particles, etc.) are weak.

3.2. Kinetic analysis in the low-frequency limit

In the low-frequency regime the average transit and bounce frequencies of ions and electrons are taken to be much higher than the mode frequencies of interest. Recall that

$$\omega_b, \omega_t \sim v_T/L_c$$

with v_T being the thermal velocity and L_c being the connection length, so that, within this approximation,

$$\omega L_c/v_T$$

can be used as a small expansion parameter. The low-frequency modes in general are of particular interest because ideal-MHD theory indicates that near marginal stability $\omega \to 0$. Earlier studies have indicated that if finite-gyroradius effects are taken into account for these modes, a finite oscillation frequency close to the ion diamagnetic drift frequency ($\omega \sim \omega_*$) will result at marginal stability [19, 20]. Hence, the requirement that $\omega L_c/v_T \ll 1$ with $\omega \sim \omega_*$ implies that only long-perpendicular-wavelength modes are of interest here. This property, together with the assumption that the ballooning modes are sufficiently localized in the extended poloidal variable, then allows the use of the small-argument expansion of the Bessel functions in the solutions for the perturbed distributions.

Noting that in the low-frequency regime I_a^b (defined by Eq.(2.26)) is a first-order quantity (i.e. of order $\omega L_c/v_{Ti}$), the expression for $\hat{h}$ governing the circulating particle responses and given in Eq.(2.27) reduces in lowest order to

$$\frac{1}{2}(\hat{h}_+ + \hat{h}_-) = \frac{e}{T} F_m \left(1 - \frac{\omega_*^T}{\omega}\right) \hat{\psi}_\parallel \tag{3.7}$$

with

$$\hat{\psi}_\parallel \equiv \frac{i\omega}{2c}\left[\int_{-\infty}^{\chi} d\chi' JB\, \hat{A}_\parallel - \int_{\chi}^{\infty} d\chi' JB\, \hat{A}_\parallel\right] \tag{3.8}$$

For trapped particles, the exact form for h given by Eq.(2.29) reduces in similar fashion to

$$\frac{1}{2}(\hat{h}_+ + \hat{h}_-) = \frac{e}{T} F_m \left[\left(1 - \frac{\omega_*^T}{\omega}\right)\hat{\psi}_\parallel + \frac{\omega - \omega_*^T}{\omega - \langle\omega_D\rangle}\langle \chi \rangle\right] \tag{3.9}$$

where

$$\chi \equiv \hat{\phi} - \left(1 - \frac{\omega_D}{\omega}\right)\hat{\psi}_\parallel + \frac{v_\perp^2}{2\Omega}\frac{\delta\hat{B}_\parallel}{c} \tag{3.10}$$

and the angular brackets represent the trapped-particle orbit average, e.g.

$$\langle \hat{\phi} \rangle = \frac{\int_{\chi_1}^{\chi_2} d\chi\, JB\, \hat{\phi}\,(\chi)/|v_\parallel|}{\int_{\chi_1}^{\chi_2} d\chi\, JB/|v_\parallel|} \tag{3.11}$$

To obtain Eq.(3.9) from Eq.(2.29), the approximations $\cos I_a^b \to 1$ and $\sin I_a^b \to I_a^b$ were made, and an integration by parts was carried out for the term involving $\hat{A}_\parallel$.

Substitution of Eqs (3.7) and (3.9) into the quasi-neutrality condition given by Eq.(2.31) yields

$$(1+\tau)\;(\hat{\psi}_\parallel - \hat{\phi}) + \sum_j \frac{T_e}{T_j}\int_{T_r} d^3v\, \frac{F_{mj}}{n_o}\left(\frac{\omega - \omega_*^T}{\omega - \langle\omega_D\rangle}\right)_j \langle \chi_j \rangle = 0 \tag{3.12}$$

where the velocity space integration is over trapped particles only, $\tau \equiv T_e/T_i$, and the j subscripts denote particle species. For later comparison with the ideal-MHD ballooning-mode equations, it will be of interest to consider the limit in which

$$|\omega_D/\omega| \ll 1$$

Using this as an expansion parameter and neglecting terms of order

$$(\omega_D/\omega)\;\delta\hat{B}_\parallel$$

and

$$(\omega_D/\omega)^2\,\hat{\psi}_\parallel$$

then reduces Eq.(3.12) to

$$(1+\tau)\left[\hat\psi_\parallel - \hat\phi - \frac{1}{2}\int_{T_r} d\lambda B(1-\lambda B)^{-1/2}\left\langle \hat\psi_\parallel - \hat\phi\right\rangle\right]$$

$$+\frac{3}{4}\tau\frac{\omega_{*p}}{\omega}\int d\lambda B(1-\lambda B)^{-1/2}\left[\left\langle\hat\psi_\parallel-\hat\phi\right\rangle\left\langle\frac{\bar\omega_{Di}}{\omega}\right\rangle\right.$$

$$\left.-\left\langle\psi_\parallel\right\rangle\left\langle\frac{\bar\omega_{Di}}{\omega}\right\rangle-\left\langle\lambda B\widetilde{\delta B}_\parallel\right\rangle\right]=0 \qquad (3.13)$$

with

$$\lambda \equiv \mu/E,\quad \omega_{*p}\equiv\omega_{*i}(1+\eta_i)-\omega_{*e}(1+\eta_e)$$

$$\bar\omega_{Di}\equiv\omega_{Di}(m_iE/T_i)^{-1}=[2\omega_\kappa(1-\lambda B)+\lambda B\omega_B]$$

$$\omega_\kappa\equiv[\vec n\times(\vec n\cdot\nabla\vec n)\vec k_\perp(T/m\Omega)_i$$

$$\omega_B=(\vec n\times\nabla B\cdot\vec k_\perp)(T/m\Omega B)_i$$

and

$$\widetilde{\delta B}_\parallel\equiv\frac{T_i}{M_i\Omega_i}\frac{\delta\hat B_\parallel}{c} \qquad (3.14)$$

In addition, since

$$\vec n.\nabla\vec n=(1/B^2)\nabla[(B^2/2)+4\pi\nabla P]$$

it is useful to note that

$$(\beta_i/2)\omega_{*p}=\omega_\kappa-\omega_B \qquad (3.15)$$

Equation (3.13) can lead to two different classes of eigenmodes. For

$$\omega^2\gg\omega_*\omega_D$$

the approximate solution is

$$\hat\phi=\hat\psi_\parallel$$

resulting in a predominantly electromagnetic mode with a vanishing parallel electric field, $\hat E_\parallel=0$. On the other hand, if

$$\omega^2\sim\epsilon^{1/2}\omega_*\omega_D$$

(with $\epsilon^{1/2}$ being a measure of the fraction of trapped particles), it is clear that, at sufficiently low beta, Eq.(3.13) reduces to the familiar eigenmode equation governing electrostatic collisionless trapped-particle instabilities [21].

The appropriate form of the radial current equation in the low-frequency regime is obtained by using Eqs (3.7) and (3.9) with Eq.(2.34). This yields

$$\widetilde{\delta B}_\parallel=\frac{\beta_i}{2}\left\{\frac{\omega_{*p}}{\omega}\hat\psi_\parallel\right.$$

$$\left.-\sum_j\frac{e_j}{e_i}\int_{T_r}d^3v\left[\frac{F_m}{n_o}\left(\frac{\omega-\omega_*^T}{\omega-\langle\omega_D\rangle}\right)\left(\frac{v_\perp}{v_T}\right)^2\langle X\rangle\right]_j\right\} \qquad (3.16)$$

with

$$\beta_i\equiv 8\pi n_oT_i/B^2$$

If ω_D/ω is again used as a smallness parameter and terms of order

$$\left\langle\omega_D/\omega\right\rangle\left\langle\widetilde{\delta B}_\parallel\right\rangle$$

and

$$\left\langle\omega_D/\omega\right\rangle\left\langle\hat\phi-\hat\psi_\parallel\right\rangle$$

are neglected, then Eq.(3.16) can be expressed as

$$\widetilde{\delta B}_\parallel=\frac{\beta_i}{2}\left\{\frac{\omega_{*p}}{\omega}\left[\hat\psi_\parallel+\frac{3}{4}\int_{T_r}d\lambda B\lambda B(1-\lambda B)^{-1/2}\left\langle\hat\phi-\hat\psi_\parallel\right\rangle\right]\right.$$

$$-\frac{15}{8}\int_{T_r}d\lambda B\lambda B(1-\lambda B)^{-1/2}$$

$$\left.\times\left\langle\hat\psi_\parallel\frac{\bar\omega_{di}}{\omega}+\lambda B\widetilde{\delta B}_\parallel\right\rangle\sum_j\frac{T_j}{T_i}\left[1-\frac{\omega_*}{\omega}(1+2\eta)\right]_j\right\} \qquad (3.17)$$

As noted in Section 2, the parallel-current equation can be obtained by combining Eqs (2.32) and (2.35). In the low-frequency, long-wavelength regime of interest here, Eqs (3.7) and (3.9) are used in Eq.(2.35). Re-writing Eq.(2.32) in the form

$$b\hat A_\parallel=\frac{T_i}{m_i\Omega_i^2}\frac{4\pi}{c}\hat j_\parallel \qquad (3.18)$$

with

$$b \equiv k_\perp^2 \rho_i^2 / 2$$

and ignoring terms of order

$$(\omega_D/\omega) b \hat{\psi}_\parallel \ , \ b\delta\tilde{B}_\parallel$$

and

$$b(\hat{\phi}-\hat{\psi}_\parallel)$$

then leads to the result

$$\frac{L_c^2}{JB^2} \frac{\partial}{\partial \chi}\left(\frac{b}{J}\frac{\partial}{\partial \chi}\hat{\psi}_\parallel\right)$$

$$+ \left(\frac{\omega}{\omega_A}\right)^2 \left\{ \frac{\omega_{*p}(\omega_\kappa+\omega_B)}{\omega^2}\hat{\psi}_\parallel + \frac{\omega_{*p}}{\omega}\delta\tilde{B}_\parallel \right.$$

$$- \sum_j \frac{T_i}{T_j} \int_{Tr} d^3v \frac{\omega_{Dj}}{\omega}\left[\frac{\omega-\omega_*^T}{\omega-\langle\omega_D\rangle}\left\langle X\right\rangle \frac{F_m}{n_o}\right]_j$$

$$\left. + \left[1-\frac{\omega_{*i}}{\omega}(1+\eta_i)\right] b\hat{\phi}\right\} = 0 \tag{3.19}$$

where

$$\omega_A^2 \equiv v_A^2/L_c^2, \quad v_A^2 \equiv B^2/4\pi n_o M_i$$

and use has been made of Eq.(3.8) re-written as

$$\frac{1}{JB}\frac{\partial}{\partial \chi}\hat{\psi}_\parallel = \frac{i\omega}{c}\hat{A}_\parallel \tag{3.20}$$

In the limit $\omega_D/\omega \ll 1$, the trapped particle contribution in Eq.(3.19) can be expanded to give

$$\frac{L_c^2}{JB^2} \frac{\partial}{\partial \chi}\left(\frac{b}{J}\frac{\partial}{\partial \chi}\hat{\psi}_\parallel\right)$$

$$+ \left(\frac{\omega}{\omega_A}\right)^2 \left\{ \frac{\omega_{*p}(\omega_\kappa+\omega_B)}{\omega^2}\hat{\psi}_\parallel + \frac{\omega_{*p}}{\omega}\delta\tilde{B}_\parallel \right.$$

$$+ \frac{3}{4}\frac{\omega_{*p}}{\omega}\int_{Tr} d\lambda B(1-\lambda B)^{-1/2}\left\langle\hat{\phi}-\hat{\psi}_\parallel\right\rangle \frac{\bar{\omega}_{Di}}{\omega}$$

$$- \sum_j \frac{T_i}{T_j}\left[1-\frac{\omega_*}{\omega}(1+2\eta)\right]_j$$

$$\times \frac{15}{8}\int_{Tr} d\lambda B(1-\lambda B)^{-1/2}\frac{\bar{\omega}_{Dj}}{\omega}\left[\left\langle\hat{\phi}-\hat{\psi}_\parallel\right\rangle\left\langle\frac{\bar{\omega}_{Dj}}{\omega}\right\rangle\right.$$

$$\left. + \left\langle\hat{\psi}_\parallel \frac{\bar{\omega}_{Dj}}{\omega}\right\rangle + \left\langle\lambda B\ \delta\tilde{B}_\parallel\right\rangle_j\right]$$

$$\left. + b\hat{\phi}\left[1-\frac{\omega_{*i}}{\omega}(1+\eta_i)\right]\right\} = 0 \tag{3.21}$$

Summarizing, in the low-frequency ($\omega < \omega_{bi}, \omega_{ti}$), long-wavelength ($b \ll 1$) regime, the general set of one-dimensional integro-differential equations governing ballooning modes is given by Eqs (3.12), (3.16), and (3.19). If the finite-gyroradius terms (terms containing b explicity) are ignored and if a change from the Coulomb gauge used here to the choice, $A_\parallel = 0$, is made, then the set of equations derived by Rosenbluth and Sloan [17] to analyse finite-beta effects on the collisionless trapped-particle instability can be recovered. Specifically, the relationship between the potentials of Ref.[17] (denoted by $(\phi, \vec{A})_{RS}$ with $(A_\parallel)_{RS} = 0$) and the corresponding ones in the present paper where $\nabla\cdot\vec{A}_\perp = 0$ is given by

$$\begin{aligned} \phi &= (\phi)_{RS} - (A_b)_{RS}\,\omega/k_b c \\ A_\parallel &= (i/JB)(\partial/\partial\chi)\left[(A_b)_{RS}/k_b\right] \end{aligned} \tag{3.22}$$

Noting that $\nabla \times (\vec{A}_\perp)_{RS} = 0$ emerges as a solubility condition in the ordering,

$$k_\perp \rho_j \sim O(1)$$

and using Eq.(3.21) then transforms the result for the perturbed distribution function of Rosenbluth and Sloan (Eq.(19) of Ref.[17]) into Eqs (3.7) and (3.9) of the present paper.

For the remainder of this section attention will be focused on Eqs (3.13), (3.17), and (3.21). These are of course the simpler forms obtained from Eqs (3.12), (3.16) and (3.19) in the limit $|\omega_D/\omega| \ll 1$. By using beta as a subsidiary expansion parameter they can be combined into a single integro-differential equation which contains kinetic modifications to the MHD ballooning modes.

As noted in the discussion following Eq.(3.13), the approximate solution of interest to the quasi-neutrality condition when

$$\omega^2 \gg \omega_* \omega_D$$

is

$$\hat{\phi} = \hat{\psi}_{\parallel}$$

This result can be used in Eq.(3.17). The right-hand side of this equation can be further simplified by noting that $\delta B_{\parallel}$ term there can be treated by iteration with β as an expansion parameter. The radial current equation then reduces to

$$\widetilde{\delta B}_{\parallel} = \frac{(\omega_\kappa - \omega_B)}{\omega} \hat{\psi}_{\parallel} - \frac{\beta_i}{2} \frac{15}{8} \int_{Tr} d\lambda B \lambda B (1-\lambda B)^{-\frac{1}{2}} \left\langle \hat{\psi}_{\parallel} \frac{\omega_\kappa}{\omega} \right\rangle^{-} \sum_j \frac{T_j}{T_i} \left[1 - \frac{\omega_*}{\omega} (1+2\eta)\right]_j \qquad (3.23)$$

where use has been made of Eq.(3.15). Finally, substituting these results into the parallel-current equation, Eq.(3.21), leads to a single integro-differential equation for $\hat{\phi}$ (or $\hat{\psi}_{\parallel}$) given by

$$\frac{L_c^2}{JB^2} \frac{\partial}{\partial \chi} \left(\frac{b}{J} \frac{\partial}{\partial \chi} \hat{\phi}\right) + \left(\frac{\omega}{\omega_A}\right)^2 \left\{ \frac{2\omega_{*p}\omega_\kappa}{\omega^2} \hat{\phi} + b\left[1 - \frac{\omega_{*i}}{\omega} (1+\eta_i)\right] \hat{\phi} - \sum_j \frac{T_j}{T_i} \left[1 - \frac{\omega_*}{\omega}(1+2\eta)\right]_j \frac{15}{8} \int_{Tr} d\lambda B (1-\lambda B)^{-\frac{1}{2}} \left\langle \hat{\phi} \frac{\omega_\kappa}{\omega} \right\rangle \frac{\omega_\kappa}{\omega} \right\} = 0 \qquad (3.24)$$

In the absence of the finite-gyroradius term

$$b(\omega_{*i}/\omega)(1+\eta_i)$$

and the trapped-particle contributions, Eq.(3.24) reduces exactly to Eq.(3.5), the MHD-ballooning-mode equation in the analogous limit,

$$(\omega/\omega_s)^2 \ll 1$$

Of the two kinetic effects included in Eq.(3.24) the trapped-particle terms tend to be relatively weak since they are of order

$$|\epsilon^{\frac{1}{2}} \omega_D/\omega|$$

smaller than the pressure-gradient driving term [22]. In the absence of trapped particles the finite-gyroradius term is clearly a stabilizing contribution, and at marginal stability leads to a mode frequency given by

$$\omega = \omega_{*i}(1+\eta_i)/2$$

but when trapped particles are included the finite-Larmor-radius effect may be de-stabilizing. Since the condition

$$|\omega L_c/v_{Ti}| \ll 1$$

was assumed in arriving at the equations in this section, the results apply only to perturbations with sufficiently low toroidal mode numbers, i.e. $n < n_{crit}$ with n_{crit} determined by

$$|\omega_{*i}(1+\eta_i)L_c/2v_{Ti}| < 1$$

Taking $L_c \sim Rq$ and

$$L_n \equiv (d\ell n n_0/dr)^{-1} \sim a$$

with a being the minor radius of the torus, then leads to

$$n_{crit} \simeq \frac{a^2}{R\rho_i q^2} \qquad (3.25)$$

For typical tokamak parameters, n_{crit} falls in the range 10 to 20. In the next section the intermediate-frequency regime, where $n > n_{crit}$, will be analysed.

3.3. Kinetic analysis in the intermediate-frequency limit

Since higher toroidal mode numbers are of interest in the intermediate-frequency regime

$$\omega_{bi}, \omega_{ti} < \omega < \omega_{be}, \omega_{te}$$

it is appropriate to retain the Bessel functions in their complete rather than expanded form when dealing with the ion responses. With

$$\omega > \omega_{bi}, \omega_{ti}$$

trapped-ion effects can be neglected, and

$$|\omega L_c/v_{T_i}|^{-1}$$

becomes the relevant expansion parameter. In this limit, the appropriate lowest-order form for the perturbed ion distribution function is

$$\frac{1}{2}(\hat{h}_+ + \hat{h}_-)_i = \frac{|e|}{T_i} F_{mi} \left(\frac{\omega-\omega_{*i}^T}{\omega-\omega_{Di}}\right)\left(J_o\hat{\phi} + J_1 \frac{v_\perp}{k_\perp}\frac{\hat{\delta B}_\parallel}{c}\right) \tag{3.26}$$

This leads to a perturbed density response of the form

$$\frac{n_i}{n_o} = -\frac{|e|}{T}\left[\left(1-Q\right)\hat{\phi} + Q'\,\tilde{\delta B}_\parallel\right] \tag{3.27}$$

where

$$Q \equiv \int d^3v\, \frac{F_{mi}}{n_o} J_o^2 \left(\frac{\omega-\omega_{*i}^T}{\omega-\omega_{Di}}\right) \tag{3.28}$$

and

$$Q' \equiv \int d^3v\, \frac{F_{mi}}{n_o}\left(\frac{d}{db}J_o^2\right)\left(\frac{\omega-\omega_{*i}^T}{\omega-\omega_{Di}}\right) \tag{3.29}$$

The finite gyroradius effects due to the electrons are again taken to be small, and since $\omega < \omega_{be}, \omega_{te}$, the results for the circulating- and trapped-electron distributions given in Eqs (3.7) and (3.9) can be used to give a perturbed-electron-density response of the form

$$\frac{n_e}{n_o} = \frac{|e|}{T_e}\left[\hat{\phi} - \left(1-\frac{\omega_{*e}}{\omega}\right)\hat{\psi}_\parallel - \frac{1}{n_o}\int d^3v F_{me}\left(\frac{\omega-\omega_{*e}^T}{\omega-\langle\omega_{De}\rangle}\right)\langle X_e\rangle\right] \tag{3.30}$$

Hence, the quasi-neutrality equation becomes

$$\left(1-\frac{\omega_{*e}}{\omega}\right)\hat{\psi}_\parallel = \left(1-\tau-\tau Q\right)\hat{\phi} + \tau Q'\tilde{\delta B}_\parallel - \frac{1}{n_o}\int_{Tr} d^3v\, F_{m_e}\left(\frac{\omega-\omega_{*e}^T}{\omega-\langle\omega_{De}\rangle}\right)\langle X_e\rangle \tag{3.31}$$

Using Eqs (3.7), (3.9) and (3.26) in Eqs (2.34) and (2.35) leads to the radial current equation in the form,

$$\tilde{\delta B}_\parallel = \frac{\beta_i}{2}\left\{Q\hat{\phi} - R\tilde{\delta B}_\parallel + \left[1-\frac{\omega_{*e}}{\omega}(1+\eta_e)\right]\hat{\psi}_\parallel + \frac{1}{n_o}\int_{Tr} d^3v F_{m_e}\left(\frac{\omega-\omega_{*e}^T}{\omega-\langle\omega_{D_e}\rangle}\right)\langle X_e\rangle\left(\frac{v_\perp}{v_{T_e}}\right)^2\right\} \tag{3.32}$$

with

$$R \equiv \frac{2}{b}\int d^3v\, \frac{F_{m_i}}{n_o}\left(\frac{\omega-\omega_{*i}^T}{\omega-\omega_{Di}}\right)\left(\frac{mv_\perp^2}{2T}\right)_i^2 J_1^2 \tag{3.33}$$

and to the parallel-current equation in the form

$$\frac{L_c^2}{JB^2}\frac{\partial}{\partial\chi}\left(\frac{b}{J}\frac{\partial}{\partial\chi}\hat{\psi}_\parallel\right) = \left(\frac{\omega}{\omega_A}\right)^2\left\{\left(Q-1-\frac{\omega_{*e}}{\omega\tau}\right) - \left[1-\frac{\omega_{*e}}{\omega}(1+\eta_e)\right]\left(\frac{\omega_\kappa+\omega_B}{\omega}\right)\hat{\psi}_\parallel - \left[Q'+1-\frac{\omega_{*e}}{\omega}(1+\eta_e)\right]\tilde{\delta B}_\parallel + \frac{1}{\tau}\int_{Tr} d^3v\left[\frac{F_m}{n_o}\frac{\omega_D}{\omega}\left(\frac{\omega-\omega_*^T}{\omega-\langle\omega_D\rangle}\right)\langle X\rangle\right]_e\right\} \tag{3.34}$$

where use has been made of Eqs (3.18) and (3.20). The last term in Eq.(2.35) is formally of order $(\omega_{ti}/\omega)^2$ and has been neglected here. Equations (3.32) and (3.34), together with Eq.(3.31), comprise the general set of coupled one-dimensional integro-differential equations governing ballooning modes in the intermediate-frequency regime.

If trapped-particle effects are ignored, the three kinetic ballooning-mode equations can be readily combined into a single differential equation as follows. First note that with this approximation Eq.(3.32) reduces to

$$\tilde{\delta B}_\parallel = \frac{(\beta_i/2)\left\{\left[1-\frac{\omega_{*e}}{\omega}(1+\eta_e)\right]\hat{\psi}_\parallel + Q'\hat{\phi}\right\}}{1+\left(\frac{\beta_i}{2}\right)R} \tag{3.35}$$

Combining with Eq.(3.31) then gives

$$\hat{\psi}_{\parallel} = \hat{\phi}\left[(1+\tau-\tau Q)\left(1+\frac{\beta_i}{2}R\right)+\tau Q'^2\frac{\beta_i}{2}\right]$$

$$\times\left[\alpha_{0e}\left(1+\frac{\beta_i}{2}R\right)-\alpha_{1e}\tau Q'\frac{\beta_i}{2}\right]^{-1} \tag{3.36}$$

with

$$\alpha_{\ell j} = \left[1-\frac{\omega_{*j}}{\omega}(1+\ell\eta_j)\right]$$

Hence, $\widetilde{\delta B}_{\parallel}$ in terms of $\hat{\phi}$ alone is just

$$\widetilde{\delta B}_{\parallel} = \hat{\phi}\,\frac{\beta_i}{2}\left[\alpha_{0e}Q'+\alpha_{1e}(1+\tau-\tau Q)\right]$$

$$\times\left[\alpha_{0e}\left(1+\frac{\beta_i}{2}R\right)-\alpha_{1e}\tau Q'\frac{\beta_i}{2}\right]^{-1} \tag{3.37}$$

Substituting Eqs (3.36) and (3.37) into the parallel-current equation, Eq.(3.34), with the trapped-particle contributions suppressed, then gives a single differential eigenmode equation of the form

$$\frac{L_c^2}{JB^2}\frac{\partial}{\partial\chi}\left(\frac{b}{J}\frac{\partial\hat{\psi}_{\parallel}}{\partial\chi}\right) = \frac{\omega^2}{\omega_A^2}\hat{\psi}_{\parallel}K \tag{3.38}$$

where

$$K = \left\{\left(Q-1-\frac{\omega_{*e}}{\omega\tau}\right)\left[\alpha_{0e}\left(1+\frac{\beta_i}{2}R\right)-\alpha_{1e}\tau Q'\frac{\beta_i}{2}\right]\right.$$

$$\left.-\frac{\beta_i}{2}(Q'+\alpha_{1e})\left[\alpha_{0e}Q'+\alpha_{1e}(1+\tau-\tau Q)\right]\right\}$$

$$\times\left[(1+\tau-\tau Q)\left(1+\frac{\beta_i}{2}R\right)+\tau Q'^2\frac{\beta_i}{2}\right]^{-1}$$

$$-\alpha_{1e}\frac{(\omega_K+\omega_B)}{\omega} \tag{3.39}$$

Note that Eq.(3.38) allows for ion drift resonance effects which are contained in the terms involving Q, Q′, and R. In addition, neither β nor b have been assumed small in arriving at this result.

The kinetic ballooning-mode equation just derived for the intermediate-frequency regime can now be compared with the analogous MHD result given in Eq.(3.6). First recall that to arrive at Eq.(3.6) it was assumed that $\beta \ll 1$ and $(\omega/\omega_s)^2 \gg 1$. As noted earlier, the condition on ω here corresponds to the intermediate-frequency regime since ω_s, ω_{bi}, and ω_{ti} are all roughly of the same magnitude. In considering the MHD limit of Eq.(3.38), it is appropriate to treat b and $|\omega_D/\omega|$ as small parameters. Taking $\beta \ll 1$, together with these approximations, then reduces the kinetic equation to

$$\frac{L_c^2}{JB^2}\frac{\partial}{\partial\chi}\left(\frac{b}{J}\frac{\partial}{\partial\chi}\hat{\phi}\right) = -\left(\frac{\omega}{\omega_A}\right)^2\hat{\phi}\left[b\alpha_{1i}+2\frac{\omega_\kappa\omega_{*p}}{\omega^2}\right.$$

$$\left.-\left(\frac{\omega_\kappa}{\omega}\right)^2\left(7\alpha_{2i}+4\tau\alpha_{1i}\alpha_{1e}/\alpha_{0e}\right)\right] \tag{3.40}$$

The MHD result given in Eq.(3.6) corresponds exactly to this form in the limit $|\omega_*/\omega| \ll 1$ if

$$\gamma = \left[(7/4)\,T_i + T_e\right] / (T_e + T_i) \tag{3.41}$$

with γ being the ratio of specific heats.

Note that, in the limit $\omega \gg \omega_*$, the terms proportional to ω_κ^2 provide additional stability compared with the low-frequency regime. However, if ω_*/ω effects are retained this may not be so in general when ω at marginal stability must be determined by solution of Eq.(3.40). Thus, for example, if

$$\omega = \frac{\omega_{*pi}}{2}$$

the marginal value in the low-frequency regime, the additional terms may be de-stabilizing!

If the trapped-particle contributions are retained, it becomes considerably more complicated to obtain a single eigenmode equation from Eqs (3.31), (3.32), and (3.34). Nevertheless, by making the same approximations as those leading to Eq.(3.40), the resulting kinetic ballooning-mode equation becomes

$$\frac{L_c^2}{JB^2}\frac{\partial}{\partial\chi}\left(\frac{b}{J}\frac{\partial}{\partial\chi}\hat{\phi}\right) = -\left(\frac{\omega}{\omega_A}\right)^2\hat{\phi}\left[b\alpha_{1i}+2\frac{\omega_\kappa\omega_{*p}}{\omega^2}\right.$$

$$\left.-\left(\frac{\omega_\kappa}{\omega}\right)^2\left(7\alpha_{2i}+4\tau\alpha_{1i}\alpha_{1e}/\alpha_{0e}\right)\right]$$

$$+\left(\frac{\omega}{\omega_A}\right)^2\tau\left(\frac{\omega_\kappa}{\omega}\right)\int_{Tr} d\lambda B\lambda B(1-\lambda B)^{-1/2}\left\langle\frac{\omega_\kappa}{\omega}\hat{\phi}\right\rangle$$

$$\times\left\{\frac{15}{8}\alpha_{2e}+\frac{1}{2}\frac{\alpha_{1e}}{\alpha_{0e}}\left(\alpha_{1i}-3\alpha_{1e}\right)\right\} \qquad (3.42)$$

Thus, the last term in Eq.(3.42) introduced by the trapped-particle effects changes the governing eigenmode equation into an integro-differential equation. However, since this term is of order $|\epsilon^{1/2}\omega_D/\omega|$ smaller than the pressure-gradient driving term the effects here are again relatively weak.

4. CONCLUSIONS

To investigate the influence of specifically kinetic effects on the ideal-MHD ballooning modes with high toroidal mode number n, the Vlasov-Maxwell equations have been solved by using the familiar gyroradius expansion in $\epsilon \equiv \rho_i/L$. Since these modes have the property of long wavelengths parallel to the magnetic field but short perpendicular wavelengths, the ordering, $n \sim 1/\epsilon$, is appropriate and suggests an eikonal representation for the perturbations. The attendant problems of periodicity in a sheared toroidal field are solved by employing the ballooning transformation. This procedure reduces the stability problem, in lowest order in ϵ, to one of solving three coupled one-dimensional integro-differential equations on each flux surface. Corrections to these lowest-order local eigenvalues together with the radial structure of the eigenmodes can be determined in next order.

The kinetic ballooning-mode equations can be considerably simplified in two limits: the low-frequency regime,

$$\omega < \omega_{bi}, \omega_{ti}$$

and the intermediate-frequency regime,

$$\omega_{bi}, \omega_{ti} < \omega < \omega_{be}, \omega_{te}$$

In ideal-MHD theory, the fact that one is particularly interested in marginal stability ($\omega \to 0$) suggests that a kinetic treatment of the low-frequency regime alone might be appropriate. It is, however, expected that the inclusion of finite-ion-gyroradius effects will lead to a finite oscillation frequency, $\omega \sim \omega_*$, at marginal stability, so that the relevant regime will depend on the perpendicular wavenumber. In particular,

$$\omega \lessgtr \omega_{bi}, \omega_{ti} \quad \text{if} \quad n \lessgtr \frac{a}{\rho_i}\frac{a}{Rq^2}$$

and the transition may well occur for quite moderate values of n. For comparison with these predictions, the limits

$$\omega \lessgtr \omega_s$$

of the ideal-MHD ballooning-mode equations must be considered since

$$\omega_{bi}, \omega_{ti} \sim \omega_s$$

In the low-frequency regime the three fundamental equations, Eqs (3.12), (3.16), and (3.19), are naturally closely related to those derived by Rosenbluth and Sloan [17] in their study of finite-β modifications of the electrostatic trapped-particle modes. Introducing an expansion in $\beta < 1$ and taking $|\omega_D/\omega| < 1$ (which requires $a/R < 1$ for $\omega \sim \omega_*$) reduces the system of kinetic equations to the single integro-differential equation given as Eq.(3.24). In the situation $a/R < 1$, the integral terms arising from trapped particles are relatively weak. If these are ignored, then Eq.(3.24) reduces, in the limit $|\omega_*/\omega| \ll 1$, to the MHD-result, Eq.(3.5), obtained in the analogous low-frequency regime, $\omega < \omega_s$. When the terms of order ω_*/ω are retained, the general effect is stabilizing with marginal stability occurring when $\omega = \omega_{*pi}/2$. Specifically, $\omega^2_{MHD} \to \omega^2_{MHD} + \omega^2_{*pi}/4$. It should also be noted that the trapped-particle terms, which are stabilizing when $|\omega_*/\omega| \ll 1$ (as follows from the kinetic energy principle [11]) become de-stabilizing when $\omega = \omega_{*pi}/2$. Finally, it is of interest to point out that field compression effects associated with $\delta B_\parallel$ convert the magnetic drifts entirely into curvature drifts in Eq.(3.24).

At higher values of n, the appropriate frequency regime to study is the intermediate range where stability is governed by the three coupled integro-differential equations, Eqs (3.31), (3.32), and (3.33). If the trapped-particle integral terms are ignored, then this system of equations can be reduced to a single second-order differential equation, Eq.(3.38), which accounts for finite-gyroradius and ion-drift-resonance effects without approximations. In the limit $\beta < 1$

and $|\omega_D/\omega| < 1$, this simplifies to Eq.(3.40). If $|\omega_*/\omega| \ll 1$, then this equation further reduces to the MHD-result, Eq.(3.6), obtained in the analogous frequency regime, $\omega > \omega_s$, for an appropriate choice of the adiabatic index γ. Note that, in this MHD-limit, the intermediate-frequency-regime equation differs from the low-frequency result through the addition of stabilizing terms proportional to ω_κ^2. However, as emphasized in Section 3.3, these terms can change sign and may become de-stabilizing for $\omega \sim \omega_*$. Hence, the specific effect of the ω_κ^2 terms depends on the eigenvalue ω determined by Eq.(3.40). By modifying the results of Ref.[17] to the intermediate regime and making a number of simplifying assumptions, Chu et al. [20] have obtained an equation similar in structure to Eq.(3.40). However, in assessing the importance of the ω_κ^2 terms it was assumed that at marginal stability the eigenvalue is given by $\omega = \omega_{*p_i}/2$. This is valid only if the ω_κ^2 terms are treated as small corrections. If the trapped-particle correction terms (for $a/R < 1$) are included, then Eq.(3.40) is modified to the integro-differential equation, Eq.(3.42). As a final point it should be noted that the role of the field compression ($\delta B_\parallel$) effects is again to ensure that only curvature drifts appear in the kinetic ballooning-mode equation.

Although the relation of the kinetic results based on the Vlasov-Maxwell equations to the ideal-MHD results has been stressed in this paper, it is also important to emphasize that the original sets of three coupled one-dimensional integro-differential equations [Eqs (3.12), (3.16), and (3.19) for the low- and Eqs (3.31), (3.32), and (3.33) for the intermediate-frequency regimes] fully describe the effects of finite ion gyroradius, trapped particles, and ion drift resonances on the MHD ballooning modes. By taking $\beta < 1$ as an expansion parameter, the radial-current equations governing $\delta B_\parallel$ can be iterated to reduce these equations to sets of two coupled equations. If trapped-particle terms are ignored, then the results can be further simplified to single second-order differential equations. For example, Eq.(3.38), obtained in the intermediate-frequency regime, can be readily used to assess the influence of finite ion gyroradius and ion drift resonances. Note that the result here does not require the gyroradius parameter b to be small and that the drift resonance effects are emphasized by the secularities in the radial wavenumber (Eq. (2.12)).

To summarize, it has been shown in this paper that, despite the apparently very different physical basis for their existence, both ideal-MHD and collisionless kinetic ballooning-mode theories lead to closely related fluid-like equations. Furthermore, the equations necessary for a systematic investigation of the role of specifically kinetic effects, such as finite ion gyroradius, trapped particles, and ion drift resonances, have been derived.

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