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To cite this article: P J Catto *et al* 1981 *Plasma Phys.* **23** 639

Plasma Physics, Vol. 23, No. 7, pp. 639 to 650, 1981
Printed in Northern Ireland

0032-1028/81/070639-12$02.00/0
Institute of Physics and Pergamon Press Ltd.

GENERALIZED GYROKINETICS

P. J. CATTO and W. M. TANG*
Science Applications, Inc., Plasma Research Institute, 934 Pearl Street, Boulder, CO 80302, U.S.A.
and
D. E. BALDWIN
Lawrence Livermore Laboratory, Livermore, CA 94550, U.S.A.

(*Received* 9 *October* 1980; *and in revised form* 6 *January* 1981)

Abstract—By retaining the magnetic moment μ to higher order in the gyroradius over scale length expansion and employing a gyrokinetic change of variables a full finite β derivation of the gyrokinetic equation is presented within the eikonal ansatz for arbitrary magnetic fields and μ dependent unperturbed distribution functions.

1. INTRODUCTION

THE RETENTION of kinetic effects on tearing modes and other MHD disturbances, as well as the study of drift-Alfvén fluctuations, has become an important area of research in both toroidal and tandem mirror geometries. For such modes the gyroradius ρ is typically smaller than the shortest unperturbed scale length L_0 and the wave frequency ω is low compared to the smallest cyclotron frequency. In addition, the parallel wavenumber $k_{\|}$ is typically small compared to the perpendicular wavenumber $k_{\perp}$. Consequently, gyrokinetic treatments employing an eikonal ansatz provide the most expeditious means of obtaining the appropriate reduced kinetic equations while still retaining the full finite gyroradius modifications.

Since the classic work of RUTHERFORD and FRIEMAN (1968) and TAYLOR and HASTIE (1968), gyrokinetic derivations have usually been restricted to electrostatic (RUTHERFORD and FRIEMAN, 1968; TAYLOR and HASTIE, 1968; JAMIN, 1971; CATTO, 1978) or low beta (NEWBERGER, 1976; HITCHCOCK and HAZELTINE, 1978) perturbations and/or axisymmetric configurations (JAMIN, 1971; NEWBERGER, 1976). Only the recent work of ANTONSEN and LANE (1980) gives the full finite beta gyrokinetic equations for arbitrary magnetic field geometry. Here we present an alternate derivation which we believe to be somewhat more transparent and algebraically less cumbersome. Unlike the technique of ANTONSEN and LANE (1980), our derivation is not an extension of the techniques of RUTHERFORD and FRIEMAN (1968) or TAYLOR and HASTIE (1968).

The technique used here is an extension of CATTO's (1978) gyrokinetic change of variables which permits the treatment of magnetic moment dependent unperturbed distribution functions F_0. The effects of such F_0 are retained by employing the magnetic moment μ to one order higher in ρ/L_0 than is customary. As a result, we are able to *explicitly* obtain the unperturbed distribution function to next order in ρ/L_0 and thereby show that the indefinite form (RUTHERFORD and FRIEMAN, 1968) must correspond to expansion of μ about its lowest order value $\mu_0 = v_{\perp}{}^2/2B$, as well as the diamagnetic correction from the expansion of its spatial dependence about the guiding center location.

* Permanent address: Princeton University, Plasma Physics Laboratory, Princeton, NJ 08544, U.S.A.

Working with the higher order magnetic moment has other advantages as well. Most importantly, it allows one to write the guiding center velocity $\bar{v}_{\|}$ in a form containing the next order ρ/L_0 corrections. This ability to retain the distinction between $\bar{v}_{\|}$ and the parallel particle velocity $v_{\|}$ allows a proper treatment of the ρ/L_0 corrections to $v_{\|}$. In particular, we demonstrate that the parallel velocity correction found by BRAGINSKII (1956); BOGOLIUBOV and MITROPOLSKY (1961); BAÑOS (1967); HAZELTINE (1973), which is normally neglected as small, turns out to be automatically absorbed by the higher order expressions for μ and energy.

In Section 2 we determine the gyrokinetic variables having the preceding properties. Section 3 then uses these variables to derive the gyrokinetic equation, with algebraic details being relegated to the Appendix.

2. GYROKINETIC VARIABLES

Starting with the unperturbed Vlasov operator

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + [\Omega\mathbf{v}\times\hat{\mathbf{n}} - (Ze/M)\nabla\Phi_0(\mathbf{r})]\cdot\nabla_v \tag{1}$$

we desire to perform a gyrokinetic change of variables from $\mathbf{r}, \mathbf{v}$ to the guiding center variable $\mathbf{R} = \mathbf{R}(\mathbf{r}, \mathbf{v})$, the energy

$$E = (v^2/2) + (Ze/M)\Phi_0(\mathbf{r}) \tag{2}$$

the magnetic moment $\mu = \mu(\mathbf{r}, \mathbf{v})$, and the gyrophase ϕ. We employ the notation $B = |\mathbf{B}|$, $\hat{\mathbf{n}} = \mathbf{B}/B$, $\Omega = ZeB/Mc$, $v = |\mathbf{v}|$, and

$$\mathbf{v} = v_{\|}\hat{\mathbf{n}} + v_{\perp}(\hat{\mathbf{e}}_1 \cos\phi + \hat{\mathbf{e}}_2 \sin\phi) \tag{3}$$

with $v_{\|} = \hat{\mathbf{n}}\cdot\mathbf{v}$, $\mathrm{v}_{\perp} = |\mathbf{v}_{\perp}|$, $\mathbf{v}_{\perp} = \hat{\mathbf{n}}\times(\mathbf{v}\times\hat{\mathbf{n}})$, and the unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{n}}$ forming an orthogonal system in which $\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2 = \hat{\mathbf{n}}$. The quantities Z and M are the species charge number and mass; e and c are the magnitude of the charge on an electron and the speed of light. The magnetic field $\mathbf{B} = \mathbf{B}(\mathbf{r})$ is an arbitrary function of $\mathbf{r}$ satisfying the Maxwell equations.

Using $\nabla_v E = \mathbf{v}$, $\nabla E = (Ze/M)\nabla\Phi_0$, and noting that $\hat{\mathbf{e}}_1\cdot\mathbf{v} = v_{\perp}\cos\phi$ and $\hat{\mathbf{e}}_2\cdot\mathbf{v} = v_{\perp}\sin\phi$ allows us to find

$$\begin{aligned}\nabla_v\phi &= v_{\perp}^{-2}\hat{\mathbf{n}}\times\mathbf{v}\\ \nabla\phi &= \nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 - (v_{\|}/v_{\perp}^2)\nabla\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\times\mathbf{v}\end{aligned} \tag{4}$$

so that $\mathrm{d}/\mathrm{d}t$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{\phi}\frac{\partial}{\partial\phi} + \dot{\mu}\frac{\partial}{\partial\mu} + \dot{\mathbf{R}}\cdot\bar{\nabla} \tag{5}$$

with $\bar{\nabla} \equiv \partial/\partial\mathbf{R}$ (so that $\bar{\nabla}\mathbf{R} = \boldsymbol{I}$, the unit dyad), and

$$\dot{\phi} \equiv -\Omega - \frac{Ze}{Mv_{\perp}^2}\hat{\mathbf{n}}\times\mathbf{v}\cdot\nabla\Phi_0 + \mathbf{v}\cdot\nabla\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}} - \frac{v_{\|}}{v_{\perp}^2}\mathbf{v}\cdot\nabla\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\times\mathbf{v} \tag{6}$$

$$\dot{\mu} \equiv \mathbf{v}\cdot\nabla\mu + [-(Ze/M)\nabla\Phi_0 + \Omega\mathbf{v}\times\hat{\mathbf{n}}]\cdot\nabla_v\mu \tag{7}$$

$$\dot{\mathbf{R}} \equiv \mathbf{v}\cdot\nabla\mathbf{R} + [-(Ze/M)\nabla\Phi_0 + \Omega\mathbf{v}\times\hat{\mathbf{n}}]\cdot\nabla_v\mathbf{R}. \tag{8}$$

When the gyroradius ρ is small compared to the shortest unperturbed scale length L_0, and the time variation is slow compared to the cyclotron motion, then to lowest order $\mathrm{d}/\mathrm{d}t \approx -\Omega\partial/\partial\phi$. Consequently, the distribution function operated on by the unperturbed Vlasov operator is independent of ϕ to lowest order. As a result, using the periodicity in ϕ of this function, the next order expression for $\mathrm{d}/\mathrm{d}t$ may be gyroaveraged using

$$\langle Q\rangle = (2\pi)^{-1}\int_0^{2\pi} \mathrm{d}\phi\, Q(R, E, \mu, \phi, t) \tag{9}$$

to obtain

$$\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\right\rangle = \frac{\partial}{\partial t} + \langle\dot{\mu}\rangle\frac{\partial}{\partial\mu} + \langle\dot{\mathbf{R}}\rangle\cdot\bar{\boldsymbol{\nabla}}. \tag{10}$$

It is important to note that the gyroaverage operation defined by equation (9) is to be performed holding $\mathbf{R}$, E, μ, and t fixed, rather than $\mathbf{r}$, E, μ, and t.

The magnetic moment is an adiabatic invariant, which enables us, in principle, to determine an expression for μ such that $\dot{\mu} = 0$ to the requisite order in ρ/L_0 (Kruskal, 1962). In addition, we desire to choose $\mathbf{R}$ such that

$$\langle\dot{\mathbf{R}}\rangle = \bar{v}_{\|}(\mathbf{R}, E, \mu)\hat{\mathbf{n}}(\mathbf{R}) + \mathbf{v}_d \tag{11}$$

with $\mathbf{v}_d$ the magnetic plus electric drifts

$$\mathbf{v}_d = \hat{\mathbf{n}}\times\left[\frac{c}{B}\boldsymbol{\nabla}\Phi_0 + \frac{v_\perp{}^2}{2\Omega}\boldsymbol{\nabla}\ln B + \frac{v_\|{}^2}{\Omega}\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\hat{\mathbf{n}}\right] \tag{12}$$

and $\bar{v}_\|\hat{\mathbf{n}}$ the parallel guiding center velocity, which will be defined presently. By employing μ to one order higher in ρ/L_0 than the lowest order form $\mu_0 = v_\perp{}^2/2B(\mathbf{r})$, and noting that the appropriate choice for the guiding center variable $\mathbf{R}$ (Catto, 1978) is

$$\mathbf{R} = \mathbf{r} + \frac{1}{\Omega}\mathbf{v}\times\hat{\mathbf{n}}, \tag{13}$$

the desired form of equation (10) will be obtained, namely

$$\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\right\rangle = \frac{\partial}{\partial t} + (\bar{v}_\|\hat{\mathbf{n}} + \mathbf{v}_d)\cdot\bar{\boldsymbol{\nabla}}. \tag{14}$$

It is not necessary to solve for ρ/L_0 corrections to μ_0 in order to derive the gyrokinetic equation. However, when μ_0 is employed (Rutherford and Frieman, 1968) rather than $\mu_0 + \mu_1$, one must employ essentially the same equation for the ρ/L_0 correction to the lowest order unperturbed distribution function. Consequently, using $\mu_0 + \mu_1$ has the virtue of *explicitly* displaying the dependence of the unperturbed distribution function on $\mu_0 + \mu_1$.

Taking $\mathbf{R}$ to be given by equation (13) and writing

$$\mu = \mu_0 + \mu_1(\mathbf{r}, \mathbf{v}) \tag{15}$$

with $\mu_0(\mathbf{r},\mathbf{v})=v_\perp{}^2/2B$ and $\mu_1/\mu_0 \sim O(\rho/L_0)$, and noting that

$$\begin{aligned}\nabla_v\mu_0 &= B^{-1}\mathbf{v}_\perp \\ \nabla\mu_0 &= -(\mu_0/B)\nabla B-(v_\parallel/B)\nabla\hat{\mathbf{n}}\cdot\mathbf{v} \\ \nabla R &= \boldsymbol{I}-\nabla(\Omega^{-1}\hat{\mathbf{n}})\times\mathbf{v} \\ \nabla_v\mathbf{R} &= \Omega^{-1}\boldsymbol{I}\times\hat{\mathbf{n}}\end{aligned} \tag{16}$$

gives

$$\begin{aligned}\dot{\mu}_0 &= -\mu_0\mathbf{v}\cdot\nabla\ln B-(v_\parallel/B)\mathbf{v}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{v}-(Ze/M)\mathbf{v}_\perp\cdot\nabla\Phi_0 \\ \dot{\mathbf{R}} &= v_\parallel\hat{\mathbf{n}}+(c/B)\hat{\mathbf{n}}\times\nabla\Phi_0-\mathbf{v}\cdot\nabla(\Omega^{-1}\hat{\mathbf{n}})\times\mathbf{v}.\end{aligned} \tag{17}$$

Evaluating $\langle\dot{\mu}_0\rangle$ and $\langle\dot{\mathbf{R}}\rangle$ by noting that $\mathbf{r}\approx\mathbf{R}$ may be employed everywhere except for the $v_\parallel\hat{\mathbf{n}}$ term in $\dot{\mathbf{R}}$, we find

$$\begin{aligned}\langle\dot{\mu}_0\rangle &= O(\mu_0\rho^2/\Omega L_0{}^2) \\ \langle\dot{\mathbf{R}}\rangle &= \langle v_\parallel\rangle\hat{\mathbf{n}}(\mathbf{R})+\mathbf{v}_d+\frac{v_\perp{}^2}{2\Omega}\hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}}.\end{aligned} \tag{18}$$

To obtain equations (18) we employ $\langle\mathbf{v}\rangle=\langle v_\parallel\rangle\hat{\mathbf{n}}(\mathbf{R})\approx v_\parallel\hat{\mathbf{n}}+O(v_\parallel\rho/L_0)$ and

$$\langle\mathbf{v}\mathbf{v}\rangle=\tfrac{1}{2}v_\perp{}^2(\boldsymbol{I}-\hat{\mathbf{n}}\hat{\mathbf{n}})+v_\parallel{}^2\hat{\mathbf{n}}\hat{\mathbf{n}} \tag{19}$$

to determine to lowest order that

$$\begin{aligned}\langle-[\mathbf{v}\cdot\nabla(\Omega^{-1}\hat{\mathbf{n}})\times\hat{\mathbf{v}}]\cdot(\boldsymbol{I}-\hat{\mathbf{n}}\hat{\mathbf{n}})\rangle &= \hat{\mathbf{n}}\times\left(\frac{v_\perp{}^2}{2B}\nabla\ln B+\frac{v_\parallel{}^2}{\Omega}\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}\right) \\ \langle-[\mathbf{v}\cdot\nabla(\Omega^{-1}\hat{\mathbf{n}})\times\mathbf{v}]\cdot\hat{\mathbf{n}}\rangle &= (v_\perp{}^2/2\Omega)\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}}, \\ \langle\mu_0\mathbf{v}\cdot\nabla\ln B+(v_\parallel/B)\mathbf{v}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{v}\rangle &= (v_\parallel\mu_0/B)\nabla\cdot\mathbf{B}=0.\end{aligned} \tag{20}$$

The parallel velocity correction $(v_\perp{}^2/2\Omega)\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}}$ in $\langle\dot{\mathbf{R}}\rangle$ arises if $\mu=\mu_0$ is employed (BRAGINSKII, 1956; BOGOLIUBOV and MITROPOLSKY, 1961; BAÑOS, 1967; HAZELTINE, 1973) because of the difference between the gyroaveraged parallel component of the *particle* velocity $\langle v_\parallel\rangle$ and the parallel component of the 'guiding center' velocity which from equations (11) and (18) is

$$\bar{v}_\parallel\equiv\langle v_\parallel\rangle+(v_\perp{}^2/2\Omega)\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}}. \tag{21}$$

The distinction between $\bar{v}_\parallel$, $\langle v_\parallel\rangle$ and $v_\parallel$ need not be made in higher order terms. A discussion of the need to distinguish $\langle v_\parallel\rangle$ from $\bar{v}_\parallel$ when the parallel current is non-zero (so that the magnetic field shears) is given in Appendix B of NORTHROP and ROME (1978).

In order to determine the leading ρ/L_0 correction to μ_0 we must evaluate μ_1 by demanding $\dot{\mu}=0$ to the appropriate order. Solving $\dot{\mu}=0$ iteratively in ρ/L_0, the gyrophase dependent portion of μ_1 is found from

$$\mathbf{v}\cdot\nabla\mu_0-(Ze/M)\nabla\Phi_0\cdot\nabla_v\mu_0+\Omega\mathbf{v}\times\hat{\mathbf{n}}\cdot\nabla_v\mu_1=0. \tag{22}$$

Using $\nabla\mu_0$ from equation (16)

$$\nabla_v\mu_0=B^{-1}\mathbf{v}_\perp,\quad \Omega\mathbf{v}\times\hat{\mathbf{n}}\cdot\nabla_v\mu_1\approx-\frac{\partial\mu_1}{\partial\phi}$$

and integrating in ϕ yields

$$\mu_1 = -B^{-1}\mathbf{v}_d \cdot \mathbf{v}_\perp - (v_\parallel/4\Omega B)(\mathbf{v}_\perp \times \hat{\mathbf{n}} \cdot \nabla\hat{\mathbf{n}} \cdot \mathbf{v}_\perp + \mathbf{v}_\perp \cdot \nabla\hat{\mathbf{n}} \cdot \mathbf{v}_\perp \times \hat{\mathbf{n}}) + \langle\mu_1\rangle \quad (23)$$

where $\partial\langle\mu_1\rangle/\partial\phi = 0$. The gyrophase independent piece of μ_1, $\langle\mu_1\rangle$, can then be determined by the constraint that the gyroaverage of the next order correction to $\dot{\mu}$ vanish, namely

$$\langle \mathbf{v} \cdot \nabla\mu_1 - (Ze/M)\nabla\Phi_0 \cdot \nabla_v\mu_1 \rangle = 0. \quad (24)$$

Equation (24) is a rather complicated constraint equation, however. Fortunately, and quite remarkably, $\langle\mu_1\rangle$ can be easily determined by demanding that $\bar{v}_\parallel$ also satisfy

$$\tfrac{1}{2}\bar{v}_\parallel^2 = E - (Ze/M)\Phi_0(\mathbf{R}) - \mu B(\mathbf{R}) \quad (25)$$

with E defined by equation (2). Expanding $\Phi_0(\mathbf{r})$ and $B(\mathbf{r})$ about $\mathbf{R}$

$$\tfrac{1}{2}v_\parallel^2 = E - (Ze/M)\Phi_0(\mathbf{r}) - \mu_0 B(\mathbf{r})$$

may then be written as

$$\tfrac{1}{2}v_\parallel^2 = \tfrac{1}{2}\bar{v}_\parallel^2 + \mu_1 B(\mathbf{R}) + (\mu_0/\Omega)\mathbf{v}\times\hat{\mathbf{n}} \cdot \nabla B + (c/B)\mathbf{v}\times\hat{\mathbf{n}} \cdot \nabla\Phi_0. \quad (26)$$

Gyroaveraging equation (26), noting that $v_\parallel$ and $\bar{v}_\parallel$ are equal to lowest order in ρ/L_0, and using equation (21) gives

$$\begin{aligned}\langle\mu_1\rangle &= \langle v_\parallel^2 - \bar{v}_\parallel^2\rangle/2B(\mathbf{R}) \approx (v_\parallel/B)(\langle v_\parallel\rangle - \bar{v}_\parallel) \\ &= -(v_\parallel\mu_0/\Omega)\hat{\mathbf{n}} \cdot \nabla\times\hat{\mathbf{n}}.\end{aligned} \quad (27)$$

The full expression for μ_1, equations (23) and (27), is in agreement with the Vlasov result of HASTIE, TAYLOR and HAAS (1967) and the single particle result of NORTHROP (1963) and by direct substitution can be verified as being the solution of equation (24) (an extremely tedious calculation!). It is not obvious why demanding that the two definitions of $\bar{v}_\parallel$, equations (21) and (25), be the same is equivalent to the constraint equation, equation (24); however, the same remarkable simplification has also been observed in single particle descriptions by NORTHROP and ROME (1978). Because $\bar{v}_\parallel$ rather than $\langle v_\parallel\rangle$ enters in equation (25), the 'parallel velocity correction' is property accounted for without explicitly appearing.

3. DERIVATION OF GYROKINETIC EQUATION

Employing the vector and scalar potentials $\mathbf{A}$ and Φ to write the perturbed magnetic and electric fields, the linearized Vlasov equation becomes

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{Ze}{M}\left[\nabla\Phi + \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \frac{1}{c}\mathbf{v}\times(\nabla\times\mathbf{A})\right] \cdot \nabla_v F \quad (28)$$

with $\mathrm{d}/\mathrm{d}t$ the unperturbed Vlasov operation defined in equation (1), and f and F the perturbed and unperturbed distribution functions. We note that F satisfies the unperturbed Vlasov equation ($\partial/\partial t = 0$) when $\hat{\mathbf{n}} \cdot \bar{\nabla} F = 0$ and

$$F = F(E, \mu, \mathbf{R}) \quad (29)$$

with E defined by equation (2), μ the solution of equation (7) satisfying $\dot{\mu}=0$ to the appropriate order, and $\mathbf{R}$ the solution of $\hat{\mathbf{n}}\times\dot{\mathbf{R}}=0$ to the desired order with $\dot{\mathbf{R}}$ defined by equation (8). In the next paragraphs we show that the expressions for $\mathbf{R}$ and μ as given by equations (13), (15), (23) and (27) are of sufficient accuracy to describe F and derive the linearized gyrokinetic equation.

When $\mathbf{R}$ and μ are known to sufficient accuracy, then we may employ the operator of equation (1) and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial E}\right)=0=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \mu}\right),\quad \frac{\mathrm{d}}{\mathrm{d}t}(\bar{\nabla}F)\sim 0(v_d F/L_0{}^2) \tag{30}$$

to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi\frac{\partial F}{\partial E}\right)\doteq\frac{\partial F}{\partial E}\left(\frac{\partial \Phi}{\partial t}+\mathbf{v}\cdot\nabla\Phi\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(\Phi-v_\|A_\|/c)}{B}\frac{\partial F}{\partial \mu}\right]\doteq\frac{1}{B}\frac{\partial F}{\partial \mu}\left[\frac{\partial}{\partial t}(\Phi-v_\|A_\|/c)+\mathbf{v}\cdot\nabla(\Phi-v_\|A_\|/c)\right.$$
$$\left.-(\Phi-v_\|A_\|/c)\mathbf{v}\cdot\nabla\ln B+(Ze/Mc)A_\|\hat{\mathbf{n}}\cdot\nabla\Phi_0\right]$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{B}\mathbf{A}\times\hat{\mathbf{n}}\cdot\nabla F\right)\doteq\frac{1}{B}\left[\frac{\partial}{\partial t}(\mathbf{A}\times\hat{\mathbf{n}})+\mathbf{v}\cdot\nabla(\mathbf{A}\times\hat{\mathbf{n}})-\mathbf{A}\times\hat{\mathbf{n}}(\mathbf{v}\cdot\nabla\ln B)\right]\cdot\nabla F$$

where $A_\|\equiv\hat{\mathbf{n}}\cdot\mathbf{A}$. Defining h via

$$f=h+\frac{Ze}{M}\left\{\Phi(\mathbf{r},t)\frac{\partial F}{\partial E}+[\Phi(\mathbf{r},t)-v_\|A_\|(\mathbf{r},t)/c]\frac{1}{B}\frac{\partial F}{\partial \mu}\right\}+\frac{1}{B}\mathbf{A}(\mathbf{r},t)\times\hat{\mathbf{n}}\cdot\bar{\nabla}F. \tag{31}$$

Equation (28) may be rewritten as

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t}=&-\frac{Ze}{M}\frac{\partial F}{\partial E}\frac{\partial}{\partial t}\left(\Phi-\frac{1}{c}\mathbf{v}\cdot\mathbf{A}\right)-\frac{Ze}{MB}\frac{\partial F}{\partial \mu}\left[\frac{\partial}{\partial t}\left(\Phi-\frac{v_\|}{c}A_\|-\frac{B}{c}\nabla_v\mu\cdot\mathbf{A}\right)\right.\\&+(\mathbf{v}-B\nabla_v\mu)\cdot\nabla\left(\Phi-\frac{1}{c}\mathbf{v}\cdot\mathbf{A}\right)+\frac{1}{c}\mathbf{v}\cdot\nabla\mathbf{A}\cdot(\mathbf{v}-B\nabla_v\mu)\\&-\left(\Phi-\frac{v_\|}{c}A_\|\right)\mathbf{v}\cdot\nabla\ln B-\frac{A_\|}{c}\mathbf{v}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{v}-\frac{v_\|}{c}\mathbf{v}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{A}\\&\left.+\frac{Ze}{Mc}A_\|\hat{\mathbf{n}}\cdot\nabla\Phi_0\right]+\frac{Ze}{M}\left\{\nabla\left(\Phi-\frac{1}{c}\mathbf{v}\cdot\mathbf{A}\right)\cdot\nabla_v\mathbf{R}\right.\\&+\frac{1}{c}\frac{\partial}{\partial t}\left[\mathbf{A}\cdot\nabla_v\left(\mathbf{R}-\frac{1}{\Omega}\mathbf{v}\times\hat{\mathbf{n}}\right)\right]+\frac{1}{c}\mathbf{v}\cdot\nabla\mathbf{A}\cdot\nabla_v\left(\mathbf{R}-\frac{1}{\Omega}\mathbf{v}\times\hat{\mathbf{n}}\right)\\&\left.+\frac{1}{\Omega c}\mathbf{v}\cdot\nabla\hat{\mathbf{n}}\times\mathbf{A}+\frac{1}{\Omega c}(\mathbf{v}\cdot\nabla\ln B)\mathbf{A}\times\hat{\mathbf{n}}\right\}\cdot\bar{\nabla}F\end{aligned} \tag{32}$$

where we have employed

$$\Omega^{-1}\boldsymbol{I}\times\hat{\mathbf{n}}=\nabla_v(\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}).$$

In order to retain the full finite β modifications we treat $|\mathbf{v}\cdot\mathbf{A}|\sim c\,|\Phi|$. Then taking $|\partial/\partial t| \equiv \omega$, $|\hat{\mathbf{n}} \cdot \nabla| \equiv k_\parallel \ll |\hat{\mathbf{n}} \times \nabla| \equiv k_\perp$, $|\mathbf{v}| \sim v_t = (2T/M)^{1/2}$, and $|\nabla_v| \sim 1/v_t$, we consider the most general ordering

$$\omega \sim k_\perp v_t \rho / L_0 \sim k_\parallel v_t$$

where L_0 is the shortest unperturbed scale length and $k_\perp L_0 \gg 1$. Because the next order corrections to equations (13), (15), (23) and (27) are smaller by ρ/L_0, we see that these equations give $\mathbf{R}$ and μ to the desired accuracy. In addition, the $\mathbf{A} \times \hat{\mathbf{n}} \cdot \bar{\nabla} F$ term may be neglected in equation (31). As a result, equation (32) may be simplified to

$$\begin{aligned} \frac{\mathrm{d}h}{\mathrm{d}t} = & -\frac{Ze}{M}\frac{\partial F}{\partial E}\frac{\partial}{\partial t}\left(\Phi - \frac{1}{c}\mathbf{v}\cdot\mathbf{A}\right) - \frac{Ze}{MB}\frac{\partial F}{\partial \mu}\left[\frac{\partial}{\partial t}\left(\Phi - \frac{1}{c}\mathbf{v}\cdot\mathbf{A}\right)\right. \\ & + (v_\parallel \hat{\mathbf{n}} - B\nabla_v \mu_1)\cdot\nabla\left(\Phi - \frac{1}{c}\mathbf{v}\cdot\mathbf{A}\right) - \frac{B}{c}\mathbf{v}\cdot\nabla\mathbf{A}\cdot\nabla_v\mu_1 \\ & - \frac{v_\parallel}{c}\mathbf{v}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{A} - \frac{A_\parallel}{c}\mathbf{v}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{v} - \left(\Phi - \frac{v_\parallel}{c}A_\parallel\right)\mathbf{v}\cdot\nabla\ln B \\ & \left. + \frac{Ze}{M}A_\parallel \hat{\mathbf{n}}\cdot\nabla\Phi_0\right] + \frac{c}{B}\nabla\left(\Phi - \frac{1}{c}\mathbf{v}\cdot\mathbf{A}\right)\times\hat{\mathbf{n}}\cdot\bar{\nabla}F \end{aligned} \tag{33}$$

where from equations (23) and (27) we may determine $B\nabla_v\mu_1$ to be

$$\begin{aligned} B\nabla_v\mu_1 = & -\mathbf{v}_d - \frac{1}{4\Omega}\hat{\mathbf{n}}(\hat{\mathbf{n}}\times\nabla\hat{\mathbf{n}} - \nabla\hat{\mathbf{n}}\times\hat{\mathbf{n}}):\mathbf{v}_\perp\mathbf{v}_\perp \\ & - \frac{v_\parallel}{4\Omega}[(\hat{\mathbf{n}}\times\nabla\hat{\mathbf{n}} - \nabla\hat{\mathbf{n}}\times\hat{\mathbf{n}})\cdot\mathbf{v}_\perp + \mathbf{v}_\perp\cdot(\hat{\mathbf{n}}\times\nabla\hat{\mathbf{n}} - \nabla\hat{\mathbf{n}}\times\hat{\mathbf{n}})] \\ & - \frac{1}{\Omega}[\mathbf{v}_\perp\mathbf{v}_\perp\cdot\hat{\mathbf{n}}\times\nabla\ln B + \tfrac{7}{4}v_\parallel\hat{\mathbf{n}}\mathbf{v}_\perp\cdot\nabla\times\hat{\mathbf{n}} + (\tfrac{1}{2}v_\perp{}^2\hat{\mathbf{n}} + v_\parallel\mathbf{v}_\perp)\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}]. \end{aligned} \tag{34}$$

Our double dot convention is $\mathbf{ab}:\mathbf{cd} = \mathbf{a}\cdot\mathbf{d}\,\mathbf{b}\cdot\mathbf{c}$.

Performing the gyrokinetic change of variables on the d/dt on the left side of equation (33), taking $-\Omega\partial h/\partial\phi = 0$ to lowest order, and gyroaveraging the next order equation yields

$$\begin{aligned} \frac{\partial h}{\partial t} + (\bar{v}_\parallel\hat{\mathbf{n}} + \mathbf{v}_d)\cdot\bar{\nabla}h = & -\frac{Ze}{M}\frac{\partial F}{\partial E}\frac{\partial}{\partial t}\left\langle\Phi - \frac{1}{c}\mathbf{v}\cdot\mathbf{A}\right\rangle + \frac{c}{B}\left\langle\nabla\left(\Phi - \frac{1}{c}\mathbf{v}\cdot\mathbf{A}\right)\times\hat{\mathbf{n}}\right\rangle\cdot\nabla F \\ & - \frac{Ze}{MB}\frac{\partial F}{\partial\mu}\left[\frac{\partial}{\partial t}\left\langle\Phi - \frac{1}{c}\mathbf{v}\cdot\mathbf{A}\right\rangle\right. \\ & + \left\langle(v_\parallel\hat{\mathbf{n}} - B\nabla_v\mu_1)\cdot\nabla\left(\Phi - \frac{1}{c}\mathbf{v}\cdot\mathbf{A}\right)\right\rangle \\ & - (B/c)\langle\mathbf{v}\cdot\nabla\mathbf{A}\cdot\nabla_v\mu_1\rangle - (v_\parallel/c)\langle\mathbf{v}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{A}\rangle - \langle(A_\parallel/c)\mathbf{v}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{v}\rangle \\ & \left. - \langle(\Phi - v_\parallel A_\parallel/c)\mathbf{v}\cdot\nabla\ln B\rangle + (Ze/Mc)\langle A_\parallel\rangle\hat{\mathbf{n}}\cdot\nabla\Phi_0\right] \end{aligned} \tag{35}$$

where $O(\rho/L_0)$ corrections to gyroaveraged quantities may be neglected.

In order to explicitly evaluate the gyroaverage we consider the high mode number ballooning limit in which eikonal forms are appropriate for Φ and $\mathbf{A}$

$$[\Phi(\mathbf{r}, t), \mathbf{A}(\mathbf{r}, t)] = [\tilde{\Phi}(\mathbf{r}), \tilde{\mathbf{A}}(\mathbf{r})] \exp[iS(\mathbf{r}) - i\omega t] \tag{36}$$

with $\hat{\mathbf{n}} \cdot \nabla S = 0$, $\nabla S \equiv i\mathbf{k}_\perp = i\mathbf{k}_\perp(\mathbf{r})$ and $\tilde{\Phi}$ and $\tilde{\mathbf{A}}$ slow functions of $\mathbf{r}$. Because $k_\parallel \ll k_\perp$, we may then employ $\nabla\Phi \approx i\mathbf{k}_\perp\Phi$ and $\nabla\mathbf{A} \approx i\mathbf{k}_\perp\mathbf{A}$ everywhere on the right-hand side of equation (35) except in the $v_\parallel \hat{\mathbf{n}} \cdot \nabla(\Phi - \mathbf{v} \cdot \mathbf{A}/c)$ term. Expanding $S(\mathbf{r})$ about $\mathbf{R}$, and defining

$$-i\nabla S = \mathbf{k}_\perp = k_\perp(\mathbf{e}_1 \cos\alpha + \hat{\mathbf{e}}_2 \sin\alpha) \tag{37}$$

and

$$L = \Omega^{-1}\mathbf{k}_\perp \cdot \hat{\mathbf{n}} \times \mathbf{v} = (k_\perp v_\perp/\Omega) \sin(\alpha - \Omega) \tag{38}$$

we may then write

$$[\Phi(\mathbf{r}, t), \mathbf{A}(\mathbf{r}, t)] \approx [\Phi(\mathbf{R}, t), \mathbf{A}(\mathbf{R}, t)] \exp(iL). \tag{39}$$

The gyroaverages may then be performed by employing

$$\begin{aligned}
\langle \exp(iL) \rangle &= J_0(k_\perp v_\perp/\Omega) \\
\langle \mathbf{k}_\perp \cdot \mathbf{v}_\perp \exp(iL) \rangle &= 0 \\
\langle \mathbf{v}_\perp \exp(iL) \rangle &= (iv_\perp/k_\perp) J_1(k_\perp v_\perp/\Omega) \mathbf{k}_\perp \times \hat{\mathbf{n}} \\
\langle \mathbf{k}_\perp \cdot \mathbf{v}_\perp \mathbf{v}_\perp \exp(iL) \rangle &= (v_\perp \Omega/k_\perp) J_1(k_\perp v_\perp/\Omega) \mathbf{k}_\perp \\
\langle \mathbf{v}_\perp \mathbf{v}_\perp \exp(iL) \rangle &= (v_\perp \Omega/k_\perp^3) J_1(k_\perp v_\perp/\Omega) \mathbf{k}_\perp \mathbf{k}_\perp \\
&\quad + (v_\perp^2/k_\perp^2) J_1'(k_\perp v_\perp/\Omega) \mathbf{k}_\perp \times \hat{\mathbf{n}} \mathbf{k}_\perp \times \hat{\mathbf{n}} \\
\langle \mathbf{k}_\perp \cdot \mathbf{v}_\perp \mathbf{v}_\perp \mathbf{v}_\perp \exp(iL) \rangle &= -i(v_\perp^2 \Omega/k_\perp^2)[J_1'(k_\perp v_\perp/\Omega) \\
&\quad - (\Omega/k_\perp v_\perp) J_1(k_\perp v_\perp/\Omega)](\mathbf{k}_\perp \mathbf{k}_\perp \times \hat{\mathbf{n}} + \mathbf{k}_\perp \times \hat{\mathbf{n}} \mathbf{k}_\perp)
\end{aligned} \tag{40}$$

where a prime denotes a derivative with respect to argument. The gyroaverage of equation (35) then yields

$$\begin{aligned}
\frac{\partial h}{\partial t} + (\bar{v}_\parallel \hat{\mathbf{n}} + \mathbf{v}_d) \cdot \bar{\nabla} h &= \left(-\frac{Ze}{M} \frac{\partial F}{\partial E} \frac{\partial}{\partial t} + \frac{ic}{B} \mathbf{k}_\perp \times \hat{\mathbf{n}} \cdot \bar{\nabla} F \right) \\
&\quad \times \left[\left(\Phi - \frac{v_\parallel}{c} A_\parallel \right) J_0 + \frac{v_\perp}{k_\perp c} B_\parallel J_1 \right] \\
&\quad - \left[\frac{\partial}{\partial t} + (v_\parallel \hat{\mathbf{n}} + \mathbf{v}_d) \cdot \bar{\nabla} \right] \left\{ \frac{Ze}{MB} \frac{\partial F}{\partial \mu} \left[\left(\Phi - \frac{v_\parallel}{c} A_\parallel \right) J_0 + \frac{v_\perp}{k_\perp c} B_\parallel J_1 \right] \right\}
\end{aligned} \tag{41}$$

with

$$B_\parallel \equiv i\hat{\mathbf{n}} \cdot \mathbf{k}_\perp \times \mathbf{A} \tag{42}$$

and where Φ, $A_\parallel$, and $B_\parallel$ are all functions of $\mathbf{R}$, t. Equation (41) is obtained without recourse to a gauge for $\mathbf{A}$. The algebraic details necessary to obtain equation (41) from (35) are presented in the Appendix.

Because $k_\parallel \ll k_\perp$, the distinction between $v_\parallel$ and $\bar{v}_\parallel$ becomes unnecessary so that we define g as

$$g = h + \frac{Ze}{MB} \frac{\partial F}{\partial \mu} \left\{ \left[\Phi(\mathbf{R}, t) - \frac{v_\parallel}{c} A_\parallel(\mathbf{R}, t) \right] J_0 + \frac{v_\perp}{k_\perp c} B_\parallel(\mathbf{R}, t) J_1 \right\}. \tag{43}$$

Equation (41) then becomes

$$\frac{\partial g}{\partial t}+(v_{\parallel}\hat{\mathbf{n}}+\mathbf{v}_d)\cdot\bar{\nabla}g=\left(-\frac{Ze}{M}\frac{\partial F}{\partial E}\frac{\partial}{\partial t}+\frac{ic}{B}\mathbf{k}_\perp\times\hat{\mathbf{n}}\cdot\nabla F\right)\left[\left(\Phi-\frac{v_{\parallel}}{c}A_{\parallel}\right)J_0+\frac{v_\perp}{k_\perp c}B_{\parallel}J_1\right]. \tag{44}$$

Seeking an eikonal solution form for g

$$g=\tilde{g}(\mathbf{R},E,\mu)\exp\left[-i\omega t+iS(\mathbf{R})\right] \tag{45}$$

with $\nabla S(\mathbf{R})=\mathbf{k}_\perp$; and inserting equation (36) and writing $\hat{\mathbf{n}}\cdot\bar{\nabla}=\partial/\partial\bar{s}$, equation (44) becomes

$$v_{\parallel}\frac{\partial\tilde{g}}{\partial s}-i(\omega-\mathbf{k}_\perp\cdot\mathbf{v}_d)\tilde{g}=i\left(\frac{Ze\omega}{M}\frac{\partial F}{\partial E}+\frac{c}{B}\mathbf{k}_\perp\times\hat{\mathbf{n}}\cdot\nabla F\right)\left[\left(\tilde{\Phi}-\frac{v_{\parallel}}{c}\tilde{A}_{\parallel}\right)J_0+\frac{v_\perp}{k_\perp c}\tilde{B}_{\parallel}J_1\right] \tag{46}$$

where $\tilde{g}$, $\tilde{\Phi}$, $\tilde{A}_{\parallel}$ and $\tilde{B}_{\parallel}$ are all slow functions of $\mathbf{R}$. Equation (46) is the final form of the linearized gyrokinetic equation and is to solved for $\tilde{g}(\mathbf{R},E,\mu)$ which is then inserted into equations (31) and (43) to obtain f. In order to form moments of f, however, it must be written in terms of $\mathbf{r}$ rather than $\mathbf{R}$ by using equation (39) and expanding (45) about $\mathbf{r}$ to obtain

$$g(\mathbf{R},E,\mu,t)=g(\mathbf{r},E,\mu,t)\exp(-iL). \tag{47}$$

As a result, writing $f=\tilde{f}(\mathbf{r},E,\mu)\exp\left[iS(\mathbf{r})-i\omega t\right]$ and using equation (31) (with the $\mathbf{A}\times\hat{\mathbf{n}}\cdot\nabla F$ term neglected as being down by ρ/L_0), (43), (45) and (47) gives

$$\tilde{f}=\frac{Ze}{M}\frac{\partial F}{\partial E}\tilde{\Phi}+\frac{Ze}{MB}\frac{\partial F}{\partial\mu}\left\{\left(\tilde{\Phi}-\frac{v_{\parallel}}{c}\tilde{A}_{\parallel}\right)[1-J_0\exp(-iL)]-(v_\perp/k_\perp c)\tilde{B}_{\parallel}J_1\exp(-iL)\right\}$$
$$+\tilde{g}\exp(-iL) \tag{48}$$

where the distinction between $\mathbf{r}$ and $\mathbf{R}$ need not be retained in the slow eikonal amplitudes $\tilde{g}$, $\tilde{\Phi}$, $\tilde{A}_{\parallel}$, and $B_{\parallel}$. Note that the gyrophase integration implied by $\mathrm{d}v^3=2B\,\mathrm{d}E\,\mathrm{d}\mu\,\mathrm{d}\phi/|v_{\parallel}|$ will result in additional Bessel functions. All gyrokinetic techniques discriminate between particle $\mathbf{r}$ and guiding center $\mathbf{R}$ locations perpendicular to field lines. The distinction between $\mathbf{R}$ and $\mathbf{r}$ is a feature of a particle's trajectory and independent of whether an eikonal or ballooning formalism is employed. Equations (46) and (48) are in agreement with the recent results of ANTONSEN and LANE (1980).

In equation (46) we observe that $\bar{s}$ is the guiding center location along a field line (and is related to the particle location by $\bar{s}=s+\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}\cdot\nabla s$). In performing the trajectory integral solution of equation (46) it may be necessary to retain the $\bar{s}$ dependence of $J_0(k_\perp v_\perp/\Omega)$ in some limits (BELLEW and BAKSHI, 1976; and LINSKER, 1980), although it is often neglected.

A closed system of three equations in the three unknowns $\tilde{\Phi}$, $\tilde{A}_{\parallel}$, and $\tilde{B}_{\parallel}$ can then be obtained from quasi-neutrality

$$\sum Ze\int\mathrm{d}^3v\tilde{f}=0 \tag{49}$$

and the two non-zero components of Ampere's law

$$\frac{4\pi}{c}\sum Ze\int d^3v(\hat{\mathbf{n}}v_{\parallel}+k_{\perp}^{-2}\mathbf{k}_{\perp}\times\hat{\mathbf{n}}\mathbf{k}_{\perp}\times\hat{\mathbf{n}}\cdot\mathbf{v})f=\mathbf{k}_{\perp}\tilde{\mathbf{A}}-\mathbf{k}_{\perp}\mathbf{k}_{\perp}\cdot\tilde{\mathbf{A}}=k_{\perp}^{2}\tilde{A}_{\parallel}\hat{\mathbf{n}}+i\tilde{B}_{\parallel}\mathbf{k}_{\perp}\times\hat{\mathbf{n}} \tag{50}$$

where $\sum$ is used to denote a sum over all charge species. In writing Ampere's law it has not been necessary to specify a gauge, since within the eikonal formalism, the neglect of displacement current allows the final form of equation (50) to be obtained by writing $\mathbf{A}$ as in equation (A8) with the definition (42) for $B_{\parallel}$ inserted. In order to obtain $\mathbf{k}_{\perp}\cdot\tilde{\mathbf{J}}$, where $\tilde{\mathbf{J}}=\sum Ze\int d^3v\,\mathbf{v}\tilde{f}$, one employs $\boldsymbol{\nabla}\cdot\tilde{\mathbf{J}}=0$ to obtain

$$\mathbf{k}_{\perp}\cdot\tilde{\mathbf{J}}=i\mathbf{B}\cdot\boldsymbol{\nabla}(B^{-1}\hat{\mathbf{n}}\cdot\mathbf{J})=\frac{ic}{4\pi}\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{k_{\perp}^{2}}{B}\tilde{A}_{\parallel}\right). \tag{51}$$

As a result, it is often convenient to employ

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{4\pi}{cB}\sum Ze\int d^3v\,v_{\parallel}\tilde{f}\right)=\mathbf{B}\cdot\boldsymbol{\nabla}\left(\frac{k_{\perp}^{2}}{B}\tilde{A}_{\parallel}\right) \tag{52}$$

in place of the parallel component of equation (50).

4. DISCUSSION

In the preceding sections we have presented a derivation of the gyrokinetic equation which employs a gyrokinetic change of variables incorporating both the higher order expression for the magnetic moment as well as the distinction between the particle and guiding center location. Equations (46) and (48) represent our principle results, which when solved and inserted in quasi-neutrality and Ampere's law give a closed system of equations. In dealing with equations (46) and (48) it should be kept in mind that the distinction between $\mathbf{R}$ and $\mathbf{r}$ in the slow eikonal amplitude $\tilde{\Phi}$, $\tilde{\mathbf{A}}$ and $\tilde{g}$ of equations (36) and (45) need not be retained, and that F is taken to be of the form $F(E,\mu,\mathbf{R})$ with $\hat{\mathbf{n}}\cdot\bar{\boldsymbol{\nabla}}F=\partial F/\partial\bar{s}=0$.

Acknowledgements—This work was supported by the U.S. Department of Energy under contracts DE-AC03-79-ET-53057 at Science Applications, Inc. and W-7405-ENG-48 at Lawrence Livermore Laboratory. P. J. C. performed the initial portion of this research at the University of Rochester under contract EY-76-02-3497. The authors had the benefit of important discussions with T. Antonsen.

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APPENDIX

In order to obtain equation (41) from (35) by inserting (40), some vector manipulations are required in the terms multiplied by $\partial F/\partial\mu$. For simplicity we first consider only the terms in Φ in order to illustrate the method. This restriction will be removed shortly.

Inserting equation (34) into the Φ terms of equation (35) and using (40) yields

$$\begin{aligned}
i\langle(B\mathbf{k}_\perp\cdot\nabla_v\mu_1)\Phi\rangle+\langle\Phi\mathbf{v}\cdot\nabla\ln B\rangle &= -i\mathbf{k}_\perp\cdot\mathbf{v}_d\langle\Phi\rangle \\
&\quad-\frac{iv_\parallel}{4\Omega}(\hat{\mathbf{n}}\times\nabla\hat{\mathbf{n}}-\nabla\hat{\mathbf{n}}\times\hat{\mathbf{n}}):\langle(\mathbf{k}_\perp\mathbf{v}_\perp+\mathbf{v}_\perp\mathbf{k}_\perp)\Phi\rangle \\
&\quad-i\Omega^{-1}\langle\mathbf{k}_\perp\cdot\mathbf{v}_\perp\mathbf{v}_\perp\Phi\rangle\cdot\hat{\mathbf{n}}\times\nabla\ln B \qquad\text{(A1)}\\
&\quad-i\Omega^{-1}v_\parallel(\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}})\langle\mathbf{k}_\perp\cdot\mathbf{v}_\perp\Phi\rangle+\langle\Phi\mathbf{v}\rangle\cdot\nabla\ln B \\
&= -ik_\perp\cdot\mathbf{v}_d\Phi J_0+(v_\perp v_\parallel/2k_\perp\Omega)J_1\Phi(\mathbf{k}_\perp\times\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{k}_\perp\times\hat{\mathbf{n}}-\mathbf{k}_\perp\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{k}_\perp) \\
&\quad+\Phi J_0 v_\parallel\hat{\mathbf{n}}\cdot\nabla\ln B.
\end{aligned}$$

Using

$$\mathbf{k}_\perp\times\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{k}_\perp\times\hat{\mathbf{n}}+\mathbf{k}_\perp\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{k}_\perp=k_\perp{}^2\nabla\cdot\hat{\mathbf{n}}=-k_\perp{}^2\hat{\mathbf{n}}\cdot\nabla\ln B \qquad\text{(A2)}$$

and noting that

$$\mathbf{k}_\perp\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{k}_\perp=-\mathbf{k}_\perp\cdot\nabla\mathbf{k}_\perp\cdot\hat{\mathbf{n}}=-\hat{\mathbf{n}}\cdot\nabla\mathbf{k}_\perp\cdot\mathbf{k}_\perp=-k_\perp\hat{\mathbf{n}}\cdot\nabla k_\perp \qquad\text{(A3)}$$

since $\nabla\times\mathbf{k}_\perp\times 0$, we may write

$$\mathbf{k}_\perp\times\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{k}_\perp\times\hat{\mathbf{n}}-\mathbf{k}_\perp\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{k}_\perp=-k_\perp{}^2\hat{\mathbf{n}}\cdot\nabla\ln B+2k_\perp\hat{\mathbf{n}}\cdot\nabla k_\perp. \qquad\text{(A4)}$$

As a result, the desired equation is obtained, namely

$$\frac{1}{B}\left[\frac{\partial\langle\Phi\rangle}{\partial t}+\langle(v_\parallel\hat{\mathbf{n}}-B\nabla_v\mu_1)\cdot\nabla\Phi\rangle-\langle\Phi\mathbf{v}\cdot\nabla\ln B\rangle\right]\approx\left[\frac{\partial}{\partial t}+(v_\parallel\hat{\mathbf{n}}+\mathbf{v}_d)\cdot\bar{\nabla}\right]\left[\frac{\Phi(\mathbf{R},t)J_0(k_\perp v_\perp/\Omega)}{B}\right] \qquad\text{(A5)}$$

since

$$\begin{aligned}
B\hat{\mathbf{n}}\cdot\nabla[J_0(k_\perp v_\perp/\Omega)/B] &= -J_0\hat{\mathbf{n}}\cdot\nabla\ln B+J_0{}'\hat{\mathbf{n}}\cdot\nabla(k_\perp v_\perp/\Omega)\\
&= -J_0\hat{\mathbf{n}}\cdot\nabla\ln B-(k_\perp v_\perp/\Omega)J_1\hat{\mathbf{n}}\cdot\nabla\ln(k_\perp/B^{1/2})\\
&= [-J_0+(k_\perp v_\perp/2\Omega)J_1]\hat{\mathbf{n}}\cdot\nabla\ln B-(v_\perp/\Omega)J_1\hat{\mathbf{n}}\cdot\nabla k_\perp.
\end{aligned}$$

Next we consider the terms in $\mathbf{A}$. In order to simplify this portion of the calculation we first gather up the $v_\parallel A_\parallel J_0/B$ term that enters in a manner similar to the $\Phi J_0/B$ term. Employing equations (A2) and (A4), we find using equation (40) that

$$\begin{aligned}
&-i\mathbf{k}_\perp\cdot\mathbf{v}_d v_\parallel\langle A_\parallel\rangle+(i/4\Omega)(\hat{\mathbf{n}}\times\nabla\hat{\mathbf{n}}-\nabla\hat{\mathbf{n}}\times\hat{\mathbf{n}}):\langle\mathbf{v}_\perp\mathbf{v}_\perp\mathbf{k}_\perp\cdot\mathbf{v}_\perp A_\parallel\rangle\\
&\quad-(iv_\parallel{}^2/4\Omega)(\hat{\mathbf{n}}\times\nabla\hat{\mathbf{n}}-\nabla\hat{\mathbf{n}}\times\hat{\mathbf{n}}):\langle(\mathbf{v}_\perp\mathbf{k}_\perp+\mathbf{k}_\perp\mathbf{v}_\perp)A_\parallel\rangle-\nabla\hat{\mathbf{n}}:\langle\mathbf{v}_\perp\mathbf{v}_\perp A_\parallel\rangle\\
&\quad+v_\parallel{}^2\hat{\mathbf{n}}\cdot\nabla\ln B\langle A_\parallel\rangle+(Ze/M)(\hat{\mathbf{n}}\cdot\nabla\Phi_0)\langle A_\parallel\rangle+v_\parallel\langle A_\parallel\mathbf{v}_\perp\rangle\cdot\nabla\ln B\\
&\quad-(iv_\parallel/\Omega)\langle A_\parallel\mathbf{k}_\perp\cdot\mathbf{v}_\perp\mathbf{v}_\perp\rangle\cdot\hat{\mathbf{n}}\times\nabla\ln B\\
&=-i\mathbf{k}\cdot\mathbf{v}_d v_\parallel J_0+v_\parallel{}^2A_\parallel\left(J_0-\frac{k_\perp v_\perp}{2\Omega}J_1\right)\hat{\mathbf{n}}\cdot\nabla\ln B \qquad\text{(A6)}\\
&\quad+\frac{v_\perp v_\parallel{}^2}{\Omega}A_\parallel J_1\hat{\mathbf{n}}\cdot\nabla k_\perp+\tfrac{1}{2}v_\perp{}^2A_\parallel J_0\hat{\mathbf{n}}\cdot\nabla\ln B+\frac{Ze}{M}(\hat{\mathbf{n}}\cdot\nabla\Phi_0)A_\parallel J_0\\
&=-B(v_\parallel\hat{\mathbf{n}}+\mathbf{v}_d)\cdot\bar{\nabla}(v_\parallel A_\parallel J_0/B).
\end{aligned}$$

As a result, the $A_\parallel$ terms may be gathered up by using equation (A6) and

$$\langle v_\parallel{}^2\hat{\mathbf{n}}\cdot\nabla\mathbf{A}\cdot\hat{\mathbf{n}}\rangle+\langle v_\parallel{}^2\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{A}\rangle=v_\parallel{}^2\hat{\mathbf{n}}\cdot\nabla\langle A_\parallel\rangle=v_\parallel{}^2\hat{\mathbf{n}}\cdot\nabla(J_0A_\parallel). \qquad\text{(A7)}$$

Finally, we evaluate the remaining $\mathbf{A}$ terms in the $\partial F/\partial\mu$ coefficient. In order to perform this task we first note that we may write $\mathbf{A}$ as

$$\mathbf{A}=k_\perp^{-2}\mathbf{k}_\perp\mathbf{k}_\perp\cdot\mathbf{A}+k_\perp^{-2}\mathbf{k}_\perp\times\hat{\mathbf{n}}\mathbf{k}_\perp\times\hat{\mathbf{n}}\cdot\mathbf{A}+\hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\mathbf{A} \tag{A8}$$

in order to gather up terms in $\mathbf{k}_\perp\cdot\mathbf{A}$ and $B_\parallel$. In addition we use $\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}=-\hat{\mathbf{n}}\times(\nabla\times\hat{\mathbf{n}})$ and $\hat{\mathbf{n}}\times[(\nabla\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}]=\nabla\times\hat{\mathbf{n}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}}=-(\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}})\times\hat{\mathbf{n}}$ to obtain the useful expression

$$\mathbf{k}_\perp\cdot\nabla\times\hat{\mathbf{n}}=\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{k}_\perp\times\hat{\mathbf{n}}.$$

Using $\nabla\times\mathbf{k}_\perp=0$ and equations (40) again, we then may gather up the $B_\parallel$ terms from

$$\begin{aligned}
&-v_\parallel\hat{\mathbf{n}}\cdot\bar{\nabla}\langle\mathbf{A}\cdot\mathbf{v}_\perp\rangle-i\mathbf{k}_\perp\cdot\mathbf{v}_d\langle\mathbf{v}_\perp\cdot\mathbf{A}\rangle+(iv_\parallel/4\Omega)(\hat{\mathbf{n}}\times\nabla\hat{\mathbf{n}}-\nabla\hat{\mathbf{n}}\times\hat{\mathbf{n}}):\langle(\mathbf{v}_\perp\mathbf{A}+\mathbf{A}\mathbf{v}_\perp)\mathbf{k}_\perp\cdot\mathbf{v}_\perp\rangle\\
&\quad+(i7v_\parallel/4\Omega)\langle A_\parallel\mathbf{k}_\perp\cdot\mathbf{v}_\perp\mathbf{v}_\perp\rangle\cdot\nabla\times\hat{\mathbf{n}}-v_\parallel\nabla\hat{\mathbf{n}}:\langle\mathbf{A}\mathbf{v}\rangle\\
&\quad-(iv_\parallel/\Omega)(\hat{\mathbf{n}}\times\nabla\hat{\mathbf{n}}-\nabla\hat{\mathbf{n}}\times\hat{\mathbf{n}}):\langle(\mathbf{v}_\perp\mathbf{k}_\perp+\mathbf{k}_\perp\mathbf{v}_\perp)\mathbf{v}_\perp\cdot\mathbf{A}\rangle-v_\parallel\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}\cdot\langle\mathbf{v}_\perp A_\parallel\rangle\\
&=(v_\parallel\hat{\mathbf{n}}\cdot\bar{\nabla}+i\mathbf{k}_\perp\cdot\mathbf{v}_d)\frac{v_\perp}{k_\perp}J_1B_\parallel-\frac{iv_\perp v_\parallel}{k_\perp}J_1A_\parallel\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{k}_\perp\times\mathbf{A}\\
&\quad-\frac{iv_\perp v_\parallel}{k_\perp^{3}}J_1\mathbf{k}_\perp\cdot\mathbf{A}\mathbf{k}_\perp\times\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}\cdot\mathbf{k}_\perp+\frac{v_\parallel v_\perp}{k_\perp^{2}}B_\parallel\hat{\mathbf{n}}\cdot\nabla k_\perp-\frac{v_\perp v_\parallel B_\parallel}{2k_\perp}\left(J_1+\frac{k_\perp v_\perp}{\Omega}J_0\right)\hat{\mathbf{n}}\cdot\nabla\ln B\\
&=B(\mathbf{v}_d+v_\parallel\hat{\mathbf{n}})\cdot\bar{\nabla}\left(\frac{v_\perp J_1B_\parallel}{k_\perp B}\right).
\end{aligned} \tag{A9}$$

Consequently, by employing equations (A6), (A7) and (A9), (41) may be obtained from (35).