Correction to the Alfvén-Lawson criterion for relativistic electron beams

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The Alfvén-Lawson criterion for relativistic electron beams is revised. The parameter range is found, in which a stationary beam can carry arbitrarily large current, regardless of its transverse structure. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2358970]

I. INTRODUCTION

The theoretical interest in the limiting currents in relativistic beams of charged particles has revived recently due to the latest advances in high-power technology. Currently available laser systems can deliver electromagnetic pulses with intensities about $10^{21}\ \mathrm{W/cm^2}$,[1–3] which, interacting with solid targets, produce relativistic electron beams (REBs) with huge current densities of the order of $10^{12}\ \mathrm{A/cm^2}$.[4,5] The unique feature of these beams is that they can transport large energies inside a plasma where the laser radiation itself does not penetrate. This is particularly important for, among other applications, the fast ignition technique in inertial confinement fusion, which requires propagation of nearly a GA beam over a distance of a few 100 μm.[6,7] To propagate such intense currents, however, remains a challenge[8] because of the REB filamentation,[4,9–11] also related to the Weibel instability, which occurs in a plasma with anisotropic temperature.[12,13]

The filamentation instability is due to the pinching force, which the beam cannot resist if the self-generated magnetic pressure exceeds the kinetic pressure,[14,15] assuming the beam electric field is neutralized by the plasma background. This condition, first introduced by Bennett for Maxwellian plasmas,[16,17] imposes a limitation on the total current I carried by the beam. As shown by Alfvén,[18] for a stationary monoenergetic beam of particles with charge e, mass m, normalized velocity $\beta_0 = v_0/c$ (here c is the speed of light), and $\gamma_0 = (1-\beta_0^2)^{-1/2}$, the limitation reads $I \lesssim 1.65 I_{\mathrm{A}}$, where

$$I_{\mathrm{A}} = \frac{mc^3}{e}\beta_0\gamma_0 \tag{1}$$

is called the Alfvén current, and $mc^3/e \approx 17$ kA for electrons. [The Alfvén result applies to the case when the current density profile $\mathbf{j} = \mathbf{z}^0 j_z(r)$ is step-like; $\mathbf{z}^0$ is the unit vector along the beam. Depending on the actual $j_z(r)$, the limiting current differs from I_{A} by a geometrical factor;[19–21] see also Sec. IV.]

For incomplete charge neutralization, the expression for the maximum current $I_{\max}$ must be revised taking into account the self-generated electrostatic field of the beam. The corresponding modification of the Alfvén limit was analyzed by Lawson[22–24] under the assumption that the background ion density is proportional to the beam electron density with the constant coefficient f. The approximate expression for $I_{\max}$ was later summarized by Hammer and Rostoker[21] and reads

$$I_{\mathrm{AL}} = I_{\mathrm{A}}\frac{\beta_0^2}{\beta_0^2 - 1 + f}, \tag{2}$$

where we use "AL" for "Alfvén-Lawson." (A modified version of this formula was also proposed in Ref. 25; see Sec. IV C.) For f approaching γ_0^{-2}, Eq. (2) suggests that the maximum current can significantly exceed I_{A}. However, I_{AL} is still limited for given f (unless $f = \gamma_0^{-2}$), so the Alfvén-Lawson criterion allows only a finite current to be carried by a beam.[26] Hence, Eq. (2) predicts limited potential for increasing $I_{\max}$ by means of the electrostatic force, and, as a result, little effort has been applied to study this possibility so far. Importantly though, Eq. (2) is only approximate, so novel techniques to boost the stationary beam maximum current could flow from searching for a more precise expression for $I_{\max}$.

The objective of this paper is to show that boosting the maximum current by means of radial electrostatic fields is, in fact, considerably more promising compared to what follows from the Alfvén-Lawson limit, at least within the *same*, single-particle model originally used to derive Eq. (2). We do not address the self-consistent problem here;[21,27–33] however, we show that, as opposed to the order-of-magnitude calculation suggested in Refs. 21–24, the rigorous single-particle solution for $I_{\max}$ allows unlimited currents for f ranging in the interval of finite width below γ_0^{-2}.

The paper is organized as follows. In Sec. II, we review the existing techniques for beating the Alfvén limit and show how, in principle, radial electrostatic fields can support unlimited currents in relativistic beams. In Sec. III, we introduce a fully nonlinear single-particle formulation of the problem. In Sec. IV, we apply this formulation to derive the criterion of a solid beam. In Sec. V, we summarize our main results. Supplementary calculations are given in the Appendix.

II. DRIFT INTERPRETATION

Within a single-particle approach, the Alfvén criterion for a high-current REB can be explained as follows.[18] The beam-generated azimuthal magnetic field, zero on the beam axis z, grows with the radial coordinate r. The beam particles are therefore unmagnetized close to the axis and magnetized on the periphery, so they travel along z freely at small r and undergo drift motion at large r. Assuming the beam is completely neutralized ($f=1$) and r is large compared to the gyroradius r_g, the drift velocity is given by[34,35]

$$\mathbf{V}_m = \frac{c\mathbf{B}}{\gamma_0 e B^2} \times \left(\mu \nabla B + \frac{p_\parallel^2}{m} \boldsymbol{\kappa} \right), \tag{3}$$

with the index m standing for "magnetic." Here the two terms describe, respectively, the gradient drift and the curvature drift; $\mu = p_\perp^2/2mB$ is the adiabatically conserved magnetic moment, $p_\perp$ and $p_\parallel$ are the components of the kinetic momentum $\mathbf{p}$ transverse and parallel to the magnetic field $\mathbf{B}$, $\kappa = (\mathbf{B}\cdot\nabla)\mathbf{B}/B^2$ is the vector curvature of the magnetic field lines. Assuming that all the beam particles are initially launched strictly along z with $p_{z,0} > 0$ and there is no external magnetic field, one has $\mathbf{B} = \boldsymbol{\theta}^0 B_\theta$, where θ is the azimuthal angle and $B_\theta > 0$ (we take $e > 0$ for clarity). In this case, $p_\parallel = 0$, so the magnetized particles undergo only meridional motion (i.e., that at fixed θ), the curvature drift is zero, and the gradient drift is in $-z$ direction. (The full zoology of particle trajectories is discussed in Refs. 18, 31, and 36.) The magnetized particles hence provide negative contribution to the beam current, so, in fact, the positive current along z is possible only within $r \lesssim r_g$. Therefore, a solid beam with r exceeding the particle gyroradius cannot exist; thus, the maximum current in a single filament corresponds to $R \sim r_g$, where R is the beam radius. Assuming a constant current density for $r < R$, this estimate readily yields the Alfvén limit (1).

To increase the maximum current, a number of options are available. First of all, one can increase the gyroradius, $r_g = \mathcal{E}_0 \beta_0 / eB$, by increasing the particle energy $\mathcal{E} = mc^2 \gamma_0$. In addition to using larger γ_0 by accelerating the beam to a higher velocity, an alternative method involving effective increase of the particle mass is possible. The technique can be accomplished by propagating a relativistically intense laser pulse along the beam. If the laser frequency ω_w far exceeds the particle gyrofrequency, ponderomotive interaction with the laser field E_w supplies a particle with an effective mass $m_{\text{eff}} = m(1+a^2)^{1/2}$, where $a = eE_w/mc\omega_w$. Apart from the mass change, the particle gyromotion equations remain unchanged.[37–39] Hence, the Alfvén current increases by the factor $(1+a^2)^{1/2}$, which can be of the order of 10 or even larger given the parameters of the existing laser facilities.[1–3]

Beams with $I \gg I_{\text{A}}$[40] are also possible with small r_g, if the particle drift is reversed at $r > r_g$. Unlike the gradient drift, the curvature drift described by Eq. (3) is in $+z$ direction, and the ratio of the corresponding velocities roughly equals $\tan^2 \chi$, where χ is the pitch angle. Therefore, regardless of R, a beam can propagate with a fixed radius if particles are launched with $\chi \lesssim \pi/4$; i.e., primarily in the azimuthal direction, as also predicted in Ref. 41. This approach does not necessarily imply that magnetic field along z is generated, since equal number of particles can rotate in both $+\theta$ and $-\theta$ directions producing no azimuthal current in total.[41] However, the presence of such field, either self-generated or external, makes it even easier to propagate a solid beam[15,21,33] and can utilize the same effect.[42] The additional field $\mathbf{z}^0 B_z$ alters the *projection* of the initial particle momentum $\mathbf{p}_0$ on the field line: for a particle launched along z, $p_\parallel$ is now nonzero and equals $p_0 B_z / B$, where $B = (B_\theta^2 + B_z^2)^{1/2}$ is the total field; correspondingly, $p_\perp = p_0 B_\theta / B$. The ratio of the gradient drift velocity and the curvature drift velocity then equals B_θ / B_z, meaning that a solid beam is possible if $B_z \gtrsim B_\theta$. The same criterion was also obtained in Ref. 21 from other considerations.

Another effect, which can compensate for the negative gradient drift, is due to a conservative radial force on the particles, $\mathbf{F} = \mathbf{r}^0 F_r$. In this regard, the electrostatic field, either external[43] or beam-generated,[20–24] might be useful assuming it is not fully compensated by the background ($f < 1$). Alternatively, an average ponderomotive force can be applied (say, by propagating an electromagnetic wave along the beam). In this case, the average effect of the wave on the particles could be in producing an effective potential Φ, so $F_r = -\Phi'(r)$.[44,45]

The analytical treatment of the radial forces is equivalent in the two cases, so, for clarity, we will simply assume $F_r = eE_r$. The radial electrostatic field $E_r > 0$ produces the $\mathbf{E} \times \mathbf{B}$ drift $V_e \sim c\hat{\beta}$ in $+z$ direction, $\hat{\beta} \equiv E_r / B_\theta$.[34,35] The gradient drift in $-z$ direction is $V_m \sim cp_\perp^2 / 2me\gamma_0 r B_\theta$, or $V_m \sim c\beta_0 r_g / 2r$; therefore, to have a solid beam, $\hat{\beta}/\beta_0 \gtrsim r_g / 2r$ is required. For a positive net drift at arbitrary r in the drift domain $r > r_g$, it is then sufficient to have $\hat{\beta} \gtrsim \beta_0/2$, assuming $\hat{\beta}$ does not change drastically with r. In addition, we must require that all particles reside within $r \leq R$ at any time, which is most easily violated for the edge particles (initially placed $r_0 = R$) as they exhibit cyclotron rotation. For a particle launched along z with $p_{z,0} > 0$, the initial longitudinal momentum in the drift frame (where the electric field vanishes) cannot be negative then, which gives $\hat{\beta} < \beta_0$. Hence, the general condition for propagating a solid beam with an arbitrary R reads

$$\beta_0/2 \lesssim \hat{\beta} < \beta_0, \tag{4}$$

where the coefficient $1/2$ is, of course, approximate.

Equation (4) predicts that, in order to boost the limiting current of a beam, the electric field strength must only fall within a certain range; if it does, the beam radius (and hence the net current) can be arbitrarily large. In Secs. III and IV, we re-derive this result [and Eq. (4)] more accurately.

III. BASIC EQUATIONS

Following the argument from Sec. II, consider a relativistic particle beam with a given current density $\mathbf{j} = \mathbf{z}^0 j_z(r)$ and radius R, so that the radial velocity at $r = R$ must be equal to zero, and the radial acceleration at the beam edge must be zero or negative. The "solid beam" condition is to have $j_z(r) > 0$ for all $r < R$; thus, we will require all the beam particles to travel on average in the same $+z$ direction (assuming $e > 0$), regardless of their initial radial location r_0. The minimum r_0 for which the condition $\langle v_z(r_0) \rangle > 0$ is violated, gives the maximum radius of a solid beam $R_{\max}$:

$$\langle v_z(R_{\max}) \rangle = 0. \tag{5}$$

After Ref. 18, we will assume particles to have the same initial momentum $\mathbf{p}_0 = \mathbf{z}^0 mc\beta_0\gamma_0$ at each r_0. To find $R_{\max}$ we must then determine the sign of $\langle v_z \rangle$ as a function of r_0 at fixed β_0.

Rewrite the particle average velocity as

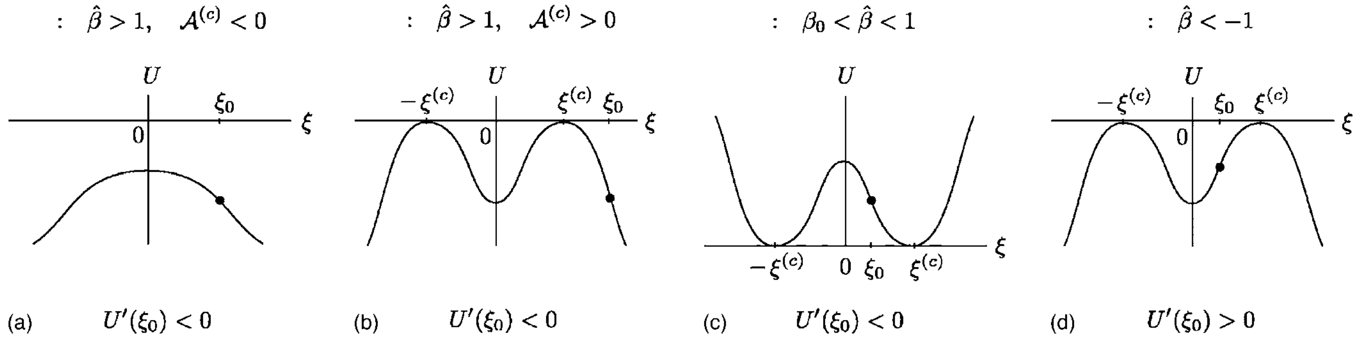

FIG. 1. The effective potential $U(\xi)$ [Eq. (9)] in various parameter domains: (a) $\hat{\beta}>1$, $\mathcal{A}^{(c)}<0$; (b) $\hat{\beta}>1$, $\mathcal{A}^{(c)}>0$; (c) $\beta_0<\hat{\beta}<1$; (d) $\hat{\beta}<-1$. Here, $\mathcal{A}^{(c)}=\mathcal{A}_0-\delta\mathcal{A}$, with $\delta\mathcal{A}$ given by Eq. (10); ξ_0 is the characteristic initial coordinate.

$$\langle v_z\rangle=\bar{p}_z/m\bar{\gamma}, \tag{6}$$

where the bar denotes averaging over the particle proper time τ (not to mix with averaging over t denoted by angle brackets), $d\tau=dt/\gamma$, and $\gamma=\mathcal{E}/mc^2$ is the normalized energy, $\bar{\gamma}>0$. With τ serving as the new time, the particle Hamiltonian derived in the Appendix reads

$$\mathcal{H}=\frac{p_r^2}{2m}+\frac{1}{2m}\left(P_z-\frac{e}{c}A_z\right)^2+\frac{m^2c^4-(\mathfrak{E}-e\phi)^2}{2mc^2}. \tag{7}$$

Here, $\mathcal{H}$ remains zero along each particle trajectory regardless of the initial conditions, $\mathbf{P}=\mathbf{p}+(e/c)\mathbf{A}$ is the canonical momentum, $\mathfrak{E}=mc^2\gamma+e\phi$ is the full particle energy, and ϕ is the electrostatic potential. After Ref. 18, we also assume no longitudinal fields; hence p_θ remains zero; i.e., particles exhibit purely meridional motion.

Since $\partial_t\mathcal{H}\equiv 0$ and $\partial_z\mathcal{H}\equiv 0$, the quantities $\mathfrak{E}$ and P_z are conserved, so we can put Eq. (7) in the form

$$\mathcal{H}=\tfrac{1}{2}\mathcal{P}_\xi^2+U(\xi), \tag{8a}$$

$$U=\tfrac{1}{2}(\mathcal{A}-\mathcal{A}_0+\mathcal{P}_0)^2-\tfrac{1}{2}(\varphi-\varphi_0+\gamma_0)^2, \tag{8b}$$

where we omitted an insignificant constant and used the following dimensionless notation: $\mathcal{P}_\xi=p_r/mc$, $\xi=r/\rho$, $\mathcal{P}_0=\gamma_0\beta_0$, $\mathcal{A}=-eA_z/mc^2$, $\varphi=-e\phi/mc^2$; the index "0" denotes the initial values. The quantity ρ is arbitrary here, but we also assume the new time to be normalized accordingly on ρ/c. After Ref. 31, we take $\hat{\beta}\equiv E_r/B_\theta$ independent of ξ for simplicity, so $\varphi/\mathcal{A}=\hat{\beta}$, where we choose φ and $\mathcal{A}$ equal to zero at $\xi=0$. Omitting a constant again, one then gets

$$U=\frac{1}{2\hat{\gamma}^2}(\mathcal{A}-\mathcal{A}^{(c)})^2, \tag{9}$$

where $\hat{\gamma}=(1-\hat{\beta}^2)^{-1/2}$, $\mathcal{A}^{(c)}=\mathcal{A}_0-\delta\mathcal{A}$, and

$$\delta\mathcal{A}=\mathcal{P}_0\hat{\gamma}^2\left(1-\frac{\hat{\beta}}{\beta_0}\right). \tag{10}$$

Since $\langle v_z\rangle=c\bar{\mathcal{P}}_z/\bar{\gamma}$, where

$$\bar{\mathcal{P}}_z=\mathcal{P}_0+\bar{\mathcal{A}}-\mathcal{A}_0, \tag{11}$$

knowing the particle radial dynamics will allow us to calculate the sign of the average velocity and hence to derive the criterion of a solid beam.

IV. CRITERION OF A SOLID BEAM

A. Parameter domains

First, consider the strong field case, i.e., $\hat{\beta}>1$, so the effective potential U [Eq. (9)] is negative. (The case $\hat{\beta}<-1$ is discussed in Sec. IV B.) Since $\mathcal{A}(\xi)$ is monotonic, two cases can be distinguished. If $\mathcal{A}^{(c)}<0$, $U(\xi)$ has a single maximum at $\xi=0$ [Fig. 1(a)]; if $\mathcal{A}^{(c)}>0$, $\xi=0$ corresponds to a minimum, whereas the potential also has two maxima at $\xi=\pm\xi^{(c)}$, such as $\mathcal{A}(\xi^{(c)})=\mathcal{A}^{(c)}$ [Fig. 1(b)]. (We consider only meridional motion here; hence, although ξ is a radial coordinate, it can formally take both positive and negative values corresponding, respectively, to $\theta=0$ and $\theta=\pi$.) Because $\delta\mathcal{A}>0$, one has $\mathcal{A}_0>\mathcal{A}^{(c)}$, so ξ_0 falls on the repulsive slope of $U(\xi)$ in both cases. This violates our assumption that $U'(\xi_0)\geq 0$ for edge particles; hence, beams with $\hat{\beta}>1$ cannot propagate with a constant radius, apparently due to electrostatic repulsion.

Supposing a weaker radial field, i.e., $\hat{\beta}^2<1$, then $U(\xi)$ is positive. At $\hat{\beta}>\beta_0$, one has $\delta\mathcal{A}<0$ and $\mathcal{A}^{(c)}>0$. The two minima of $U(\xi)$ are then located at $\xi=\pm\xi^{(c)}$ [Fig. 1(c)]. Since $\mathcal{A}^{(c)}>\mathcal{A}_0$ in this case, ξ_0 again falls on the repulsive slope of $U(\xi)$; hence, beams with $\hat{\beta}>\beta_0$ cannot have a fixed R either, again due to electrostatic repulsion rather than magnetic pinching. (Certainly, there could be solutions with an oscillating beam radius here, but we traditionally associate the current limit with stationary beams only.) Consider now the case $-1<\hat{\beta}<\beta_0$, so $\delta\mathcal{A}>0$ (Fig. 2). At sufficiently small ξ_0, one has $\mathcal{A}^{(c)}<0$; $U(\xi)$ is then monotonous, with the minimum at $\xi=0$ corresponding to a center on the phase plot $(\xi,\mathcal{P}_\xi)$. As ξ_0 passes the point where $\mathcal{A}_0=\delta\mathcal{A}$, a bifurcation occurs: a saddle appears at $\xi=0$ and two centers are born at $\xi=\pm\xi^{(c)}$. A separatrix forms in this case to separate the trajectories encompassing only one of the centers from those encompassing all the three equilibria. Note that the phase space itself is modified here as the initial conditions change.

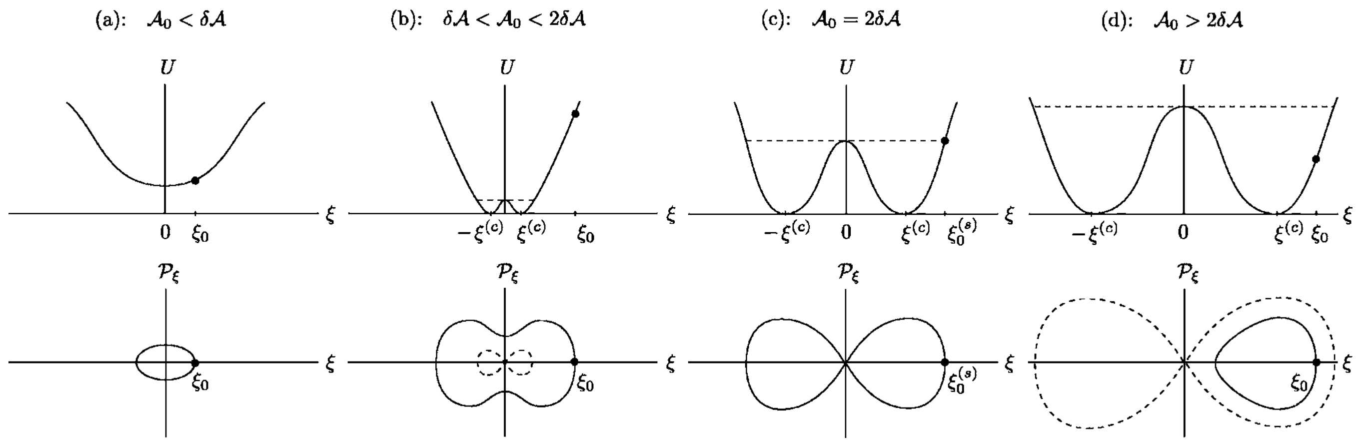

FIG. 2. Modification of the effective potential $U(\xi)$ [Eq. (9)] and the meridional oscillations phase space $(\xi, \mathcal{P}_\xi)$ as the particle initial coordinate ξ_0 increases $(-1 < \hat{\beta} < \beta_0)$. Here, $\mathcal{A}_0 \equiv \mathcal{A}(\xi_0)$, and $\delta\mathcal{A}$ is given by Eq. (10); dashed is the separatrix trajectory.

The separatrix coordinate $\xi^{(s)} = \xi^{(s)}(\xi_0)$ grows more quickly than ξ_0, so, counterintuitively, the particle trajectory approaches the separatrix from *outside* with the increase of ξ_0. For the trajectory to follow the separatrix, ξ_0 must equal $\xi_0^{(s)}$ such that $\xi^{(s)}(\xi_0^{(s)}) = \xi_0^{(s)}$. By definition, $U(\xi_0^{(s)}) = U(0)$, which yields $\mathcal{A}_0^{(s)} = 2\delta\mathcal{A}$. [We use the index (s) to denote the particular "separatrix" values.] The phase-space trajectories corresponding to $\mathcal{A}_0 > \mathcal{A}_0^{(s)}$ reside within the separatrix, meaning that the particle does not cross z axis in this case.

On the separatrix, the particle spends only finite time in the domain $\mathcal{A} > 0$ but infinitely large time at $\mathcal{A} = 0$; hence, $\bar{\mathcal{A}}^{(s)} = 0$. Using $\mathcal{A} \geq 0$, one can then write for $\mathcal{A}_0 < \mathcal{A}_0^{(s)}$:

$$\bar{\mathcal{P}}_z > \mathcal{P}_0 - \mathcal{A}_0 > \mathcal{P}_0 - \mathcal{A}_0^{(s)} = \bar{\mathcal{P}}_z^{(s)}. \tag{12}$$

On the other hand, for $\mathcal{A}_0 > \mathcal{A}_0^{(s)}$, the conservation of $\mathcal{H}$ provides $|\mathcal{A} - \mathcal{A}^{(c)}| \leq \delta A$; thus,

$$\bar{\mathcal{P}}_z = \bar{\mathcal{P}}_z^{(s)} + (\bar{\mathcal{A}} - \mathcal{A}^{(c)}) + \delta\mathcal{A} > \bar{\mathcal{P}}_z^{(s)}. \tag{13}$$

We conclude, therefore, that $\bar{\mathcal{P}}_z \geq \bar{\mathcal{P}}_z^{(s)}$ for all ξ_0, with the equality satisfied at, and only at, $\xi_0 = \xi_0^{(s)}$. Then, depending on the sign of

$$\bar{\mathcal{P}}_z^{(s)} = \mathcal{P}_0\left[1 - 2\hat{\gamma}^2\left(1 - \frac{\hat{\beta}}{\beta_0}\right)\right], \tag{14}$$

two different cases can be distinguished, as we discuss below.

B. Limited currents

Suppose $\bar{\mathcal{P}}_z^{(s)} < 0$, which corresponds to

$$-1 < \hat{\beta} < \beta_0 \frac{\gamma_0}{\gamma_0 + 1}. \tag{15}$$

(Recall that we require $\hat{\beta} > -1$ from other considerations.) In this case, $2\delta\mathcal{A} > \mathcal{P}_0$, so $\xi_0^{(s)} > \xi^{(a)}$, such that $\mathcal{A}(\xi^{(a)}) = \mathcal{P}_0$. Since $\bar{\mathcal{P}}_z(\xi_0 = \xi_0^{(s)}) < 0$, $\bar{\mathcal{P}}_z(\xi_0 < \xi^{(a)}) = [\mathcal{A}(\xi^{(a)}) - \mathcal{A}_0] + \bar{A} > 0$, and $\bar{\mathcal{P}}_z(\xi_0)$ is a continuous function, there must exist ξ_*, $\xi^{(a)} < \xi_* < \xi_0^{(s)}$, such that $\bar{\mathcal{P}}_z(\xi_*) = 0$. Then, by definition, ξ_* is just the normalized $R_{\max}$, so, under Eq. (15), the maximum beam radius is always limited.

Consider how this limitation affects the maximum beam current I. Ampère's law yields

$$I = \frac{R}{\Delta} c |A(R)| \leq \frac{R}{\Delta} \frac{mc^3}{e} \mathcal{A}(\xi_*), \tag{16}$$

where $\Delta = 2A(R)/A'(R)$ is the spatial scale of the current density profile at the edge of the beam, so the maximum current is

$$I_{\max} = \frac{R}{\Delta} \frac{mc^3}{e} \mathcal{A}(\xi_*). \tag{17}$$

Since $\mathcal{P}_0 < \mathcal{A}(\xi_*) < 2\delta\mathcal{A}$, one has

$$I_1 < I_{\max} < I_2, \tag{18a}$$

$$I_1 = \frac{R}{\Delta} I_{\mathrm{A}}, \tag{18b}$$

$$I_2 = 2\hat{\gamma}^2\left(1 - \frac{\hat{\beta}}{\beta_0}\right) I_1. \tag{18c}$$

(Interestingly, as also follows from the derivation, $I_{\max} > I_1$ applies to *all* cases when stable transverse oscillations are present, including the case $\hat{\beta} < -1$; see below.)

The factor R/Δ equals 1 for $j_z(r)$ uniform at $r < R$. [For $\hat{\beta} = 0$, Eq. (18) then predicts $I_{\mathrm{A}} < I_{\max} < 2I_{\mathrm{A}}$, in agreement with the precise Alfvén limit $I_{\max} \approx 1.65 I_{\mathrm{A}}$, which applies in this case.] If the current is concentrated at the edge of the beam, one has $R/\Delta \gg 1$; in this case, both R and I can be made arbitrarily large by adjusting the beam geometry.[18,19,21,29] Nevertheless, Eq. (18) proves that, for a *given* set of parameters, $I_{\max}$ is always limited.

Let us estimate $I_{\max}$ within the interval defined by Eq. (18). At $I_2 \sim I_1$, one immediately gets $I_{\max} \sim I_1$, so the case when the upper and the lower limits differ significantly is of interest for us. Since ξ_* approaches $\xi_0^{(s)}$ only if $I_1 \to I_2$, all solid-beam trajectories must be far from the separatrix when

$I_2 \gg I_1$. The coefficient $\eta \equiv 1 - \bar{A}/\mathcal{A}_0$ is of the order of 1 in this case. Hence, $\bar{\mathcal{P}}_z = \mathcal{P}_0 - \eta\mathcal{A}_0$ is positive when $\mathcal{P}_0 \gtrsim \mathcal{A}_0$, so $I_{\max} \sim I_1$ again.

The above results can also be extended to the case of a strong attractive radial field, $\hat{\beta} < -1$. The effective potential U [Eq. (9)] is negative here, and so is $\delta\mathcal{A}$; hence, $\mathcal{A}^{(c)} > 0$. Then, $U(\xi)$ has a local minimum at $\xi = 0$ and two local maxima at $\xi = \pm\xi^{(c)}$ [Fig. 1(d)]. Since $\mathcal{A}_0 < \mathcal{A}^{(c)}$, the initial particle location ξ_0 always falls on the attractive slope of the potential, approaching the separatrix from inside at $\mathcal{A}_0/\delta\mathcal{A} \to \infty$. Hence, stable meridional oscillations are observed in this case, so $I_{\max} > I_1$, as shown above. Suppose $I_{\max} \gg I_1$, i.e., there exists ξ_0, such that $\bar{\mathcal{P}}_z > 0$ and $\mathcal{A}_0 \gg \mathcal{P}_0$. Apparently, this is possible only in the vicinity of a separatrix where η scales as $\ln^{-1}(\mathcal{A}_0/\delta\mathcal{A})$. Since $\mathcal{P}_0 > \eta\mathcal{A}_0$ in this case, one must have $\epsilon = \delta\mathcal{A}/\mathcal{P}_0 \gg 1$ (so $\epsilon \sim 1/\hat{\beta}$), and $\mathcal{A}_0 \lesssim \mathcal{P}_0 \ln(1/\epsilon)$. The current carried by the beam cannot then exceed $I_1 \ln \hat{\beta}$ by the order of magnitude. Omitting the slow logarithmic factor, one again obtains $I_{\max} \sim I_1$.

Summarizing, both for the parameter domain (15) and $\hat{\beta} < -1$, one has

$$I_{\max} \sim I_A \frac{R}{\Delta}. \tag{19}$$

Therefore, a radial field cannot substantially increase the limiting current if

$$\hat{\beta} < \beta_0 \frac{\gamma_0}{\gamma_0 + 1}. \tag{20}$$

C. Unlimited currents

Suppose now $\bar{\mathcal{P}}_z^{(s)} > 0$, which corresponds to

$$\beta_0 \frac{\gamma_0}{\gamma_0 + 1} < \hat{\beta} < \beta_0. \tag{21}$$

(Recall that beams with $\beta_0 < \hat{\beta}$ cannot be solid, and hence are not considered below.) Since $\bar{\mathcal{P}}_z > 0$ for all ξ_0 in this and only this case, Eq. (21) is the necessary and sufficient condition for Eq. (5) to have no solution. Therefore, if Eq. (21) is satisfied [cf. Eq. (4)], the beam radius is unlimited, and, according to Eq. (17), the beam can carry an arbitrarily large current.

Assuming E_r is the beam-produced electrostatic field, Eq. (21) can also be expressed in terms of the fractional neutralization coefficient f. The corresponding condition will be different from the one suggested earlier,[21] that is, $f = \gamma_0^{-2}$ [Eq. (2)], and will rather take the form $f_1 < f < f_2$.[46] To calculate f_1 and f_2 one must solve the self-consistent problem, which is out of the scope of the present study. A rough estimate can be obtained assuming, after Ref. 23, that all particles have $\langle v_z \rangle = c\beta_0$, so $\hat{\beta} = (1-f)/\beta_0$. In this case, the requirement on f to result in $I_{\max} = \infty$ is

$$\gamma_0^{-2} < f < \gamma_0^{-1}. \tag{22}$$

Our results contradict those presented in Ref. 25, where it was proposed to determine the maximum current of a partially neutralized beam using a transformation to the frame where the electric field vanishes. In Ref. 25, it was suggested that the Alfvén limit can be calculated in the moving frame in the normal manner, and a transformation back to the laboratory frame would accomplish the derivation. Doing so, however, would be incorrect because of the following reasons. (i) First of all, the Alfvén limit cannot be calculated in the moving frame in the normal manner. Unlike in the laboratory frame, a nonzero current of background charges (ions) is present in the moving frame, which creates negative magnetic neutralization[21] neglected in Ref. 25. (In other words, the sign of the net current density and that of the average velocity of beam particles generally are not the same here.) Hence the traditional calculation of the Alfvén limit does not apply. (ii) Second of all, contrary to Ref. 25, the Alfvén limit is not relativistically invariant and in principle cannot be calculated using just frame transformation. The Alfvén criterion says that the beam must remain solid; i.e., all particles with $r_0 < R$ must have $\langle v_z \rangle > 0$. However, $\langle v_z \rangle$ is frame-dependent, so even though in a particular frame the beam might be broken into filaments, the filamentation (i.e., partial current reversal) might not occur in other frames. Since the Alfvén limit, as formulated within the single-particle approach, is not an issue of stability, this fact does not contradict the relativity principle. Note also that Ref. 25 predicts $I_{\max} = 0$ at $\hat{\beta} = \beta_0$, although clearly $I_{\max}$ must be infinite because the net force on beam particles is zero in this case, so equilibrium beams of arbitrary radii (and hence currents) can propagate. Given all of the above, Ref. 25 apparently contains incorrect results.

V. DISCUSSION

Being based on the single-particle approach, our work represents a preliminary step toward an expanded study of self-consistent relativistic beam equilibria beyond the Alfvén-Lawson limit. On one hand, the paper suggests a new insight into the classification of beam particle orbits, which is necessary for understanding the microscopic structure of such equilibria. On the other hand, since the field profiles are not specified in our calculations (except that we take $\hat{\beta} = \text{const}$ for simplicity), the solutions we offer are flexible enough to explain particle dynamics in the self-consistent problem as well, unlike those, e.g., in Refs. 18 and 22–25. To actually solve the self-consistent problem, the suggested line of reasoning cannot be expanded; yet it is not the purpose of our study. Instead, we show that the *existing* Alfvén-Lawson limit is incorrect even within the same single-particle approach originally used for its derivation. Remarkably though, our explanation on why the currents above this limit are possible applies to the self-consistent problem just as well.

Our main results can be summarized as follows. We found that boosting the maximum current by means of radial electrostatic fields is considerably more promising compared to what follows from the Alfvén-Lawson criterion for beams with fractional electrostatic neutralization. We show that the Alfvén-Lawson criterion incorrectly reflects the dependence of the maximum beam current $I_{\max}$ on the neutralization coefficient f. Contradicting the original estimate,[21–24] which

TABLE I. The maximum current $I_{\max}$, which can be carried by a solid relativistic electron beam versus the normalized magnitude of the transverse radial field E_r, $\hat{\beta} \equiv E_r/B_\theta$, under the assumption $\hat{\beta}$=const. Here, $\beta_0 = v_0/c$ is the normalized velocity of the beam particles at their maximum excursion from the beam axis, $\gamma_0 = (1-\beta_0^2)^{-1/2}$, R is the beam radius, and Δ is the spatial scale of the current density profile at the edge of the beam. The maximum beam current is limited due to magnetic pinching at low $\hat{\beta}$, and due to electrostatic repulsion at large $\hat{\beta}$. The dependence on $\hat{\beta}$ can also be considered as that on the fractional neutralization coefficient f. The coefficient f is formally calculated here using $\hat{\beta} = (1-f)/\beta_0$, as if all particles traveled with the average velocity $\langle v_z \rangle = c\beta_0$.

$\hat{\beta} \equiv E_r/B_\theta$	$\hat{\beta} < \beta_0[\gamma_0/(\gamma_0+1)]$	$\beta_0[\gamma_0/(\gamma_0+1)] < \hat{\beta} < \beta_0$	$\beta_0 < \hat{\beta}$
f	$f > \gamma_0^{-1}$	$\gamma_0^{-1} > f > \gamma_0^{-2}$	$\gamma_0^{-2} > f$
$I_{\max}$	$\sim (R/\Delta) I_A$	∞	0

predicts that $I_{\max}(f)$ is a continuous function, we find, within the same single-particle model, that current boosting by means of the electrostatic field is a threshold effect (Table I). We show that weak radial fields do not alter $I_{\max}$ substantially; on the contrary, sufficiently strong fields (within a certain parameter range) allow propagation of arbitrarily large currents, regardless of the beam transverse structure. Above this range, the magnetic field is too weak to pinch the beam, and it is the electrostatic repulsion, which does not allow an equilibrium beam with a constant radius.

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APPENDIX: HAMILTONIAN REPRESENTATION

Consider a dynamical system with the action

$$dS = \mathbf{P} \cdot d\mathbf{Q} - H dt, \tag{A1}$$

where $H(\mathbf{Q},\mathbf{P},t)$ is the Hamiltonian, $(\mathbf{Q},\mathbf{P})$ is the canonical pair, and t is time. Define $dS' = d(S+Ht) - \mathcal{H} d\tau$, where $\mathcal{H}$ and τ are some functions of $\mathbf{Q}$, $\mathbf{P}$, t, and the energy $\mathfrak{E} = H(\mathbf{Q},\mathbf{P})$:

$$dS' = \mathbf{P} \cdot d\mathbf{Q} + t dH - \mathcal{H} d\tau. \tag{A2}$$

In the form (A2), S' can be treated as the action of the "extended" system (see, e.g., Ref. 47), where $\mathcal{H}$ is the new Hamiltonian, τ is the new "time," and $(\mathbf{Q},\mathbf{P})$ and $(\mathfrak{E},t)$ are two canonical pairs. Require that, for each set of initial conditions, $\mathcal{H}$ is chosen appropriately to remain constant along the system trajectory: $\mathcal{H} = \mathcal{H}_0$. (Most generally, $\mathcal{H}$ can also be a function of τ.) The difference between dS' and dS is a full differential then, $dS' - dS = d(Ht - \mathcal{H}_0 \tau)$, so the original system is equivalent to the extended system, and its behavior can be described by the canonical equations

$$\frac{d\mathbf{r}}{d\tau} = \frac{\partial \mathcal{H}}{\partial \mathbf{P}}, \quad \frac{d\mathbf{P}}{d\tau} = -\frac{\partial \mathcal{H}}{\partial \mathbf{r}}, \tag{A3a}$$

$$\frac{d\mathfrak{E}}{d\tau} = \frac{\partial \mathcal{H}}{\partial t}, \quad \frac{dt}{d\tau} = -\frac{\partial \mathcal{H}}{\partial \mathfrak{E}}. \tag{A3b}$$

Let us apply this formalism to the relativistic particle motion.[48–50] Suppose electric and magnetic fields of the form $\mathbf{E} = -\nabla\phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$, so the Hamiltonian H is given by

$$H = \sqrt{m^2c^4 + \left(\mathbf{P} - \frac{e}{c}\mathbf{A}\right)^2 c^2} + e\phi, \tag{A4}$$

where $\mathbf{P} = \mathbf{p} + (e/c)\mathbf{A}$ is the canonical momentum, and $\mathfrak{E} = mc^2\gamma + e\phi$ is the full energy. Choose τ to be the particle proper time, i.e., $d\tau = dt/\gamma$, so

$$\frac{dt}{d\tau} = -\frac{\partial \mathcal{H}}{\partial \mathfrak{E}} = \frac{\mathfrak{E} - e\phi}{mc^2}. \tag{A5}$$

Integrating with respect to $\mathfrak{E}$, one gets $\mathcal{H} = -(\mathfrak{E} - e\phi)^2/2mc^2 + h$. The unknown function h, independent of $\mathfrak{E}$, is obtained by requiring $\mathcal{H} \equiv 0$ for each trajectory, so

$$\mathcal{H} = \frac{1}{2m}\left(\mathbf{P} - \frac{e}{c}\mathbf{A}\right)^2 + \frac{m^2c^4 - (\mathfrak{E} - e\phi)^2}{2mc^2}. \tag{A6}$$

The Hamiltonian (A6) is equivalent to that of a classical particle, which travels in the same magnetic field $\mathbf{B}$, yet "sees" a new effective potential

$$\phi_{\text{eff}} = \frac{mc^2}{2}\left[1 - \left(\frac{\mathfrak{E} - \phi}{mc^2}\right)^2\right]. \tag{A7}$$

(Remember that the effective time is now τ rather than t.) In the nonrelativistic limit, one has $\phi_{\text{eff}} \approx p_0^2/2m + e(\phi - \phi_0)$, so Eq. (A6) becomes $\mathcal{H} \approx H - H_0$, where the index "0" denotes the initial values.

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