Relativistic effects in energy extraction from alpha particles

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The use in a tokamak of the recently reported relativistic two-gyrostream instability is investigated. The concept is evaluated with respect to the extraction of energy from relativistic ions in an inhomogeneous magnetized plasma by means of an electrostatic wave. For application to energetic alpha particle channeling in a tokamak fusion reactor, the relativistic two-gyrostream instability effects turn out to be relatively minor. © *1997 American Institute of Physics.* [S1070-664X(97)03401-0]

I. INTRODUCTION

The amplification of plasma waves at the expense of energetic alpha particles holds considerable promise for greatly improved operation of tokamak fusion reactors. In the absence of such plasma waves, the alpha particles slow down primarily through collisions with electrons. These collisions take place on a relatively long time scale, at least compared to collisionless processes; hence, there is opportunity for tapping the energy of these particles through an interaction with plasma waves. The tapped power is then channeled for use in other processes, such as to drive current or to increase the fusion reactivity.

While there have been a number of wave candidates for achieving this collisionless slowing down of the alpha particles, it remains important to identify new wave candidates, particularly if new effects can be utilized. The impetus for the present study is the possibility of using the relativistic two-gyrostream instability, recently explored by Chen.[1–3]

Wave interactions that tend to drive the alpha particles in velocity space only tend not to extract most of the recoverable energy.[4,5] Waves that tend to release the α particle power by diffusing the α particles in both energy and space tend to be much more effective at recovering substantial amounts of energy.[6–8] Waves in the ion cyclotron frequency range, such as the short-wavelength mode-converted ion Bernstein wave[9] have been proved in tokamaks,[10–12] and are thought to be particularly effective for energy recovery. In certain regimes, these waves can diffuse α particles along a path both in space and energy, such that energetic α particles at the plasma center diffuse towards the plasma periphery as they lose energy. The wave would then be convectively unstable.

Other wave candidates include spatially localized modes in the ion cyclotron range of frequencies.[13,14] Such waves are already thought to be responsible for ion cyclotron emission from fusion products.[15–17] However, it has been thought that, to achieve the most substantial channeling of the α particle power, it would be necessary to employ more than one kind of wave. In fact, there appear to be advantages in employing one wave in the ion cyclotron range of frequencies and one lower frequency wave.[18]

The present work is motivated by the possibility of accomplishing the channeling of power from the energetic α particles also through the recently reported two-gyrostream instability discovered by Chen, which relies on a weakly relativistic effect.[1–3] This effect was explored in numerical simulations of homogeneous plasmas. The question to be settled is whether this relativistic anomalous slowing down effect can be put to good use, either by itself or in conjunction with other waves, in a tokamak geometry, where the magnetic fields are necessarily not homogeneous. The conclusion of this paper is that, unfortunately, interesting though the possibility was, these relativistic effects appear to be relatively minor in importance for application to extraction of power from alpha particles in a tokamak.

The paper is organized as follows: In Sec. II, the relativistic equations of motion are written and solved for a particle that interacts with an electrostatic wave in a magnetized homogeneous plasma. The two-gyrostream instability explored by Chen,[1] namely, that an instability may result in a uniform plasma from the relativistic gyrophase bunching, is recovered. In Sec. III, the analysis is generalized to an inhomogeneous plasma, where the relativistic effects are shown to be of relatively lesser importance. Section IV compares the diffusion in particle energy to the directed drag on the particle energy. Section V shows that the relation between the guiding center displacement and the energy change of the particle holds without modification in the relativistic case. Section VI summarizes the main conclusions.

II. EQUATIONS OF MOTION

In a tokamak, an α particle interacting with a mode converted ion Bernstein wave repeatedly encounters a wave region, which is a vertical slab. In such a case, the α particle streams poloidally and toroidally along a magnetic surface. As it circles the magnetic axis poloidally, the α particle encounters the region where the wave is intense. Due to the (major) radial dependence of the magnetic field in a tokamak, the gyrofrequency of the α particle varies significantly over its poloidal orbit. In addition, the horizontal wave number of the mode converted ion Bernstein wave is large and rapidly varying as a function of horizontal position from the resonance. In the presence of the tokamak poloidal magnetic field, it follows that the parallel wave number is also rapidly

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varying along the particle orbit. As a result, the particle streams in and out of resonance as it passes through the wave field.

To describe a particle streaming in and out of resonance as it passes through the wave field, suppose that the magnetic field is uniform, $\mathbf{B}=\hat{e}_z B_0$, but consider an electrostatic wave of finite extent in the direction of the static field, e.g., with potential

$$\psi=\psi_0(z)\sin(k_x x+\xi(z)-\omega t). \tag{1}$$

The change in the particle energy E as a result of the work done by the electric field is

$$\frac{dE}{dt}=-q\mathbf{v}\cdot\nabla\psi, \tag{2}$$

which can be written also as

$$\frac{dE}{dt}=-q\,\frac{d\psi}{dt}+q\,\frac{\partial\psi}{\partial t}, \tag{3}$$

since $d/dt\equiv\partial/\partial t+\mathbf{v}\cdot\nabla$, where $\mathbf{v}$ is the particle velocity. The change in the particle energy over the complete finite region in which ψ_0 is nonzero is then

$$\Delta E=\int^t dt'\; q\,\frac{\partial\psi}{\partial t'}=-\int^t dt'\; q\omega\psi_0\,\cos(k_x x+\xi(z)-\omega t'), \tag{4}$$

where x and z are functions of t.

The energy transferred in one transit can be calculated by solving the relativistic equations of motion,

$$\frac{dp_x}{dt}=\frac{q\epsilon_x}{mc}+\frac{\Omega p_y}{\gamma}, \tag{5}$$

$$\frac{dp_y}{dt}=\frac{q\epsilon_y}{mc}-\frac{\Omega p_x}{\gamma}, \tag{6}$$

$$\frac{dp_z}{dt}=\frac{q\epsilon_z}{mc}, \tag{7}$$

for the dimensionless momentum $\mathbf{p}\equiv\gamma\mathbf{v}/c$, where c is the velocity of light in vacuum, and $\gamma\equiv(1-\mathbf{v}\cdot\mathbf{v}/c^2)^{-1/2}$. Also, $\Omega\equiv qB_0/mc$, where q and m are the particle charge and mass. Since the electric field of the electrostatic wave,

$$\boldsymbol{\epsilon}=-\nabla\psi, \tag{8}$$

has no y component, the canonical momentum in that direction,

$$P_i\equiv p_y+x\,\frac{\Omega}{c}, \tag{9}$$

is conserved. Using Eq. (9) to write the coordinate x of the location of the particle in terms of p_y, we can write Eqs. (5) and (6) as

$$\frac{dp_t}{dt}+ip_t\left(\frac{\Omega}{\gamma}-\frac{\Omega}{\gamma_i}-\frac{d\phi}{dt}\right)=\frac{q\epsilon_x}{mc}\exp\left[i\left(\frac{\Omega t}{\gamma_i}+\phi\right)\right], \tag{10}$$

where

$$p_t\exp\left[-i\left(\frac{\Omega t}{\gamma_i}+\phi\right)\right]\equiv p_x+ip_y, \tag{11}$$

and γ_i is the value of γ before the particle interacts with the wave.

We can solve Eq. (11) by assuming that the electric field is weak and expanding the particle coordinates and momenta in a power series with

$$\Delta E^{(J)}=q\int^t dt'\left(\frac{\partial\psi}{\partial t'}\right)^{(j)}, \tag{12}$$

where j denotes the order of the expansion. The change in energy to first order in the amplitude of the electric field is

$$\Delta E^{(1)}=-\omega q\int^t dt'\;\psi_0(t')\cos(k_x x^{(0)}+\xi(z)^{(0)}-\omega t'), \tag{13}$$

where, to zeroth order, the functions, $p_t=p_{ti}$, $\phi=\phi_i$ and $\gamma=\gamma_i$, are constant and so remain at their initial (subscript i) values. Using Eqs. (9) and (11), we find

$$x^{(0)}=\frac{cP_i}{\Omega}+\frac{cp_{ti}}{\Omega}\sin\left(\frac{\Omega t}{\gamma_i}+\phi_i\right). \tag{14}$$

Substituting this expression into Eq. (13), and expanding the integrand as an infinite sum of Bessel functions, we obtain

$$\Delta E^{(1)}=-\sum_n\;\frac{\omega q}{2}\int^t dt'\;\psi_0(t')J_n(\lambda)\exp[i\alpha(t')+r]+\text{c.c.}, \tag{15}$$

where

$$\alpha(t)\equiv(n\Omega/\gamma_i-\omega)t+\xi(z(t))^{(0)}, \tag{16}$$

$$\lambda\equiv\frac{k_x cp_{ti}}{\Omega}, \tag{17}$$

and

$$r\equiv\frac{k_x cP_i}{\Omega}+n\phi_i. \tag{18}$$

The main contribution to the sum in Eq. (15) comes from the resonant harmonic, for which $d\alpha/dt=0$. In the following we need keep only such resonant terms. Note, however, from Eq. (15), that if the particle experiences a succession of kicks, each time entering the wave region with random phase r, on average $\Delta E^{(1)}$ is zero. The diffusion in energy due to these kicks is second order in the electric field. For application to tokamaks, we assume that the particles indeed enter the wave region with random phase, either because of decorrelations introduced in the particle motion outside the wave region, or because the wave itself, being injected from the tokamak periphery, is itself decorrelated. Decorrelations in the wave can result either intentionally from randomly phasing the injected wave, or inadvertently through the propagation of the injected wave through random density fluctuations before reaching the resonant region of interaction. In any event, competing with the diffusion of the energy at second order in the electric field are coherent, or drag terms, also appearing to this order and which we now calculate.

Since there are many wavelengths within the wave region we neglect the dependence of ψ_0 on z, and, since $k_x x^{(1)}$, $\xi(z)^{(1)} \ll \pi$, we can write

$$\left(\frac{\partial \psi}{\partial t}\right)^{(2)} = \omega \psi_0 [k_x x^{(1)} + \xi(z)^{(1)}] \sin(k_x x^{(0)} + \xi(z)^{(0)} - \omega t). \tag{19}$$

Expanding Eqs. (9) and (11), we get

$$x^{(1)} = -\frac{ic}{2\Omega} (p_t^{(1)} + i p_{ti} \phi^{(1)}) \exp\left[i \left(\frac{\Omega t}{\gamma_i} + \phi \right) \right] + \text{c.c.}, \tag{20}$$

and, integrating Eq. (10) and keeping only the resonant harmonic n, we have

$$p_t^{(1)} - i p_{ti} \phi^{(1)} = -\frac{qk}{2mc} \int^t dt' \ \psi_0(t') [J_{n-1} \exp[i(\alpha(t') + r)] + J_{n+1} \times \exp[-i(\alpha(t') + r)]] + i \frac{p_{ti} \Omega}{\gamma_i^2} \int^t dt' \gamma^{(1)}. \tag{21}$$

The last term in Eq. (21) arises from our retaining relativistic effects. Similarly, we calculate

$$z^{(1)} = -b \frac{v_{zi}}{\gamma_i} \int^t dt' \ \gamma^{(1)} - \frac{q}{m \gamma_i} J_n(\lambda) \int^t dt' \int^{t'} dt'' \times \psi_0(t'') \frac{d\xi(z(t''))}{dz} \cos(\alpha(t'') + r), \tag{22}$$

where again we keep only the resonant harmonic.

Substituting the first order terms into Eq. (19), keeping the resonant terms only, and averaging over the phase r, we get

$$\left\langle \left(\frac{\partial \psi}{\partial t}\right)^{(2)} \right\rangle = -\frac{\omega q \psi_0}{2 m \gamma_i} \left\{ -\frac{n k_x}{v_{ti}} J_n(\lambda) J_n(\lambda)' \exp[i\alpha(t)] \int^t dt' \ \psi_0(t') \exp[-i\alpha(t')] + i \left(\frac{n\Omega}{\gamma_i} + \frac{d\xi}{dz} (z(t))^{(0)} v_{zi} \right) \times \frac{\omega}{2c^2} J_n^2 \exp(i\alpha(t)) \int^t dt' \int^{t'} dt'' \psi_0(t'') \exp(-i\alpha(t'')) - \frac{i}{2} \frac{d\xi}{dz} (z(t))^{(0)} J_n^2 \exp(i\alpha(t)) \times \int^t dt' \int^{t'} dt'' \psi_0(t'') \frac{d\xi}{dz} (z(t''))^{(0)} \exp(-i\alpha(t'')) \right\} + \text{c.c.} \tag{23}$$

Here, the second term arises from the relativistic effect. Substituting Eq. (23) into Eq. (4) gives the phase-averaged energy change to second order, and the main contribution to the integral comes from the region of stationary phase.

Note that while the first order energy change phase-averaged vanishes, it contributes diffusively to second order. On the other hand, there is a nonvanishing phase-averaged contribution in second order, which represents over time a constant deceleration (or acceleration, for some energies), sometimes called the ''drag'' term.

Note also that the phase is stationary over the characteristic width of $\alpha(t)$ near $d\alpha/dt = 0$, or $1/(v_{zi}\sqrt{d^2\xi/dz^2})$. If the characteristic width of $\psi_0(z)$, i.e., the wave region, is l, then the condition, $l\sqrt{d^2\xi/dz^2} \gg 1$, means that the time spent near the stationary phase point is small compared to the transit time through the wave region, and the effective wave–particle interaction is localized to this small region. This is the case that we expect for the mode converted ion Bernstein wave, where the wave number varies rapidly near the ion-hybrid resonance layer.

In the opposite limit, namely where $d^2\xi/dz^2 \to 0$, the wave–particle interaction is not localized. This is the limit of an homogeneous plasma. In this limit, say $\xi = k_0 z$, then from Eq. (15) we have

$$\Delta E^{(1)} = \frac{i \omega q \psi_0 J_n}{2\delta} [\exp(i(\delta t + r)) - \exp(ir)] + \text{c.c.}, \tag{24}$$

where we defined the constant resonance mismatch as

$$\delta \equiv \frac{n\Omega}{\gamma_i} - \omega + k_0 v_{zi}. \tag{25}$$

(When not written explicitly, the argument of all Bessel functions is λ.) Similarly, using Eq. (23) in Eq. (12), we get

$$\Delta E^{(2)} = \frac{\omega (q\psi_0)^2}{m \gamma_i} \left\{ \frac{n k_x}{v_{ti} \delta^2} J_n J_n' [1 - \cos(\delta t)] + \left(\frac{\omega^2}{c^2} - k_0^2 \right) \times \frac{J_n^2}{\delta^3} \left[1 - \cos(\delta t) - \frac{\delta t \ \sin(\delta t)}{2} \right] \right\}. \tag{26}$$

The relativistic effect is contained in the terms proportional to ω^2/c^2, which is the result calculated, in the limit $d\xi/dz \to 0$ and for $k_0 = 0$, by Chen.[1–3] As noted in Ref. 2, for certain values of δ and other parameters, $\Delta E^{(2)}$ can be positive and larger in magnitude due to the relativistic effect than when only the nonrelativistic effects are retained. In the absence of other effects, $\Delta E^{(2)} > 0$ for some range of parameters means that certain waves are unstable in this limit.

To see this, note that the relativistic effect can dominate in the limit $\delta \to 0$, but δt finite. The ratio of this term to the nonrelativistic term in Eq. (26) is

$$(\omega^2/k_x^2 c^2) \Omega t_{\text{tr}} (J_n \lambda / n J_n'). \tag{27}$$

The number of gyroperiods Ωt_{tr} is much larger than unity. This term appears because the destabilizing term is propor-

tional to the inverse of a higher power of the resonant parameter δ. In the configuration described in Refs. 7 and 8 the length of the region along the magnetic field lines in which the particles interact with the wave is about 400 cm. For α particles of 3.5 MeV, the velocity amplitude is $9\cdot10^8$ cm s^{-1}. For half of the velocity parallel to the magnetic field, the transit time is $8.8\cdot10^{-7}$ s. In a magnetic field of 50 kG, $\Omega=2.4\cdot10^8$ s^{-1} for alphas, so that, for alphas, $\Omega t_{\rm tr}\approx200$. For protons of 14 MeV, the velocity magnitude is $3.6\cdot10^9$ cm s^{-1}, so that $\Omega t_{\rm tr}\approx100$. The ratio (27) is a product of this large number and a small number $(\omega^2/k_x^2c^2)$, while $J_n\lambda/nJ_n'$ is of order λ. For alphas $\lambda\approx10$ and for protons $\lambda\approx20$. The number $(\omega^2/k_x^2c^2)$, however, is $3.5\cdot10^{-6}$ for $\omega=1.7\cdot10^8$ and $k_x=3$ cm^{-1}, the typical parameters in Ref. 7. The ratio (27) turns out to be approximately $7\cdot10^{-3}$ for both alphas and protons. Thus unfortunately, it seems that for most particles the ratio (27) is smaller than unity, and therefore even in an homogeneous plasma the relativistic effect does not seem to be dominant.

III. INHOMOGENEOUS PLASMA

To consider the case of an inhomogeneous plasma, assume again a localized wave–particle interaction. Changing the variable of integration from t to s, where $s^2\equiv\xi_0 v_{zi}^2 t^2$, we find that the contribution to the integrals in Eqs. (13) and (23) is mainly from $|s|<1$, or from a region around the resonant point of a length smaller than $1/\xi_0^{1/2}$. The magnitude of ξ_0 is of order $\xi_0\approx\Delta\xi/l^2$, so that the size of the resonance region is $\Delta z_{\rm res}\approx l/(\Delta\xi)^{1/2}$. The contribution to the energy transfer is, by assumption only from a small resonant region. Even if the wave amplitude ψ_0 changes on a scale length that is comparable to the scale on which ξ changes, we may approximate ψ_0 by its value in the small resonant region, a value that is approximately constant. Thus the stationary phase approximation, where the variation of the wave number is more important than the variation of the amplitude, applies.[19]

From Eq. (13) we find the energy change to first order,

$$\Delta E^{(1)}=-\frac{\omega q\psi_0 J_n\pi^{1/2}}{2\xi_0^{1/2}v_{zi}}\exp\left[i\left(\frac{\pi}{4}+r\right)\right]+\text{c.c.} \tag{28}$$

Using

$$\frac{d\xi}{dz}(z(t))^{(0)}=k_0+2\xi_0 v_{zi}t, \tag{29}$$

Eq. (23) then becomes

$$\begin{aligned}\left\langle\left(\frac{\partial\psi}{\partial t}\right)^{(2)}\right\rangle=&-\frac{\omega q\psi_0^2}{2m\gamma_i}\left\{-\frac{nk_x}{v_{ti}\xi_0^{1/2}v_{zi}}J_nJ_n'\exp[is^2]\int^s ds'\,\exp[-i(s')^2]\right.\\&+\frac{i}{2\xi_0 v_{zi}^2}J_n^2\left(b\,\frac{\omega^2}{c^2}-k_0^2\right)\exp(is^2)\int^s ds'\int^{s'}ds''\,\exp(-i(s'')^2)\\&-\frac{iJ_n^2k_0}{\xi_0^{1/2}v_{zi}^2}\exp(is^2)\int^s ds'\int^{s'}ds''s''\,\exp(-i(s'')^2)\\&+\frac{iJ_n^2}{\xi_0^{1/2}v_{zi}}\left(b\,\frac{\omega}{c^2}-\frac{k_0}{v_{zi}}\right)s\,\exp(is^2)\int^s ds'\int^{s'}ds''\,\exp(-i(s'')^2)-\frac{1}{2\xi_0 v_{zi}^2}\frac{\xi_1}{\xi_0^{3/2}}J_n^2\left(b\,\frac{\omega^2}{c^2}-k_0^2\right)\\&\left.\times\exp(is^2)\int^s ds'\int^{s'}ds''(s^3-(s'')^3)\exp(-i(s'')^2)\right\}+\text{c.c.}\end{aligned} \tag{30}$$

The average change in energy is

$$\langle\Delta E^{(2)}\rangle=q\int_{-\infty}^{\infty}ds\left\langle\left(\frac{\partial\psi}{\partial s}\right)^{(2)}\right\rangle,$$

which can be evaluated as

$$\begin{aligned}\langle\Delta E^{(2)}\rangle=&-\frac{\omega(q\psi_0)^2}{m\gamma_i\xi_0 v_{zi}^2}\left\{-\frac{nk_x}{v_{ti}}J_nJ_n'I_1-\frac{1}{2\xi_0^{1/2}v_{zi}}\right.\\&\times J_n^2\left(b\,\frac{\omega^2}{c^2}-k_0^2\right)I_2+\frac{J_n^2k_0}{v_{zi}}I_3-J_n^2\left(b\,\frac{\omega}{c^2}-\frac{k_0}{v_{zi}}\right)\\&\left.\times I_4-\frac{1}{2v_{zi}}\frac{\xi_1}{\xi_0^2}J_n^2\left(b\,\frac{\omega^2}{c^2}-k_0^2\right)I_5\right\},\end{aligned} \tag{31}$$

where

$$I_1\equiv\int_{-\infty}^{\infty}ds\int_{-\infty}^{s}ds'\,\cos[s^2-(s')^2]=\frac{\pi}{2}, \tag{32a}$$

$$I_2\equiv\int_{-\infty}^{\infty}ds\int_{-\infty}^{s}ds'\int_{-\infty}^{s'}ds''\,\sin[s^2-(s'')^2]=0, \tag{32b}$$

$$I_3\equiv\int_{-\infty}^{\infty}ds\int_{-\infty}^{s}ds'\int_{-\infty}^{s'}ds''s''\,\sin[s^2-(s'')^2]=\frac{\pi}{4}, \tag{32c}$$

$$I_4\equiv\int_{-\infty}^{\infty}dss\int_{-\infty}^{s}ds'\int_{-\infty}^{s'}ds''\,\sin[s^2-(s'')^2]=\frac{\pi}{4}, \tag{32d}$$

$$I_5 \equiv \int_{-\infty}^{\infty} ds \int_{-\infty}^{s} ds' \int_{-\infty}^{s'} ds''(s^3-(s'')^3)\cos[s^2-(s'')^2]$$
$$= -\frac{3\pi}{4}. \tag{32e}$$

Detailed evaluations of these integrals are performed in Appendix A. Using the explicit expressions for I_1-I_5, we obtain

$$\langle \Delta E^{(2)} \rangle = \frac{\omega(qk_x\psi_0)^2\pi}{2\Omega m \xi_0 v_{zi}^2}\left\{ \frac{nJ_nJ_n'}{\lambda} + J_n^2 \frac{\Omega}{2\gamma_i k_x^2}\left(b\,\frac{\omega}{c^2} - \frac{2k_0}{v_{zi}}\right) - \frac{3}{4\gamma_i v_{zi}} \frac{\xi_1}{\xi_0^2} J_n^2 \frac{\Omega}{k_x^2}\left(b\,\frac{\omega^2}{c^2} - k_0^2\right)\right\}. \tag{33}$$

Note the vanishing of the term proportional to I_2 in Eq. (31), which contains the destabilizing relativistic effects.

Using the estimate

$$\frac{\xi_1}{\xi_0^2} \approx \frac{l}{\Delta\xi}, \tag{34}$$

we find that the ratio of the third term in the curly brackets in (33) (the largest relativistic term), to the first term in the curly brackets is

$$\frac{\omega^2}{k_x^2 c^2}\frac{\Omega t_{\rm tr}}{\Delta\xi}\frac{J_n\lambda}{nJ_n'}. \tag{35}$$

If $k_0 v_{zi} \approx \Omega$, then $\Omega t_{\rm tr} \approx \Delta\xi$. For tokamak plasmas, typically $k_0 \approx 0.1\ \mathrm{cm}^{-1}$, so that $\Delta\xi \approx 40$. For the parameters in the discussion following Eq. (27), the ratio (35) becomes $1.7\cdot 10^{-4}$. Thus the relativistic effect seems to be small and the dominant drag term is the first term in the curly brackets in Eq. (33). In Appendix B, the drag for nonrelativistic particles is derived in a way similar to that used by Chen *et al.*,[20] who find the drag in the zero λ limit, in which limit we recover their result.

IV. RELATIVISTIC DRAG

Whereas the number of kicks required to extract all the particle energy coherently is $N_c = E_0/\langle \Delta E\rangle$, the number of kicks required to extract energy by diffusion is $N_d = E_0^2/\langle (\Delta E)^2\rangle$. Thus diffusion is dominant if $N_d \ll N_c$, or if

$$\frac{N_d}{N_c} = \frac{E_0\langle \Delta E\rangle}{\langle (\Delta E)^2\rangle}, \tag{36}$$

is smaller than unity. To lowest order, take $\langle (\Delta E)^2\rangle = \langle (\Delta E^{(1)})^2\rangle$ and $\langle \Delta E\rangle = \langle \Delta E^{(2)}\rangle$. In an homogeneous plasma, from Eq. (15),

$$\langle (\Delta E^{(1)})^2\rangle = 2\left(\frac{\omega q \psi_0 J_n}{2\delta}\right)^2 [1-\cos(\delta t)], \tag{37}$$

so that

$$\frac{N_d}{N_c} \approx \frac{v_{ti}^2}{c^2}\,\omega t_{\rm tr}, \tag{38}$$

which is 0.14 for alphas and 0.5 for protons for the parameters in Ref. 7.

In an inhomogeneous plasma, use Eq. (52) to get

$$\langle (\Delta E^{(1)})^2\rangle = \frac{(\omega q \psi_0 J_n)^2\pi}{2\xi_0 v_{zi}^2}, \tag{39}$$

to get

$$\frac{N_d}{N_c} \approx \frac{v_{ti}^2}{c^2}\frac{\omega t_{\rm tr}}{\Delta\xi}. \tag{40}$$

The cancellation of the relativistic term to lowest order results in reduction of the importance of the relativistic term by the factor $\Delta\xi$. Since $k_0 v_{zi} \approx \omega$, usually $\omega t_{\rm tr} \approx \Delta\xi$, so that the relativistic term has an effect smaller by v^2/c^2 than the standard diffusion term. For the parameters in Ref. 7 the magnitude of $\Delta\xi$ is about 40, and the ratio (40) becomes $3.5\cdot 10^{-3}$ for alphas and 0.0125 for protons.

It is of interest to note that in the nonrelativistic limit, and when $k_0 = 0$ and v_z is small,

$$\frac{d\langle (\Delta E)^2\rangle}{dE_0} = 2\langle \Delta E\rangle. \tag{41}$$

V. GUIDING CENTER DISPLACEMENT

Note that the important relation between guiding center displacement and the energy change of the particles holds not only nonrelativistically,[7] but, more generally, also in the relativistic regime. To see this, make a Lorentz transformation to a reference frame that moves with a velocity ω/k_x, smaller than c, in the x direction. In the moving frame, there is an additional component to the electric field

$$E_y' = -\gamma_p \frac{\omega}{k_x c} B_0, \tag{42}$$

where

$$\gamma_p^2 \equiv \left[1-\left(\frac{\omega}{k_x c}\right)^2\right]^{-1}. \tag{43}$$

In this moving frame the electric fields are static. Energy conservation then yields

$$E' + q\gamma_p\psi_0 \cos(\xi'(z') + k_x' x') + q\gamma_p \frac{\omega}{k_x c} B_0 y' = \text{const}, \tag{44}$$

where primed quantities are in the moving frame. The energy transforms as

$$E' = \gamma_p\left(E - \frac{\omega}{k_x c}\, m p_x\right), \tag{45}$$

and $y' = y$. Assume further that the wave exists in a finite domain only. Using Eq. (44), the change in energy along the particle trajectory can be written as

$$\Delta E = -\frac{\omega m \Omega}{k_x}\,\Delta\left(y - \frac{p_x}{c\Omega}\right). \tag{46}$$

The quantity in the brackets is the y coordinate of the guiding center of the particle. Thus, the relation between guiding center displacement and energy change holds more generally, namely, also in the relativistic regime.

VI. SUMMARY

The effects associated with the two-gyrostream instability do not appear to be important for the channeling of α particle power by electrostatic waves in a tokamak. While the calculations here do not take into account the full toroidal geometry of the tokamak, in general, the relativistic effects associated with the wave–particle interaction appear to be smaller in an inhomogeneous magnetic field, such as occurs in a tokamak, relative to similar effects occurring in a homogeneous magnetic field.

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APPENDIX A: EVALUATION OF SOME INTEGRALS

In this appendix we evaluate the integrals in Eq. (32). Firstly we write Eq. (32a) for I_1 as

$$I_1=\frac{1}{2}\int_{-\infty}^{\infty}ds\,\frac{df}{ds}f^*+\text{c.c.}, \tag{A1}$$

where

$$f(s)\equiv\int_{-\infty}^{s}ds'\,\exp[i(s')^2]. \tag{A2}$$

Performing the integration we obtain that

$$I_1=\tfrac{1}{2}|f(s=\infty)|^2=\frac{\pi}{2}. \tag{A3}$$

We turn to the calculation of I_2. We write it as

$$I_2=\frac{I}{2i}+\text{c.c.}, \tag{A4}$$

where

$$I\equiv\int_{-\infty}^{\infty}ds'\int_{s'}^{\infty}ds\,\exp(is^2)\int_{-\infty}^{s'}ds''\,\exp[-i(s'')^2], \tag{A5}$$

and we show that I is real. In writing (A4) we inverted the order of integration in (32b). We now split the integration in I to integration over positive and over negative values of s', change the variable of integration to be positive only, and therefore express I as

$$I=\int_0^{\infty}ds'\,g(s'), \tag{A6}$$

where

$$g(s')\equiv\int_{-s'}^{\infty}ds\,\exp(is^2)\int_{-\infty}^{-s'}ds''\,\exp[-i(s'')^2]+\int_{s'}^{\infty}ds\,\exp(is^2)\int_{-\infty}^{s'}ds''\,\exp[-i(s'')^2]. \tag{A7}$$

We write each term as a sum of two terms.

$$g(s')=\int_{-s'}^{s'}ds\,\exp(is^2)\int_{-\infty}^{-s'}ds''\,\exp[-i(s'')^2]+\int_{s'}^{\infty}ds\,\exp(is^2)\int_{-\infty}^{-s'}ds''\,\exp[-i(s'')^2]+\int_{s'}^{\infty}ds\,\exp(is^2)\int_{-\infty}^{-s'}ds''\,\exp[-i(s'')^2]+\int_{s'}^{\infty}ds\,\exp(is^2)\int_{-s'}^{s'}ds''\,\exp[-i(s'')^2]. \tag{A8}$$

The fourth term is the complex conjugate of the first term, and their sum is real. The second and the third term are identical. Each of them is a product of an integral by its complex conjugate, and are real as well. Thus

$$I_2=0. \tag{A9}$$

Write I_3 as

$$I_3=\frac{1}{4i}\int_{-\infty}^{\infty}ds\,\exp(is^2)\int_{-\infty}^{s}ds'\int_{-\infty}^{(s')^2}d(s'')^2\times\exp[-i(s'')^2]+\text{c.c.} \tag{A10}$$

Upon performing one integration we write the integral as

$$I_3=\frac{1}{4}\int_{-\infty}^{\infty}ds\,\frac{df}{ds}f^*+\text{c.c.} \tag{A11}$$

and therefore

$$I_3=\frac{\pi}{4}. \tag{A12}$$

Similarly, write I_4 as

$$I_4=\frac{1}{i}\int_{-\infty}^{\infty}ds'\int_{(s')^2}^{\infty}d(s^2)\exp(is^2)\times\int_{-\infty}^{s'}ds''\,\exp[-i(s'')^2]+\text{c.c.} \tag{A13}$$

In a way similar to the way of finding the value of I_3 we find that

$$I_4=\frac{\pi}{4}. \tag{A14}$$

Writing I_5 as

$$I_5=\frac{1}{2}\int_{-\infty}^{\infty}ds'\int_{s'}^{\infty}ds\int_{-\infty}^{s'}ds''(s^3-(s'')^3)\times\exp[i(s^2-(s'')^2)]+\text{c.c.}, \tag{A15}$$

and performing one integration, gives

$$I_5=-\frac{1}{4}\int_{-\infty}^{\infty}ds'\left[\exp(i(s')^2)\left[1+\frac{(s')^2}{i}\right]\right.$$
$$\times\int_{-\infty}^{s'}ds\ \exp(-is^2)+\exp(-i(s')^2)\left[1-\frac{(s')^2}{i}\right]$$
$$\left.\times\int_{s'}^{\infty}ds\ \exp(is^2)\right]+\text{c.c.} \tag{A16}$$

We write the first integral over s as a sum of two integrals. The sum of the second of those (on the same interval as the second integral) and the second integral are a subtraction of a number from its complex conjugate, which is imaginary. We are left with

$$I_5=-\frac{1}{4}\int_{-\infty}^{\infty}ds\ \exp(is^2)(1-is^2)\int_{-\infty}^{\infty}ds\ \exp(-is^2)+\text{c.c.} \tag{A17}$$

Using a standard technique of performing the integral in the complex plane, we obtain

$$I_5=-\frac{3\pi}{4}. \tag{A18}$$

APPENDIX B: THE NONRELATIVISTIC SMALL λ LIMIT

Here, the drag for nonrelativistic particles is calculated in a way similar to the way used by Chen *et al.*[20] to find the drag in the zero λ limit. Write Eqs. (6) and (7) as

$$\frac{dp}{dt}=\frac{q\epsilon}{mc}-i\frac{\Omega p}{\gamma}, \tag{B1}$$

where

$$p\equiv p_x+ip_y, \tag{B2}$$

and

$$\epsilon\equiv\epsilon_x+i\epsilon_y. \tag{B3}$$

In the nonrelativistic limit $\lambda=1$, and the solution of Eq. (B1) is

$$p(t_1)=\exp(-i\Omega t_1)\left[\int_{t_0}^{t_1}dt\ \frac{q\epsilon}{mc}\exp(i\Omega t)+p(t_0)\right]. \tag{B4}$$

The magnitude of the particle momentum is

$$|p(t_1)|^2=|p(t_0)|^2+\left|\int_{t_0}^{t_1}dt\ \frac{q\epsilon}{mc}\exp(i\Omega t)\right|^2$$
$$+\left[p(t_0)\int_{t_0}^{t_1}dt\ \frac{q\epsilon^*}{mc}\exp(-i\Omega t)+\text{c.c.}\right]. \tag{B5}$$

In the limit of a small λ, the average over r of the second term is zero. The first term is positive, denoting absorption. Retaining resonant terms only, we write the first term as

$$\left|\int_{t_0}^{t_1}dt\ \frac{q\epsilon}{mc}\exp(i\Omega t)\right|^2=\left(\frac{qk_x\psi_0}{2mc}\right)^2(J_{n-1}^2+J_{n+1}^2)$$
$$\times\left|\int_{t_0}^{t_1}dt\ \exp(i\alpha(t))\right|^2. \tag{B6}$$

Using Eq. (11) for the momentum, and assuming that at $t=t_0$ there is no wave perturbation, we write the second term in Eq. (B5) as

$$p^*(t_0)\int_{t_0}^{t_1}dt\ \cos(k_xx+\xi(z)-\omega t)\exp(i\Omega t)$$
$$=-p_{ti}\int_{t_0}^{t_1}dt\ k_xx^{(1)}\ \sin(k_xx^{(0)}-\omega t)\exp(i(\phi_i+\Omega t)). \tag{B7}$$

Using Eq. (21) and retaining only resonant terms, we obtain

$$p^*(t_0)\int_{t_0}^{t_1}dt\ \cos(k_xx+\xi(z)-\omega t)\exp(i\Omega t)=\frac{k_xp_{ti}c}{4\Omega i}\frac{qk_x\psi_0}{2mc}\int_{t_0}^{t_1}dt\int_{t_0}^{t}dt'\{[J_{n-1}\exp[-i(\alpha(t')+r)]$$
$$+J_{n+1}\exp[+i(\alpha(t')+r)]]$$
$$\times[J_{n-2}\exp[i(\alpha(t)+r)]-J_{n+2}\exp[-i(\alpha(t)+r)]]$$
$$-[J_{n-1}\exp[i(\alpha(t')+r)]+J_{n+1}$$
$$\times\exp[-i(\alpha(t')+r)]]J_n[\exp[i(\alpha(t)+r)]-\exp[-i(\alpha(t)+r)]]\}. \tag{B8}$$

Averaging over r and taking the limits of integration to be $t_0=-\infty$ and $t_1=\infty$, we obtain

$$p(t_0)\int_{-\infty}^{\infty}dt\ \frac{q\epsilon^*}{mc}\exp(-i\Omega t)+\text{c.c.}$$
$$=\frac{\lambda\pi}{8\xi_0v_{zi}^2}\left(\frac{qk_x\psi_0}{mc}\right)^2(J_{n-1}J_{n-2}-J_{n+1}J_{n+2}+J_{n-1}J_n$$
$$-J_nJ_{n+1}). \tag{B9}$$

Using (B6) and (B9), we now find

$$|p(t_1)|^2=|p(t_0)|^2+\frac{\pi}{4\xi_0v_{zi}^2}\left(\frac{qk_x\psi_0}{mc}\right)^2\left\{J_{n-1}^2+J_{n+1}^2\right.$$
$$\left.+\frac{\lambda}{2}\left[\frac{J_{n-1}^2\,2(n-1)}{\lambda}-\frac{J_{n+1}^2\,2(n+1)}{\lambda}\right]\right\}. \tag{B10}$$

Multiply by mc^2 to obtain the energy change,

$$\left(\frac{\pi}{\xi_0 v_{zi}^2}\right)(qk_x\psi_0)^2\,\frac{n^2 J_n J_n'}{2m\lambda}, \tag{B11}$$

whereupon, noticing that $\omega=n\Omega$, and that $k_0=0$, the energy change is seen to be identical with that given by Eq. (31). For $\lambda\rightarrow 0$, this result reduces to the result obtained in Ref. 4.

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