Stability of Muon Beams to Langmuir Waves during Ionization Cooling

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Fast cooling of muon beams will be needed for building either TeV muon colliders that might explore the electroweak gauge symmetry breaking or muon storage rings that could become ultrabright neutrino sources. Stochastic and electron cooling take longer than the muon lifetime, so attention has been focused on the potentially faster but less explored ionization cooling. Addressing recent concerns that excitation of Langmuir waves might be deleterious for ionization cooling techniques, we show that, while the hydrodynamic instability indeed might be dangerous, the waves are, in fact, stabilized through a combination of resistive and kinetic effects at a very modest emittance of the beam.

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The technique of ionization cooling [1] for $\mu^+\mu^-$ colliders [2] or muon storage rings [3] involves passing the muon beam through regions of dense material interposed between accelerating rf cavities. The thought is that, inside the material, the beam loses both transverse and longitudinal momentum, and, in the acceleration region, the longitudinal momentum is restored. Thus, the beam cools. The database for ionization cooling is relatively scant, but the process might be fast enough to produce the necessary cooling effect in the short (2.2 μsec) muon lifetime. This method is now attracting much attention (see, for instance, [4]).

It has been pointed out recently [5] that a realistic muon beam might excite collective electrostatic fields limiting the cooling. While there are regimes wherein beam-medium instabilities could develop within the cooling time, we show that, taking into account resistive and kinetic effects, the excitation of Langmuir electrostatic fields does not limit the beam cooling for parameters of interest.

This Letter is organized as follows: First, we analyze hydrodynamic regimes. We find that electrostatic waves might indeed be dangerously destabilized even in a highly resistive medium for parameters of interest. Second, we show that even a small amount of beam emittance stabilizes these electrostatic waves.

To describe hydrodynamic instabilities, consider that the longitudinal dielectric permittivity of a free-electron medium, such as a metal or plasma, traversed by a cold beam of charged particles is

$$\epsilon(\omega, \vec{k}) = 1 - \frac{\omega_e^2}{\omega(\omega + \imath\nu)} - \frac{\omega_b^2}{\gamma_b \gamma_{b\|}^2 (\omega - \vec{k}\cdot\vec{v}_b)^2}, \tag{1}$$

where $\omega_e = \sqrt{4\pi n_e e^2/m_e}$ and $\omega_b = \sqrt{4\pi n_b e_b^2/m_b}$ are plasma frequencies, n_e and n_b are concentrations, m_e and m_b are rest masses, and e and e_b are charges of the medium electrons and the beam particles, respectively, ν is the effective scattering frequency of medium electrons, v_b is the beam velocity, $\gamma_b = 1/\sqrt{1 - v_b^2/c^2}$ and $\gamma_{b\|} = 1/\sqrt{1 - (\vec{k}\cdot\vec{v}_b)^2/k^2c^2}$ are, respectively, the "full" and the "longitudinal" (with respect to the wave) relativistic factors of the beam, and ω and $\vec{k}$ are, respectively, the frequency and wave vector of an electrostatic wave that may be excited in such a system. The wave dispersion law $\omega(\vec{k})$ is determined from the equation

$$\epsilon(\omega, \vec{k}) = 0.$$

The second term on the right hand side of Eq. (1) describes the dielectric response of free electrons, taking into account a resistivity associated with their scattering. The third term, describing the dielectric response of the beam particles, could be obtained from the standard collisionless plasma response by means of appropriate Lorentz transformations of the wave frequency and the beam particle concentration (which enters into ω_b). This is a simple heuristic way of writing Eq. (1) without calculations. A formal derivation of Eq. (1) for plasmas can be found, for instance, in the textbook [6] (p. 170). Equation (1) neglects both the interbeam and the beam-medium particle collisions, since we are looking here just for instabilities that might develop on a faster time scale. Equation (1) also neglects the change in the medium electron concentration due to electrons expelled by the electrostatic field of the beam, because the beam concentration is much smaller than the electron concentration.

First, note two limiting cases: In the absence of a beam, the equation $\epsilon(\omega, \vec{k}) = 0$ gives the medium plasma oscillations, for which the dispersion relation is $\omega \approx \omega_e - \imath\nu/2$, for $\nu \ll \omega_e$. In the absence of a medium, it gives the beam collisionless plasma oscillations, for which the dispersion relation (transformed to the laboratory reference frame) is $\omega = \omega_b/\gamma_{b\|}\sqrt{\gamma_b} + \vec{k}\cdot\vec{v}_b$.

When both the medium and beam are present, a coupling of the beam and medium plasma oscillations occurs, which leads to an instability. For small $\omega_b \ll \omega_e$, the largest growth rate is reached at $\vec{k}\cdot\vec{v}_b \approx \omega_e$. Consider then the domain $|\vec{k}\cdot\vec{v}_b - \omega_e| \ll \omega_e$. Writing $\omega = \vec{k}\cdot\vec{v}_b + \delta\omega$, where $|\delta\omega| \ll \omega_e$, the dispersion

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equation $\epsilon(\omega, \vec{k}) = 0$ simplifies to

$$0 = \epsilon(\omega, \vec{k}) \approx \frac{2(\delta\omega + \vec{k}\cdot\vec{v}_b - \omega_e) - \imath\nu}{\omega_e} - \frac{\omega_b^2}{\gamma_b \gamma_{b\|}^2 \delta\omega^2}. \quad (2)$$

This equation is easily solved in two complementary cases, namely, $|\delta\omega| \gg |\vec{k}\cdot\vec{v}_b - \omega_e + \imath\nu/2|$ and $|\delta\omega| \ll |\vec{k}\cdot\vec{v}_b - \omega_e + \imath\nu/2|$.

In the limit

$$|\delta\omega| \gg |\vec{k}\cdot\vec{v}_b - \omega_e + \imath\nu/2|,$$

Eq. (2) describes the ideal hydrodynamic beam-plasma instability,

$$\delta\omega \approx (\omega_e \omega_b^2/\gamma_b \gamma_{b\|}^2)^{1/3}(\imath\sqrt{3} - 1)/2^{4/3}. \quad (3)$$

Note that Eq. (3) differs by the factor $(m_e/m_b)^{1/3}$ from the formula for ideal hydrodynamic instability of relativistic electron beam in plasma [7].

In the limit

$$|\delta\omega| \ll |\vec{k}\cdot\vec{v}_b - \omega_e + \imath\nu/2|,$$

Eq. (2) describes the resistive hydrodynamic beam-plasma instability of a kind considered, for instance, in [8],

$$\delta\omega \approx \sqrt{\omega_e \omega_b^2/\gamma_b \gamma_{b\|}^2 [2(\vec{k}\cdot\vec{v}_b - \omega_e) - \imath\nu]} \quad (4)$$

(provided the unstable branch of the square root is selected). The largest growth rate in Eq. (4),

$$\Gamma \equiv \mathrm{Im}\delta\omega \approx \sqrt{\omega_e \omega_b^2/2\gamma_b \nu}, \quad (5)$$

occurs for $\vec{k}\cdot\vec{v}_b = \omega_e$, and $\gamma_{b\|} \approx 1$, which corresponds, for a relativistic beam, to nearly transverse waves $\vec{k}\cdot\vec{v}_b/k \ll c$. The transverse waves, however, might be stabilized convectively if the beam radius is sufficiently small.

Consider, instead, the slower but less easily stabilized longitudinal wave instability, for which the largest growth rate in (4), reached at $\vec{k}\cdot\vec{v}_b = \omega_e$, is

$$\Gamma \equiv \mathrm{Im}\delta\omega \approx \sqrt{\omega_e \omega_b^2/2\gamma_b^3 \nu}. \quad (6)$$

The condition for this regime occurring may be written as $(\omega_e \omega_b^2)^{1/3} \ll \gamma_b \nu$, i.e.,

$$n_b \ll n_e (\gamma_b \nu/\omega_e)^3 m_b/m_e. \quad (7)$$

The ideal regime of the instability to longitudinal waves with $k v_b = \omega_e$, described by Eq. (3), takes place in the opposite case.

Typical parameters of Li, a possible medium for the muon ionization cooling, are as follows: the electron concentration is $n_e \approx 1.4 \times 10^{23}$ cm^{-3}, so that the electron plasma frequency is $\omega_e \approx 2.1 \times 10^{16}$ sec^{-1}; the conductivity in solid state at room temperature is $\sigma \approx 9.57 \times 10^{16}$ sec^{-1}, so that the electron scattering frequency,

$$\nu \approx \omega_e^2/4\pi\sigma,$$

is $\nu \approx 3.7 \times 10^{14}$ sec^{-1}. The scattering frequency ν becomes higher when Li is heated and melted. However, if the Li is heated to $T_e \sim 30$ eV, the ideal plasma approximation becomes reasonable, which gives $\nu \sim 2 \times 10^{16} \sim \omega_e$. Then, for even higher T_e, ν decreases roughly as $T_e^{-3/2}$. This is mentioned here just for a general description of the Li resistivity as a function of temperature, while temperatures to which Li would be heated by a realistic muon beam are in fact much smaller.

Plugging in Eq. (7) $m_b = 207 m_e$, $n_e \approx 1.4 \times 10^{23}$ cm^{-3}, and $\nu \approx 3.7 \times 10^{14}$ sec^{-1}, we find the condition $n_b \ll 10^{20}\gamma_b^3$. Thus, the ideal hydrodynamic instability, Eq. (3), is not limiting for regimes of interest in the muon ionization cooling project. On the other hand, the growth rate of the longitudinal resistive instability, Eq. (6), for the parameters above, is

$$\Gamma \approx \omega_e \sqrt{\frac{n_b}{2n_e}\frac{m_e}{m_b}\frac{\omega_e}{\nu\gamma_b^3}} \approx 2 \times 10^4 \sqrt{n_b \times \mathrm{cm}^3/\gamma_b^3}\ \mathrm{sec}^{-1}. \quad (8)$$

For $n_b \sim 10^{12}$ cm^{-3} and $\gamma_b = 2$ (i.e., $v_b \approx 2.6 \times 10^{10}$ cm/sec), we find $\Gamma \sim 7 \times 10^9$ sec^{-1}. Thus, this instability would make about 50 exponentiations within a 2-m-long muon beam.

Although the instability appears virulent, it is actually easily stabilized by kinetic effects. The kinetic instability is quite different from the hydrodynamic instability. The kinetic effects become important at a sufficient spread of muon longitudinal velocities Δv,

$$\Delta v > v_b \Gamma/\omega_e. \quad (9)$$

Since the ratio Γ/ω_e is very small (3×10^{-7} in the above numerical example), the condition (9) does not appear to be particularly restrictive for applications of interest within the muon ionization cooling project.

The mechanism of the instability suppression by the velocity spread Δv is associated with the corresponding Doppler spread of Cherenkov resonance frequencies $k\Delta v \approx \omega_e \Delta v/v_b$. The phase mixing causes wave damping with the rate $\sim k\Delta v$, which overbalances the wave excitation with the growth rate Γ when the condition (9) is satisfied.

This crude estimate can be refined based on the kinetic theory of beam-medium interaction, combining the macroscopic hydrodynamic description of a resistive free-electron medium with the kinetic description of a collisionless warm beam. Using the standard Vlasov formula for the beam dielectric response [described by the last term in the longitudinal dielectric permittivity, Eq. (1)], one can generalize Eq. (1) as follows:

$$\epsilon(\omega, \vec{k}) = 1 - \frac{\omega_e^2}{\omega(\omega + \imath\nu)} - \frac{\omega_b^2}{\omega^2} G(\omega\vec{k}/k^2), \quad (10)$$

$$G(\vec{u}) = m_b u^2 \int \frac{d^3p}{\vec{u}\cdot\vec{v} - u^2}\,\vec{u}\cdot\frac{\partial f_b}{\partial \vec{p}},$$

$$\vec{v} \equiv \frac{\vec{p}}{\sqrt{m_b^2 + p^2/c^2}},$$

where f_b is the beam particle momentum distribution function normalized to one, $\int d^3p\, f_b = 1$, and $\vec{u} = \omega\vec{k}/k^2$ is the wave phase velocity.

The function $\epsilon(\omega, \vec{k})$ is defined in the upper half-plane, $\mathrm{Im}\omega > 0$, and may be analytically extended to the whole complex plane ω. For stable waves, the dispersion surface $\omega(\vec{k})$, determined from the equation $\epsilon(\omega, \vec{k}) = 0$, is entirely located in the lower half-space $\mathrm{Im}\omega < 0$. When some of the waves are unstable, the surface $\omega(\vec{k})$ crosses the real hyperplane $\mathrm{Im}\omega = 0$ in $(\vec{k}, \omega)$ space. For real ω, the imaginary part of equation $\epsilon(\omega, \vec{k}) = 0$ takes the form

$$\frac{\omega_e^2\nu}{\omega(\omega^2 + \nu^2)} = \frac{\omega_b^2}{\omega^2}\,\mathrm{Im}G(\vec{u}), \quad (11)$$

$$\mathrm{Im}G(\vec{u}) = \pi m_b u^2 \times \int d^3p\,\delta(\vec{u}\cdot\vec{v} - u^2)\vec{u}\cdot\frac{\partial f_b}{\partial \vec{p}}.$$

Near the resonance considered above, $\omega \approx \omega_e \gg \nu$, Eq. (11) takes the form

$$\mathrm{Im}G(\vec{u}) \approx \omega_e\nu/\omega_b^2. \quad (12)$$

When $\mathrm{Im}G(\vec{u})$ is smaller than $\omega_e\nu/\omega_b^2$ for all $\vec{u}$, Eq. (12) has no solutions, so that the dispersion surface $\omega(\vec{k})$, determined from the equation $\epsilon(\omega, \vec{k}) = 0$, does not cross the hyperplane $\mathrm{Im}\omega = 0$ and is entirely located in the stable half-space $\mathrm{Im}\omega < 0$. Thus we arrive at the stability condition,

$$\mathrm{Im}G(\vec{u}) < \omega_e\nu/\omega_b^2, \quad (13)$$

which can be considered as a quantitative definition of Δv in Eq. (9). The fact that a very modest beam spread stabilizes the electrostatic plasma waves in regimes of interest, as expressed by the stability criteria, Eq. (13) or Eq. (9), is the major result of this Letter.

The stability condition can be put more usefully in several regimes of interest. It takes an especially simple form in the case of a nonrelativistic beam when

$$\mathrm{Im}G(\vec{u}) = \pi u^2 \frac{dF}{dv}\bigg|_{v=u}, \quad (14)$$

where F is the normalized ($\int dv\, F = 1$) one-dimensional distribution function of the beam particles over the velocity component v in the direction of the wave propagation $\vec{u}/u = \vec{k}/k$. For a beam with relative energy spread $\Delta\mathcal{E} \ll 1$ and angular spread $\Delta\theta \ll 1$, it follows that

$$\max_{\vec{u}} \mathrm{Im}G(\vec{u}) \sim \frac{\pi}{\Delta\theta^4 + \Delta\mathcal{E}^2}. \quad (15)$$

For the general relativistic case, a more complicated calculation leads to the estimate

$$\max_{\vec{u}} \mathrm{Im}G(\vec{u}) \sim \pi\gamma_b \frac{1 + 3(\gamma_b\Delta\theta)^2}{(\gamma_b\Delta\theta)^4 + \Delta\mathcal{E}^2}. \quad (16)$$

Note that the stability condition, Eq. (13), can be put in the form $\Gamma_{\mathrm{kin}} < \nu$, where Γ_{kin} is the collisionless growth rate formally evaluated as if in the kinetic regime (even though, for certain parameters, the kinetic regime formula might be actually not applicable in the collisionless case). This approach is valid because the effective growth rate tends to zero near the threshold of the instability. Hence, the kinetic regime growth rate applies near the threshold ($\Gamma_{\mathrm{kin}} \approx \nu$) regardless of how small the beam momenta spread is. The estimate $\Gamma_{\mathrm{kin}} < \nu$ agrees (up to the factors m_e/m_b, 3, and π) with the kinetic regime estimate for the collisionless growth rate of ultrarelativistic electron beam instability in plasma [9].

Note that Eq. (16) reduces to Eq. (15) in the nonrelativistic limit $\gamma_b - 1 \ll 1$. In the relativistic case, Eq. (16) can be justified similar to Eq. (14) as follows: The resonance condition $\vec{u}\cdot\vec{v} = u^2$ in (11) defines a hyperboloid of rotation around axis $\vec{u}/u$ in $\vec{p}$ space, namely,

$$\frac{p_\parallel^2}{q^2} - p_\perp^2 = m_b^2c^2, \quad (17)$$

$$p_\parallel = \frac{\vec{p}\cdot\vec{u}}{u}, \qquad \vec{p}_\perp = \vec{p} - p_\parallel\frac{\vec{u}}{u},$$

$$q = \frac{u}{c}\Bigg/\sqrt{1 - \frac{u^2}{c^2}} \equiv \frac{u}{c}\gamma_u.$$

To get the largest $\mathrm{Im}G$, this hyperboloid should cross substantially the domain populated by the beam particles in $\vec{p}$ space. For the most dangerous waves that propagate at small angles $\theta \lesssim \Delta\theta$ to the beam mean velocity

$$\vec{v}_b = \int d^3p\, f_b\vec{p}/\sqrt{m_b^2 + p^2/c^2},$$

it implies that the wave phase velocity $\vec{u}$ is close to the beam velocity $\vec{v}_b$. Small variations of momentum components inside the hyperboloid satisfy the relation $p_\parallel\delta p_\parallel = q^2p_\perp\delta p_\perp$, so that typically $\delta p_\parallel/p_\parallel \sim q^2\Delta\theta^2\delta p_\perp/p_\perp$.

If

$$q^2\Delta\theta^2 \ll \Delta p_\parallel/p_\parallel \sim \Delta\mathcal{E} + \Delta\theta^2,$$

the hyperboloid crosses the beam momenta distribution at nearly fixed

$$p_\parallel \approx m_b u\gamma_u \equiv p_u,$$

and

$$\mathrm{Im}G(\vec{u}) \approx \pi\gamma_u p_u^2 \frac{\partial}{\partial p_u}\int d^3p\, f_b\,\delta(p_\parallel - p_u). \quad (18)$$

This formula is a direct generalization of Eq. (14). In the relativistic case, the $\mathrm{Im}G(\vec{u})$ maximum is reached at $q \sim \gamma_b$, so that the applicability condition implies that

$$\gamma_b^2\Delta\theta^2 \ll \Delta\mathcal{E}.$$

A crude estimate of the $\mathrm{Im}G(\vec{u})$ maximum in Eq. (18) confirms Eq. (16):

$$\max_{\vec{u}} \mathrm{Im}G(\vec{u}) \sim \pi\gamma_b/\Delta\mathcal{E}^2. \quad (19)$$

If

$$q^2\Delta\theta^2 \gg \Delta p_\parallel/p_\parallel \sim \Delta\mathcal{E} + \Delta\theta^2$$

(which implies automatically an ultrarelativistic case $q \gg 1$), the hyperboloid crosses the beam momenta distribution at nearly fixed

$$p_\perp \approx m_b c\sqrt{\gamma_b^2/q^2 - 1} \equiv P_u,$$

and

$$\mathrm{Im}G(\vec{u}) \approx 2\pi p_b m_b c\left(1 + \frac{p_b^2}{q^2 P_u}\frac{\partial}{\partial P_u}\right) \times \int d^3p\, f_b\,\delta(p_\perp^2 - P_u^2). \quad (20)$$

This $\mathrm{Im}G(\vec{u})$ has its maximum at $P_u \sim p_b\Delta\theta$, which corresponds to $q^{-2} \sim \Delta\theta^2 + \gamma_b^{-2}$. The applicability condition then takes the form

$$\gamma_b^2\Delta\theta^2 \gg \Delta\mathcal{E}.$$

A crude estimate of the maximum confirms Eq. (16):

$$\max_{\vec{u}} \mathrm{Im}G(\vec{u}) \sim \pi\frac{1 + 3\gamma_b^2\Delta\theta^2}{\gamma_b^3\Delta\theta^4}. \quad (21)$$

Note also, that the integral in Eq. (11) can be calculated analytically for a Gaussian axisymmetric beam $f_b \propto \exp[-(p - p_b)^2/\Delta p^2 - \theta^2/\Delta\theta^2]$ (see [10]), and the result supports the above crude estimates.

To summarize, the stability condition Eq. (13) shows that excitation of electrostatic Langmuir waves, which is potentially one of the fastest possible instabilities, is not an important limiting factor for the presently anticipated applications of ionization cooling of muon beams. It should be understood, however, that there are other collective excitations not considered here (for instance, those of [11]), which might be pretty numerous in more complicated medium environments and in the presence of magnetic fields. To be sure that the ionization cooling for muon collider applications can take place unhindered by collective effects, it remains to analyze also other wave excitations particularly in a highly resistive (large ν) medium. In any event, muon beam scattering by even stable plasma waves should generally be taken into account for a precise quantitative description of the beam ionization cooling.

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