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Diffusion paths in resonantly driven Hamiltonian systems

I.Y. Dodin *, N.J. Fisch

Department of Astrophysical Sciences, Princeton University, Princeton, NJ 08544, USA

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ABSTRACT

A diffusive Hamiltonian flow triggered by a resonant drive is confined to a phase subspace determined by the resonance structure. The diffusion path is found for an arbitrary, possibly nonstationary discrete system by applying generalized Manley–Rowe relations to an extended Hamiltonian.

A diffusive Hamiltonian flow of particles can be triggered in phase space through the interaction with waves resonant to particle natural oscillations, such as Larmor rotation [1,2]. If the diffusion is slow compared to the oscillations, certain conservation laws persist; hence the flow dimension is smaller than that of the phase space. For a given Hamiltonian, the diffusion path is found from the motion equations; on the other hand, it is entirely determined by the resonance structure and thus could allow a universal form [3]. However, a general solution for an arbitrary number of resonances has not been reported.

In this Letter, we show, by using the generalized Manley–Rowe relations [4], how the diffusion path is deductively obtained for multiple resonances in any discrete system. To do this, we offer an extended Hamiltonian which connects the oscillation center energy [5,6] with the formally introduced particle action conjugate to the wave frequency. Unlike in Refs. [7–11], nonstationary quiver fields are allowed, and non-analytic variable transformations are avoided.

Consider a dynamical system with a Hamiltonian $H(\Gamma,t)$, where Γ is the canonical space, and t is time. Suppose two characteristic time scales and denote t with τ in slow functions and with ξ in fast functions; hence $d_\tau\xi = 1$, and $\partial_t = \partial_\xi + \partial_\tau$ [12,13]. Define an equivalent extended phase space, where $(\xi, -\mathcal{E})$ is an independent canonical pair, with $\mathcal{E} = H(\Gamma,t)$ being the energy. Then the new Hamiltonian is

$$\mathcal{H} = H(\Gamma,\xi,\tau) - \mathcal{E} + A(\xi), \tag{1}$$

where $A(\xi) = \int^\xi \partial_\tau H\, d\tilde{t}$ is approximately the averaged work, with the integral taken along the system trajectory.

Perform a canonical transformation $(\xi,-\mathcal{E}) \to (\zeta, W)$ as determined by the generating function

$$F(\xi,W) = W\xi - \int^{\xi} A(\tilde{\xi})\, d\tilde{\xi}. \tag{2}$$

This results in $\zeta = \partial_W F = \xi$, and $-\mathcal{E} = \partial_\xi F = W - A$, so the new Hamiltonian reads

$$\mathcal{H} = H(\Gamma,\xi,\tau) + W, \tag{3}$$

with $d_\tau W = -\partial_\xi \mathcal{H}$, yielding a corollary $d_t\mathcal{E} = \partial_t H$.

Suppose that H contains multiple scales in ξ; say, it is periodic in $\boldsymbol{\theta} = \boldsymbol{\omega}\xi$, $\boldsymbol{\omega}$ being a constant frequency vector. Then Eq. (3) gives an equivalent Hamiltonian

$$\mathcal{H} = H(\Gamma,\boldsymbol{\theta},\tau) + \boldsymbol{\omega}\cdot\mathbf{I}, \tag{4}$$

where $\mathbf{I}$ is the action vector conjugate to the angle $\boldsymbol{\theta}$, at $\dim\boldsymbol{\omega} = 1$ expressed as $I = (A-\mathcal{E})/\omega$.[1,2] Suppose also that the phase space is separated such that

$$\Gamma = (\mathbf{q},\mathbf{p}) \times (\boldsymbol{\varphi},\mathbf{J}), \tag{5}$$

* Corresponding author.
E-mail address: idodin@princeton.edu (I.Y. Dodin).

1 $\mathcal{H}$ is understood as the Hamiltonian accounting for additional degrees of freedom which are associated with the oscillatory field. The latter has the energy $\sum_i n_i\hbar\omega_i = W$, where $n_i = I_i/\hbar$ is the number of quanta in ith mode, and the total energy of the system is governed by $d_\tau(\mathcal{E} + \sum_i n_i\hbar\omega_i) = \partial_\tau H$.

2 For $\partial_\tau H \equiv 0$, the fact that $-\mathcal{E}/\omega$ is the action corresponding to oscillations at the quiver field frequency ω was previously shown in Refs. [7–11] using a non-analytic variable transformation.

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and the system is close to integrable in $(\boldsymbol{\vartheta}, \mathcal{J})$, where $\boldsymbol{\vartheta} = \boldsymbol{\varphi} \oplus \boldsymbol{\theta}$, and $\mathcal{J} = \mathbf{J} \oplus \mathbf{I}$, with $(\boldsymbol{\varphi}, \mathbf{J})$ being some angle-action variables; $d_\tau \boldsymbol{\varphi} = \boldsymbol{\Omega}$. This means that

$$H = H_0 + \epsilon H_\sim, \tag{6}$$

where ϵ is vanishingly small, and H_0 is a $\boldsymbol{\vartheta}$-independent 'oscillation center' Hamiltonian [5,6], which can be derived, e.g., from the averaged Lagrangian $\langle L \rangle$ [14]:

$$H_0 = \mathbf{p} \cdot \mathbf{v} - \mathcal{L} \approx \mathcal{E}, \quad \mathcal{L} = \langle L \rangle - \boldsymbol{\Omega} \cdot \mathbf{J}, \tag{7}$$

with $\mathbf{p} = \partial_{\mathbf{v}} \mathcal{L}$, and $\mathbf{v} = d_\tau \mathbf{q}$. Hence $\mathcal{J}$ is an adiabatic invariant, meaning[3]

$$d_\tau \mathcal{E} \approx \partial_\tau H_0, \qquad \mathbf{J} = \text{const}, \tag{8}$$

unless the frequency vector $\boldsymbol{\varpi} \equiv \boldsymbol{\Omega} \oplus \boldsymbol{\omega}$ allows resonances [15]. Otherwise, put the resonance condition as

$$\hat{\mathbf{R}} \cdot \boldsymbol{\varpi} = 0, \tag{9}$$

where $\hat{\mathbf{R}}$ is an integer $n \times n$ matrix, with $n = \dim \boldsymbol{\varpi}$ and rank $r < n$. At nontrivial $\hat{\mathbf{R}}$, the invariance is preserved for the *projection* of $\mathcal{J}$ on $\ker \hat{\mathbf{R}}$ [4,16]:

$$\mathcal{J}_{\text{ker}} \equiv \hat{\mathbf{P}}_{\text{ker}} \cdot \mathcal{J} = \text{const} \tag{10}$$

(here $\hat{\mathbf{P}}_{\text{ker}}$ is the projection operator); hence $n - r$ so-called Manley–Rowe relations [17,18] independent of $\epsilon \to 0$. Should those allow an expression in terms of $\mathcal{E}$ and $\mathbf{J}$,[4] a diffusion path in Γ is obtained:

$$\mathcal{J}_{\text{ker}}(\Gamma) = \text{const}. \tag{11}$$

Below we illustrate this technique on known sample problems. First, suppose a resonance

$$\nu \Omega \approx \ell \omega, \tag{12}$$

with integer ν and ℓ, for simplicity assuming a monochromatic field, a single frequency Ω, and $\partial_\tau H \equiv 0$. Rewrite Eq. (12) in the form (9), where

$$\hat{\mathbf{R}} = \begin{pmatrix} \nu & -\ell \\ 0 & 0 \end{pmatrix}, \tag{13}$$

yielding a one-dimensional $\ker \hat{\mathbf{R}}$ with a basis vector $\mathbf{g} = (\ell, \nu)^{\mathrm{T}}$. Hence $\mathbf{g} \cdot \mathcal{J} = \ell J + \nu I$ is conserved, or

$$d\mathcal{E}/dJ = \omega \ell / \nu, \tag{14}$$

which is a generalization of the diffusion paths known for wave-particle interactions [1] to any H_0.

For example, for a particle in a magnetic field $\mathbf{B} = \mathbf{e}_z B$, Eq. (14) yields $d\mathcal{E}/d\mu = (\omega\ell/\nu)(mc/q)$, where $\mu = qJ/mc$ is the magnetic moment associated with the Larmor rotation at frequency $\Omega = qB/mc$, m/q is the particle mass-to-charge ratio, and c is the speed of light. Assuming a wavevector $\mathbf{k} = \mathbf{e}_y k$, the guiding center displacement satisfies $dx = -(\ell/\nu)(kc/q\Omega)\, d\mu$ [19]. Hence one can also obtain a diffusion path in $(x, \mathcal{E})$ plane reading $d\mathcal{E}/dx = -m\Omega\omega/k$ (cf. Ref. [19]) used, for instance, in α-channeling theory [2,20].

To illustrate dealing with multiple Ω_i, suppose

$$\nu_1 \Omega_1 + \nu_2 \Omega_2 \approx \ell \omega. \tag{15}$$

This corresponds to

$$\hat{\mathbf{R}} = \begin{pmatrix} \nu_1 & \nu_2 & -\ell \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \tag{16}$$

hence two independent vectors in $\ker \hat{\mathbf{R}}$:

$$\mathbf{g}^{(1)} = (\ell, 0, \nu_1)^{\mathrm{T}}, \qquad \mathbf{g}^{(2)} = (0, \ell, \nu_2)^{\mathrm{T}}, \tag{17}$$

so Eq. (10) yields conservation laws

$$\ell J_1 + \nu_1 I = \text{const}, \qquad \ell J_2 + \nu_2 I = \text{const}, \tag{18}$$

with a corollary $\nu_2 J_1 - \nu_1 J_2 = \text{const}$. Then, like in the previous case, channeling occurs along a straight line in $(\mathbf{J}, \mathcal{E})$ space, now reading

$$dJ_1/\nu_1 = dJ_2/\nu_2 = d\mathcal{E}/\omega\ell. \tag{19}$$

Like Eq. (14), the obtained equalities are also understood from the conservation of the total number of quanta and energy in the particle-field system; see, e.g., Ref. [21] for a similar treatment.

In summary, a diffusive Hamiltonian flow triggered by a resonant drive is confined to a phase subspace determined by the resonance structure. The diffusion path is found for an arbitrary, possibly nonstationary discrete system by applying generalized Manley–Rowe relations to an extended Hamiltonian. For a particular system, the algorithm for obtaining the path is summarized as follows: (i) expressions are derived for the actions $\mathbf{J}$ and the oscillation center energy $\mathcal{E}$ [Eq. (7)], (ii) the resonance condition is put in the form (9), and (iii) the path is obtained from the fact that the projection of $\mathcal{J}$ on $\ker \hat{\mathbf{R}}$ remains constant [Eq. (10)]. For the two examples illustrating this technique, known results are reproduced.

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References

[1] B.I. Cohen, R.H. Cohen, W.M. Nevins, T.D. Rognlien, Rev. Mod. Phys. 63 (1991) 949;
A. Bécoulet, D.J. Gambier, A. Samain, Phys. Fluids B 3 (1991) 137;
L. Chen, J. Vaclavik, G.W. Hammett, Nucl. Fusion 28 (1988) 389, and references therein.
[2] N.J. Fisch, J.M. Rax, Phys. Rev. Lett. 69 (1992) 612.
[3] P.A. Sturrock, Ann. Phys. 9 (1960) 422.
[4] I.Y. Dodin, A.I. Zhmoginov, N.J. Fisch, Phys. Lett. A 372 (2008) 6094.
[5] R.L. Dewar, Phys. Fluids 16 (1973) 1102.
[6] P.L. Similon, A.N. Kaufman, D.D. Holm, Phys. Fluids 29 (1986) 1908.
[7] D.L. Bruhwiler, J.R. Cary, Phys. Rev. Lett. 68 (1992) 255.
[8] D.L. Bruhwiler, J.R. Cary, Phys. Rev. E 50 (1994) 3949.
[9] D.L. Bruhwiler, J.R. Cary, Part. Accel. 43 (1994) 195.
[10] G.M. Fraiman, I.Y. Kostyukov, Phys. Plasmas 2 (1995) 923.
[11] I.Y. Dodin, N.J. Fisch, Phys. Rev. E 74 (2006) 056404.
[12] A.J. Brizard, T.S. Hahm, Rev. Mod. Phys. 79 (2007) 421.
[13] G. Yao, R.E. Wyatt, J. Chem. Phys. 101 (1994) 1904.
[14] I.Y. Dodin, N.J. Fisch, Phys. Rev. E 77 (2008) 036402.
[15] A.J. Lichtenberg, M.A. Lieberman, Regular and Chaotic Dynamics, second ed., Springer-Verlag, New York, 1992.
[16] F.G. Gustavson, Astron. J. 71 (1966) 670.
[17] J.M. Manley, H.E. Rowe, Proc. Inst. Radio Eng. 44 (1956) 904.
[18] M.I. Rabinovich, D.I. Trubetskov, Oscillations and Waves in Linear and Nonlinear Systems, Kluwer, Boston, 1989.
[19] G.R. Smith, B.I. Cohen, Phys. Fluids 26 (1983) 238.
[20] N.J. Fisch, Phys. Rev. Lett. 97 (2006) 225001.
[21] I.Y. Dodin, N.J. Fisch, Phys. Lett. A 349 (2006) 356.

[3] For a particle in an external field, this regime corresponds to acquiring an effective rest mass, resulting in an effective average potential at nonrelativistic energies [14].

[4] I_i belong to the extended space (footnote 2); thus, at $\dim \boldsymbol{\omega} > 1$, not all combinations of I_i are expressed through $\mathcal{E}$.