Positive and negative effective mass of classical particles in oscillatory and static fields

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A classical particle oscillating in an arbitrary high-frequency or static field effectively exhibits a modified rest mass m_{eff} derived from the particle averaged Lagrangian. Relativistic ponderomotive and diamagnetic forces, as well as magnetic drifts, are obtained from the m_{eff} dependence on the guiding center location and velocity. The effective mass is not necessarily positive and can result in backward acceleration when an additional perturbation force is applied. As an example, adiabatic dynamics with $m_{\parallel}>0$ and $m_{\parallel}<0$ is demonstrated for a wave-driven particle along a dc magnetic field, $m_{\parallel}$ being the effective longitudinal mass derived from m_{eff}. Multiple energy states are realized in this case, yielding up to three branches of $m_{\parallel}$ for a given magnetic moment and parallel velocity.

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I. INTRODUCTION

A large number of problems connected with multiscale adiabatic dynamics of classical particles in oscillatory and static fields enjoy critical simplification within the guiding-center approach, which allows separating fast oscillatory motion of the particles from their slow translational motion [1–9]. Often, the average forces on a guiding center are then written in terms of effective potentials Ψ, such as ponderomotive [7,10–12], diamagnetic [13], or others [8,12]. Yet the applicability of the potential approximation is limited to, at most, nonrelativistic interactions, another drawback being the unphysical difference in fictitious fields $-\boldsymbol{\nabla}\Psi$ seen by different species.

The purpose of this paper is to offer an alternative approach that allows *embedding* the average forces into the guiding center properties through redefining the particle rest mass m_{eff}. The average acceleration is then attributed to the effective mass variations, which are naturally different for different species; hence no fictitious fields are introduced. By definition, this "object-oriented" formulation [14] is also intrinsically relativistic. Therefore it equally holds for arbitrary adiabatic interactions [15], thus proving to be more fundamental as compared to the effective-potential approach.

Previously, the effective mass m_{eff} was similarly introduced for an electron interacting with an intense laser wave in vacuum, with additional fields considered only as perturbations [16–29]. In this paper, we show that m_{eff} can be defined as well for any other multiscale dynamics of a particle in high-frequency or static fields. We offer a general formula for the effective mass and show how manipulations of m_{eff} as a function of the guiding center variables yield the average forces and particle trajectories. We also show that the effective mass is not necessarily positive and can result in backward acceleration when an additional force is applied. As an example, we explore the average motion of a laser-driven particle immersed in a dc magnetic field. Multiple energy states are realized in this case and yield up to three branches of m_{eff} and the effective longitudinal mass $m_{\parallel}$ for a given magnetic moment and parallel velocity. We show that both $m_{\parallel}>0$ and $m_{\parallel}<0$ are possible then, the negative-mass regime too allowing for adiabatic dynamics.

The paper is organized as follows. In Sec. II, we derive the general formula for m_{eff} and the guiding center Hamiltonian accounting for additional perturbation fields, if any. In Sec. III, we apply the effective mass formalism to the particle motion in a static magnetic field and rederive both the particle Hamiltonian and the magnetic drifts. In Sec. IV, we explore the average motion of a laser-driven particle in a static magnetic field and demonstrate the possibility for adiabatic dynamics at negative m_{eff} and $m_{\parallel}$. In Sec. V, we summarize our main ideas. Supplementary calculations are given in the Appendixes: In Appendix A, we obtain the general form of the drift Lagrangian employed in Sec. II. In Appendix B, we show how the effective mass formalism allows derivation of ponderomotive forces in various cases of interest.

II. EFFECTIVE MASS

Consider a particle undergoing arbitrary quasiperiodic oscillations superimposed on the average motion. In the adiabatic regime, one can map out the quiver dynamics by changing variables [1–8]; hence the guiding center is treated as a "dressed," or quasiparticle. Suppose, for now, that the background fields causing the oscillations do not vary along the trajectory. Then the associated field tensor $F_{\mu\nu}$ will not enter the averaged equations as a force. However, it will affect the motion such that, in response to *additional* perturbation fields $\tilde{F}_{\mu\nu}$, the guiding center will react as if it had a modified mass.

The effect is shown as follows. At zero $\tilde{F}_{\mu\nu}$, the average dynamics is determined only by the field tensor $F'_{\mu\nu}$ seen by the particle in the guiding-center rest frame K', further denoted by a prime. The action increment $d\mathcal{S}=\mathcal{L}\,dt$ in the laboratory frame K is then written as

$$d\mathcal{S}(A_\mu,\overline{\mathbf{v}}) = d\mathcal{S}'(F'_{\mu\nu}) + d\mathcal{G}, \tag{1}$$

where A_μ is the four-potential such that $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, $\overline{\mathbf{v}}=\langle\mathbf{v}\rangle$ is the guiding center velocity in K, $\mathbf{v}$ is the particle true velocity, $\langle\cdots\rangle$ denotes the time average, and $d\mathcal{G}$ depends on the selected gauge. Use $dt=\overline{\gamma}\,dt'$, where dt' is the time interval in K', $\overline{\gamma}=(1-\overline{v}^2/c^2)^{-1/2}$, and c is the speed of light; then the guiding center Lagrangian reads

$$\mathcal{L}=\mathcal{L}'/\bar{\gamma}+\dot{\mathcal{G}}, \tag{2}$$

where $\mathcal{L}'=d\mathcal{S}'/dt'$. Omitting an insignificant full time derivative, one can rewrite Eq. (2) as [30]

$$\mathcal{L}=-m_{\text{eff}}c^2\sqrt{1-\bar{v}^2/c^2}. \tag{3}$$

Hence $\mathcal{L}$ is formally equivalent to the Lagrangian of a free particle with an effective mass

$$m_{\text{eff}}=-\mathcal{L}'/c^2, \tag{4}$$

which is, by definition, both gauge- and Lorentz-invariant.

By definition, $\mathcal{L}'=\mathcal{L}'(F'_{\mu\nu})$, where $F'_{\mu\nu}$ can be written in terms of A'_μ; thus

$$\mathcal{L}'_{A'_\mu}=\frac{\mathcal{L}_{A_\mu}-\dot{\mathcal{G}}_{A_\mu}}{\sqrt{1-\bar{v}^2/c^2}}, \tag{5}$$

for any $\bar{\mathbf{v}}$. Consider $\bar{v}\to 0$; in this case, $A_\mu\to A'_\mu$, so

$$\mathcal{L}'(F'_{\mu\nu})=[\mathcal{L}_{A'_\mu}-(\partial G_{A'_\mu}/\partial t)]_{\bar{v}=0}, \tag{6}$$

where we removed the subindex "A'_μ" in the left-hand side, as $\mathcal{L}'$ is gauge-invariant.

In the absence of oscillations, $\mathcal{L}'$ must equal $-mc^2$, where m is the true mass; therefore

$$\mathcal{L}'=-mc^2+\mathcal{L}_{A'_\mu}(\bar{v}=0)-\mathcal{L}_{A'_\mu}(v=0). \tag{7}$$

For clarity, we assume below that $L(v=0)=-mc^2$. Then, using Eq. (A6), one can write m_{eff} as

$$m_{\text{eff}}=\frac{1}{c^2}(\mathbf{J}\cdot\boldsymbol{\nu}-\langle L\rangle)_{\bar{v}=0}, \tag{8}$$

where the right-hand side is to be evaluated in K' (hence the index "$\bar{v}=0$"); $\mathbf{J}$ are the actions and $\boldsymbol{\nu}$ are the frequencies of oscillations in canonical angles, if any, to average over (Appendix A). Therefore, apart from the latter, m_{eff} is proportional to the gauge-independent part of the averaged Lagrangian in the guiding-center rest frame [31]. Since $\mathcal{L}'$ is calculated in $K'(\bar{\mathbf{v}})$, m_{eff} is generally a function of the velocity $\bar{\mathbf{v}}$. When $F'_{\mu\nu}$ slowly varies with the guiding center coordinate $\bar{\mathbf{r}}$ or time t, m_{eff} may similarly depend on those as well, so Eq. (3) will automatically yield the average forces [Eq. (A8)].

Suppose now that a particle interacts with a perturbation field $\widetilde{F}_{\mu\nu}$ governed by $\widetilde{A}_\mu=(\widetilde{\mathbf{A}},\widetilde{\varphi})$, which is imposed over $F_{\mu\nu}$ [16,24,32]. In the adiabatic regime, the orbit is not altered on the oscillation time scale; thus,

$$\mathcal{L}=-m_{\text{eff}}c^2\sqrt{1-\bar{v}^2/c^2}+\frac{e}{c}(\bar{\mathbf{v}}\cdot\widetilde{\mathbf{A}})-e\widetilde{\varphi} \tag{9}$$

(e being the particle charge), and a nonelectromagnetic potential can be added similarly. Then, the canonical momentum equals $\overline{\mathbf{P}}=\bar{\mathbf{p}}+(e/c)\widetilde{\mathbf{A}}$, and the kinetic momentum $\bar{\mathbf{p}}$ is given by

$$\bar{\mathbf{p}}=\bar{\gamma}m_{\text{eff}}\bar{\mathbf{v}}-\frac{c^2}{\bar{\gamma}}\frac{\partial m_{\text{eff}}}{\partial\bar{\mathbf{v}}}. \tag{10}$$

Correspondingly, the Hamiltonian $\mathcal{H}=\overline{\mathbf{P}}\cdot\bar{\mathbf{v}}-\mathcal{L}$ reads

$$\mathcal{H}=\bar{\gamma}m_{\text{eff}}c^2-\frac{c^2}{\bar{\gamma}}\left(\bar{\mathbf{v}}\cdot\frac{\partial m_{\text{eff}}}{\partial\bar{\mathbf{v}}}\right)+e\widetilde{\varphi}, \tag{11}$$

and $\mathcal{E}=\mathcal{H}(\bar{\mathbf{r}},\overline{\mathbf{P}},t)$ is conserved when $\mathcal{H}$ is independent of time. Thus m_{eff} can be viewed also as the normalized quasienergy of an unperturbed ($\widetilde{F}_{\mu\nu}=0$) particle in the guiding-center rest frame, $m_{\text{eff}}=\mathcal{E}'/c^2$.

III. STATIC MAGNETIC FIELD

Let us demonstrate how the effective mass formalism applies to the problem of particle motion in a dc magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$, in which case

$$L=-\frac{mc^2}{\gamma}+\frac{e}{c}(\mathbf{v}\cdot\mathbf{A}), \tag{12}$$

where $\gamma=(1-v^2/c^2)^{-1/2}$. Assuming a smooth $\mathbf{B}$, the motion can be averaged over Larmor oscillations at frequency $\Omega=eB/mc\gamma$, so the guiding center dynamics that remains is one-dimensional, and the associated action $J=(mc/e)\mu$ is conserved [13], where $\mu=p_\perp^2/2Bm$ is the magnetic moment, and $\mathbf{p}_\perp=\gamma m\mathbf{v}_\perp$ is the relativistic kinetic momentum transverse to $\mathbf{B}$. Thus the effective mass reads $m_{\text{eff}}=[\mu B/\gamma'-\langle L\rangle']/c^2$, where the prime denotes the guiding-center rest frame K'; μ and B are Lorentz invariants, and

$$\gamma'=\sqrt{1+2\mu B/mc^2} \tag{13}$$

is constant. Since $\langle L\rangle'=-(mc^2+\mu B)/\gamma'$, one obtains

$$m_{\text{eff}}=m\sqrt{1+2\mu B/mc^2}, \tag{14}$$

which is a relativistic invariant. The guiding center momentum is then given by $\bar{\gamma}m_{\text{eff}}\bar{v}=p_\parallel$, where we used a Lorentz transformation $\gamma=\bar{\gamma}\gamma'$. Hence the Hamiltonian (11) reads [33,34]

$$\mathcal{H}=\sqrt{m^2c^4+2\mu Bmc^2+p_\parallel^2c^2}+e\widetilde{\varphi}, \tag{15}$$

yielding, after omitting an insignificant constant, the well-known nonrelativistic limit

$$\mathcal{H}=\frac{1}{2m}p_\parallel^2+\mu B+e\widetilde{\varphi}. \tag{16}$$

A more precise calculation also delivers particle drifts [1,13,34]: Allow arbitrary $\bar{v}_\parallel$, yet assume nonrelativistic $\bar{\mathbf{v}}_\perp$ so as to treat the transverse drift as a perturbation. Following Ref. [35], we write the new guiding center Lagrangian as $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{\text{int}}$, where $\mathcal{L}_0=-m_{\text{eff}}c^2/\bar{\gamma}$ is that for the unperturbed motion, and

$$\mathcal{L}_{\text{int}}=\frac{e}{c}(\bar{\mathbf{v}}\cdot\mathbf{A})-e\widetilde{\varphi} \tag{17}$$

is the interaction Lagrangian small compared to $\mathcal{L}_0$. For simplicity, assume static fields, so the guiding center quasienergy (11) is conserved. In this case, we can consider $\overline{\mathbf{v}}$ as a function of $\overline{\mathbf{r}}$; hence $\delta\mathcal{S}=0$ yields

$$(\overline{\mathbf{v}}\cdot\boldsymbol{\nabla}_*)\overline{\mathbf{p}}=\frac{e}{c}\overline{\mathbf{v}}\times\mathbf{B}-\frac{c^2}{\overline{\gamma}}\boldsymbol{\nabla}m_{\text{eff}}, \tag{18}$$

where $\boldsymbol{\nabla}$ differentiates with respect to $\overline{\mathbf{r}}$ at fixed $\overline{\mathbf{v}}$, and $\boldsymbol{\nabla}_*\equiv\boldsymbol{\nabla}+\Sigma_i(\boldsymbol{\nabla}\overline{v}_i)(\partial/\partial\overline{v}_i)$ is the full spatial derivative. Equation (18) is equivalent to

$$\overline{\mathbf{v}}\times\mathbf{B}^*=0,\quad \mathbf{B}^*=\mathbf{B}+(c/e)\boldsymbol{\nabla}_*\times\overline{\mathbf{p}}, \tag{19}$$

which can as well be put in the form

$$\overline{\mathbf{v}}=\overline{v}_\parallel\frac{\mathbf{B}+(c/e)\boldsymbol{\nabla}_*\times(p_\parallel\hat{\mathbf{b}})}{B+(c/e)p_\parallel\hat{\mathbf{b}}\cdot(\boldsymbol{\nabla}\times\hat{\mathbf{b}})}, \tag{20}$$

where $\hat{\mathbf{b}}=\mathbf{B}/B$, and $\overline{\mathbf{p}}\approx p_\parallel\hat{\mathbf{b}}$. This generalizes a similar analysis, which was proposed in Ref. [35] for $v\ll c$, to any v, such that $\overline{v}_\perp\ll c$.

Equations (19) and (20) yield the known expressions of the traditional drift approximation [4,33,36–39]. However, they also allow for an arbitrary m_{eff}, not necessarily that given by Eq. (14); thus additional strong fields, if any, are as well embedded here. Derivation of time-dependent and fully relativistic magnetic drifts [40,41] using the effective mass formalism should be possible, too, but remains out of the scope of the present paper.

IV. RELATIVISTIC WAVE FIELD OVER A STATIC MAGNETIC FIELD

The unified effective mass formulation readily yields the ponderomotive forces previously derived from other considerations (Appendix B). In this section, we contemplate another example of particle ponderomotive dynamics, which exhibits unusual properties that, to our knowledge, have not been covered in literature.

A. Basic equations

Consider a relativistic particle in a wave propagating along a static magnetic field [42–50]. Assume a smooth magnetic field $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}_{\text{dc}}$, approximately in the $\hat{\mathbf{z}}$ direction; then the vector potential $\mathbf{A}_{\text{dc}}$ can be considered a linear function of the particle displacement from the guiding center location:

$$\mathbf{A}_{\text{dc}}=\frac{1}{2}B(z)(\hat{\mathbf{z}}\times\mathbf{r}_\perp). \tag{21}$$

For simplicity, we will also assume a vacuum electromagnetic wave with circular polarization in the plane transverse to $\mathbf{B}$, so the corresponding vector potential reads

$$\mathbf{A}_{\text{w}}=\left(\frac{mc^2}{e}\right)\frac{a_0}{\sqrt{2}}(\hat{\mathbf{x}}\cos\xi-\hat{\mathbf{y}}\sin\xi), \tag{22}$$

where the invariant $a_0=eE_0/mc\omega$ is allowed to slowly vary in space and time, E_0 is the amplitude of the electric field $\mathbf{E}=-(1/c)(\partial\mathbf{A}_{\text{w}}/\partial t)$, and $\xi=\omega t-kz$ is the phase, with $k=\omega/c$. In this case, the particle motion is fully integrable, and the problem can be solved analytically.

According to Sec. II, we calculate m_{eff} for uniform fields, using Eq. (12) with $\mathbf{A}=\mathbf{A}_{\text{dc}}+\mathbf{A}_{\text{w}}$. The effective mass is determined by the averaged Lagrangian in the guiding-center rest frame K' (further denoted by a prime), which is found as follows. Since $\mathbf{A}$ depends on z and t only via $\xi(z,t)$, there exists an integral

$$u=\gamma-p_z/mc, \tag{23}$$

yielding that the following equality holds for any f [16]:

$$\langle f\rangle=\langle\gamma f\rangle_\xi/\langle\gamma\rangle_\xi. \tag{24}$$

(The subindex ξ denotes that the averaging is performed over the phase rather than time.) Take $f=L'$; then

$$\langle L\rangle'=-\frac{1}{\langle\gamma'\rangle_\xi}\left(mc^2-\frac{e}{mc}\langle\mathbf{p}'\cdot\mathbf{A}'\rangle_\xi\right). \tag{25}$$

With $f=\mathbf{v}$, Eq. (24) also yields

$$\overline{\mathbf{v}}=\langle\mathbf{p}\rangle_\xi/m\langle\gamma\rangle_\xi,\quad \langle\mathbf{p}'\rangle_\xi=0,\quad \langle\gamma'\rangle_\xi=u'. \tag{26}$$

Given that the average motion is solely in the $\hat{\mathbf{z}}$ direction, K' is now defined as the frame where $\langle p_z'\rangle_\xi=0$.

Hence the particle motion can be written as follows:

$$x=x_0+\mathcal{R}\cos\theta-\frac{a_0}{ku\sqrt{2}}\frac{\sin\xi}{(1-\sigma)}, \tag{27a}$$

$$y=y_0-\mathcal{R}\sin\theta-\frac{a_0}{ku\sqrt{2}}\frac{\cos\xi}{(1-\sigma)}, \tag{27b}$$

$$z=z_0+\frac{\rho_0\xi}{ku}+\frac{\mathcal{R}a_0}{u\sqrt{2}}\frac{\sigma\cos(\xi-\theta)}{(1-\sigma)^2}, \tag{27c}$$

$$p_x=-\mathcal{P}\sin\theta-\frac{mca_0}{\sqrt{2}}\frac{\cos\xi}{(1-\sigma)}, \tag{27d}$$

$$p_y=-\mathcal{P}\cos\theta+\frac{mca_0}{\sqrt{2}}\frac{\sin\xi}{(1-\sigma)}, \tag{27e}$$

$$p_z=mc\rho_0-\frac{\mathcal{P}a_0}{u\sqrt{2}}\frac{\sin(\xi-\theta)}{(1-\sigma)}. \tag{27f}$$

Here $\rho_0=u\overline{v}/(c-\overline{v})$ is the normalized phase-averaged momentum; $\mathcal{R}\equiv\mathcal{P}/m\Omega_0$, $\mathcal{P}$, and $\theta=\sigma\xi+\theta_0$ denote the gyroradius, the transverse momentum, and the phase of the *free* gyromotion superimposed on the wave-induced oscillations, σ being a Lorentz invariant:

$$\sigma=\frac{\Omega_0}{\omega u}=\frac{\Omega_0/\gamma}{\omega-kv_z}; \tag{28}$$

$\mu\equiv\mathcal{P}^2/2Bm$ is the invariant action of this gyromotion conserved under adiabatic perturbations [cf. Eq. (B6)]; x_0, y_0, z_0, $\overline{v}$, $\mathcal{P}$, and θ_0 are determined by initial conditions.

To find u, substitute $\gamma=u+\rho_z$ into $\gamma^2=1+\rho_\perp^2+\rho_z^2$, where $\boldsymbol{\rho}\equiv\mathbf{p}/mc$, and average over ξ using $\langle\rho_z\rangle_\xi=\rho_0$:

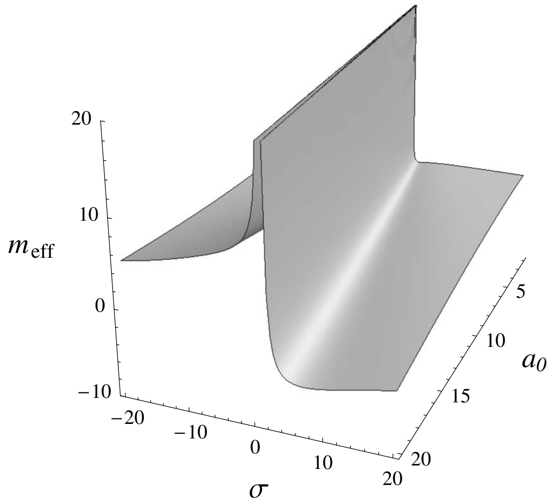

FIG. 1. Effective mass $m_{\rm eff}$ of a wave-driven particle in a magnetic field [Eq. (31), with $s=0$; m units] vs a_0 and σ.

$$(u+\rho_0)^2 = 1+\rho_0^2+\langle\rho_\perp^2\rangle_\xi. \tag{29}$$

Then,

$$u = h\sqrt{1+s^2+\frac{a_0^2}{2(1-\sigma)^2}}, \quad h=\sqrt{\frac{c-\bar{v}}{c+\bar{v}}}, \tag{30}$$

where $\sigma=\sigma(u)$, $s^2\equiv 2\mu B/mc^2$ is an invariant, and $\frac{1}{4}mc^2a_0^2$ equals the zero-B nonrelativistic ponderomotive potential $\Phi=e^2E_0^2/4m\omega^2$ [Eq. (B3)].

Combining Eqs. (25)–(30), one gets

$$\frac{m_{\rm eff}}{m} = \left[1+s^2+\frac{a_0^2(2-\sigma)}{4(1-\sigma)^2}\right]\left[1+s^2+\frac{a_0^2}{2(1-\sigma)^2}\right]^{-1/2}, \tag{31}$$

which is a covariant form of $m_{\rm eff}$ for the effective mass expressed as a function of Lorentz invariants. Equation (31) yields Eqs. (14)–(20) at $a_0=0$, Eqs. (B13) and (B14) at $B=0$, Eqs. (B9) and (B10) at $v/c\ll 1$, and Eqs. (B3)–(B5) at $v/c\ll 1$ together with $B=0$. From Eq. (31), it also readily follows that $m_{\rm eff}(a_0<4)>0$; yet $m_{\rm eff}(a_0>4)<0$ at least for some $\sigma>1$ (Fig. 1). Other properties of $m_{\rm eff}$ are discussed in Secs. IV B and IV C.

B. Tristability

In the presence of both relativistic effects and nonzero B, the cyclotron resonance is essentially nonlinear and permits multiple energy states at given $\bar{v}$ and μ. To see this, rewrite Eq. (30) as

$$\mathcal{U}(u) \equiv 2(u-\sigma_0)^2(h^{-2}u^2-s^2-1)-a_0^2u^2=0, \tag{32}$$

where $\sigma_0=\Omega_0/\omega$ [51]. Equation (32) is a fourth-order algebraic equation; thus it allows up to four values of u, which also can be found analytically [[52], Sec. 1.8-5]. (Explicit solutions are not shown here because of their complexity.) Since $\mathcal{U}(0)<0$ and $\mathcal{U}(\pm\infty)=+\infty$, two solutions always exist, one of them being unphysical ($u_4<0$). Further consideration

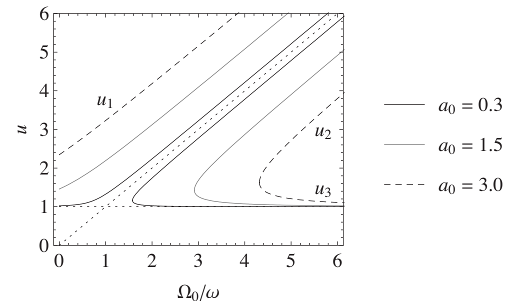

FIG. 2. Solution of Eq. (32) for u vs $\sigma_0\equiv\Omega_0/\omega$: at $\sigma_0\to\infty$, one has $u_{1,2}\sim\sigma_0\pm a_0h/\sqrt{2}$, and $u_3\sim h\sqrt{1+s^2}$, where u_i corresponds to different branches; $u(0)=h\sqrt{1+s^2+a_0^2/2}$. For a given σ_0, u_2 and u_3 appear simultaneously behind the nonlinear resonance $u=\sigma_0$ (dotted); the condition is yielded by Eq. (33). Shown is the case $\bar{v}=0$ (so $u=\langle\gamma\rangle_\xi$), $s=0$ for $a_0=0.3$, $a_0=1.5$, and $a_0=3$.

of the signs in Eq. (32) shows [[52], Sec. 1.6-6(c)] that, apart from degenerate cases, there exist either one or three positive roots, $u_1>u_2>u_3$. Therefore one or three energy states are possible (Fig. 2), allowing for hysteretic effects [53–56], which also have quantum analogies in solid-state physics [57–59].

The condition for multiple u_i reads

$$\sigma_0>\sigma_c\equiv h[(1+s^2)^{1/3}+(a_0^2/2)^{1/3}]^{3/2}. \tag{33}$$

Therefore assuming that three branches exist for a given s, there must exist the same number of energy states for $s=0$. The latter energy states correspond to three equilibria in the momentum space ($p_\perp\cos\psi, p_\perp\sin\psi, p_z$), where $\psi=\xi+\chi$, χ is the gyrophase. Apart from the degenerate case when u_2 approaches u_3, the particle trajectory (27) is a continuous function of the initial conditions for each branch. Thus assuming negligible dissipation (Sec. IV C), all three equilibria are stable here (Fig. 3), unlike for a one-dimensional (1D) nonlinear oscillator [60], as well as in contrast to the 3D cyclotron resonance in a quasistatic field [56] or any wave with a parallel refraction index $n_\parallel$ other than unity [61–64].

The difference from Refs. [56,61–65] is understood from the angle-action equations for the transverse oscillations, which now are governed by the Hamiltonian

$$\mathfrak{H} = (\sigma-1)\mathfrak{J}-\lambda\sqrt{\mathfrak{J}}\cos\psi. \tag{34}$$

Here $\mathfrak{J}=\rho_\perp^2/2\sigma$ is the action conjugate to the angle ψ, and $\lambda=a_0/\sqrt{\sigma}$; the effective time is ξ, $d\xi=(\omega u/\gamma)dt$. Since $\rho_z=(1-u^2+2\sigma\mathfrak{J})/2u$ [Eq. (27)], and $d\mathbf{r}/d\xi=\mathbf{p}/m\omega u$, the particle motion stability in the $(\psi,\mathfrak{J})$ space is equivalent to that in the 6D phase space $(\mathbf{r},\mathbf{p})$. [We define ψ through χ being the cylindrical phase in the momentum space rather than that in the coordinate space. In this case, Eq. (34) is exact; otherwise, a similar yet approximate equation follows, in agreement with Refs. [61,63] for $n_\parallel=1$, $n_\perp=0$, and $v/c\ll 1$.]

For given σ and $\lambda(\sigma)$, only one equilibrium, namely, a center, is allowed on the phase plane (ψ mod $2\pi,\mathfrak{J}$) and located at $\mathfrak{J}=[\lambda/2(1-\sigma)]^2$, $\psi=0$ $(\sigma>1)$ or $\psi=\pi$ $(\sigma<1)$. However, as $\sigma=\sigma(u_i)$, there exist up to three different phase

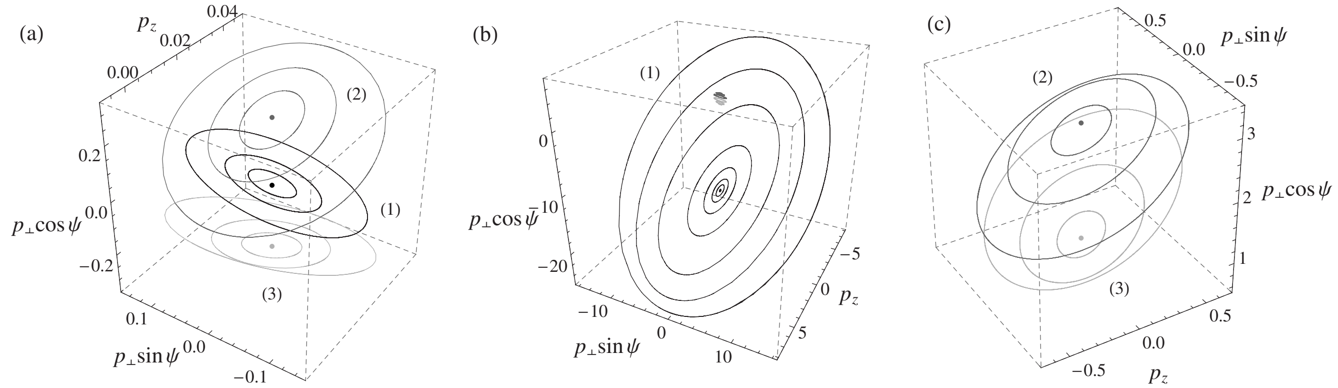

FIG. 3. Particle trajectories in the momentum space (mc units) with the same ρ_0 but different s: (a) $\rho_0=0.02$, $a_0=10^{-4}\sqrt{2}$, $\sigma_0=1.008$; (b) and (c) $\rho_0=0$, $\sigma_0=8.3$, $a_0=5\sqrt{2}$; (c) close-up of (b). Numbers in brackets denote the branches 1 (black), 2 (dark gray), and 3 (light gray), correspondingly. All three equilibria are centerlike and result in stable oscillations.

planes, and hence the equal number of center points ($i=1,2,3$). This situation is different from that, e.g., in Refs. [62,63], where several equilibria are bound to coexist on a *single* phase plot: as multiple centers are topologically impossible on a plane without a saddle [66], the intermediate-energy equilibrium is unstable and cannot be observed there. Since the topological constraint does not apply in our case, all the three equilibria are now stable and equally realized. This results in unusual particle dynamics, which we discuss in Sec. IV C.

C. Longitudinal mass

As the three energy states correspond to different effective masses, a guiding center behaves differently depending on which $m_{\rm eff}$ is selected; even the *sign* of the particle acceleration in response to perturbation forces can vary. To see this, rewrite the average motion equation (A8) as

$$m_\parallel \frac{d\bar{v}}{dt} = F_\parallel, \tag{35}$$

where $m_\parallel = \partial\bar{p}/\partial\bar{v}$, or

$$m_\parallel = \frac{\partial}{\partial\bar{v}}\left(\bar{\gamma} m_{\rm eff}\bar{v} - \frac{c^2}{\bar{\gamma}}\frac{\partial m_{\rm eff}}{\partial\bar{v}}\right) \tag{36}$$

is the effective longitudinal mass [67,68],

$$F_\parallel = -\frac{\partial}{\partial\bar{z}}\left[\bar{\gamma} m_{\rm eff}c^2 - \frac{c^2}{\bar{\gamma}}\left(\bar{v}\frac{\partial m_{\rm eff}}{\partial\bar{v}}\right) + e\tilde{\varphi}\right] - \frac{\partial\bar{p}}{\partial t} \tag{37}$$

is the perturbation force, and $\bar{p}=\bar{p}(\bar{z},\bar{v},t)$ [Eq. (10)].

A tedious yet straightforward derivation yields

$$m_\parallel = m\bar{\gamma}^3\frac{\Gamma_2^{3/2}}{\Gamma_3}, \quad \Gamma_n = 1 + s^2 + \frac{a_0^2}{2(1-\sigma)^n}, \tag{38}$$

Γ_2 coinciding with u'^2. In the absence of the laser field ($a_0=0$), Eq. (38) reads $m_\parallel = m_{\rm eff}\bar{\gamma}^3 > 0$ [Eq. (14)], as one would expect for a particle with $m_{\rm eff}$ independent of $\bar{v}$ [67]. Given a nonzero a_0, one as well has $m_{\parallel 1}>0$ because $\sigma<1$ at the first branch, as seen from Fig. 2. It is also seen from Fig. 2 that $u_3 > (\sigma_0/\sigma_c)u_c$, where $u_c \equiv u(\sigma_c)$,

$$u_c = h(1+s^2)^{1/3}[(1+s^2)^{1/3} + (a_0^2/2)^{1/3}]^{1/2};$$

thus $m_{\parallel 3}>0$, correspondingly. However, $u_2 < (\sigma_0/\sigma_c)u_c$, yielding $m_{\parallel 2}<0$ for any $\bar{v}$ and s (Fig. 4).

Since also the oscillation orbit is stable (Sec. IV B), a particle residing at the second branch will exhibit unusual behavior in response to perturbation forces $F_\parallel$, including gravitational and electrostatic potentials. Unlike a "normal" particle with a positive mass, a particle with $m_\parallel<0$ will accelerate adiabatically in the direction *opposite* to $F_\parallel$ (Fig. 5). Alternatively, should the unperturbed particle exhibit bounce oscillations in z (e.g., due to inhomogeneity of E or B), $F_\parallel$ will shift the equilibrium point in the direction determined by the sign of $F_\parallel/m_\parallel$, with stable bouncing to persist for either sign of $m_\parallel$ (Fig. 6). Merely a dissipative instability is possible for $m_\parallel<0$ (e.g., $m_\parallel\dot{\bar{v}} = -\eta\bar{v}$ yields $\bar{v}\propto e^{\eta t/|m_\parallel|}$); yet it develops on a time scale different from that of the oscillations and, for weak damping, remains insignificant until large t.

Transferring particles between the different mass branches also allows for a current drive effect distinguishable from the traditional methods, which rely on wave-induced diffusion to

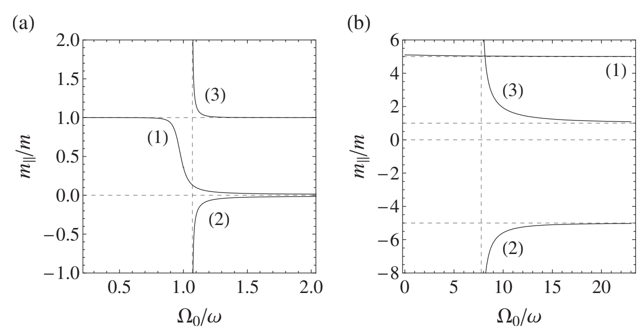

FIG. 4. Longitudinal mass $m_\parallel$ in units m vs $\sigma_0 \equiv \Omega_0/\omega$ for $s=0$, $\bar{v}=0$: (a) weakly relativistic case, $a_0=10^{-2}\sqrt{2}$ and (b) strongly relativistic case, $a_0=5\sqrt{2}$. The numbers in brackets denote branches corresponding to u_1, u_2, and u_3; $m_{\parallel 1}(\sigma_0=0)=\sqrt{1+a_0^2/2}$. The horizontal dashed lines mark zero and asymptotes at $\sigma_0\to\infty$: $m_{\parallel 1,2}\to\pm a_0/\sqrt{2}$, $m_{\parallel 3}\to 1$. The vertical dashed asymptote also marks the transition between the regimes with single and multiple branches; for conditions see the caption of Fig. 2.

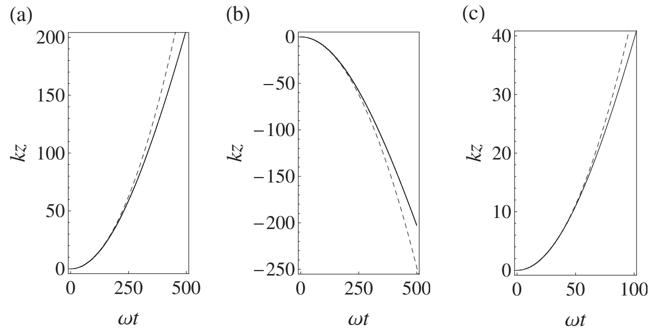

FIG. 5. $z(t)$ for a particle with initial $\bar{v}=0$, $s=0$ adiabatically accelerated along a magnetic field by a perturbation force $F_{\parallel}=10^{-2}mc\omega$; $\sigma_0=8.3$, $a_0=5\sqrt{2}$. The sign of acceleration varies depending on the initial energy state 1–3 corresponding to (a) $m_{\parallel 1}>0$, (b) $m_{\parallel 2}<0$, and (c) $m_{\parallel 3}>0$ (see Fig. 4). The solid line is the numerical data; dashed lines are analytic fits $z(t)=F_{\parallel}t^2/2m_{\parallel}$, where $m_{\parallel}$ is given by Eq. (38) with $\bar{v}=0$; z and t are measured in units k^{-1} and ω^{-1}, respectively.

higher kinetic energies [69]. The effect is explained as follows. Stationary fields conserve the particle quasienergy (11) for a given $m_{\parallel}$ branch and therefore do not permit acceleration along a closed loop. However, should $m_{\parallel}$ be changed nonadiabatically along the loop, the overall work performed can be nonzero; hence, even curl-free fields such as those due to electrostatic or ponderomotive potentials will be able to produce a continuous energy gain. Similar effects were previously discussed in Refs. [12,61,63,70–75]. With the effective-mass formalism, these effects can now be explained within a unified approach.

V. CONCLUSIONS

We showed that a classical particle oscillating in an arbitrary high-frequency or static field effectively exhibits a modified mass $m_{\rm eff}$ derived from the particle averaged Lagrangian [Eq. (8)]. We obtained relativistic ponderomotive and diamagnetic forces, as well as magnetic drifts, from the $m_{\rm eff}$ dependence on the guiding center location and velocity. The effective mass is not necessarily positive and can result in backward acceleration when an additional perturbation force is applied.

As an example, we explored the average motion of a laser-driven particle immersed in a dc magnetic field. Multiple energy states are realized in this case and yield up to three branches of $m_{\rm eff}$ and the effective longitudinal mass $m_{\parallel}$ for a given magnetic moment and parallel velocity (Fig. 4). We showed that both $m_{\parallel}>0$ and $m_{\parallel}<0$ are possible then, the latter regime too allowing for adiabatic dynamics. From other contexts, such negative masses are known to be capable of driving intriguing effects like absolute negative conductivity [76,77], negative mass instability [78–86], and related phenomena [87–89]. Yet the effects that may flow from the variable sign of $m_{\parallel}$ (or $m_{\rm eff}$) particularly for laser-driven particles in a magnetic field remain to be studied.

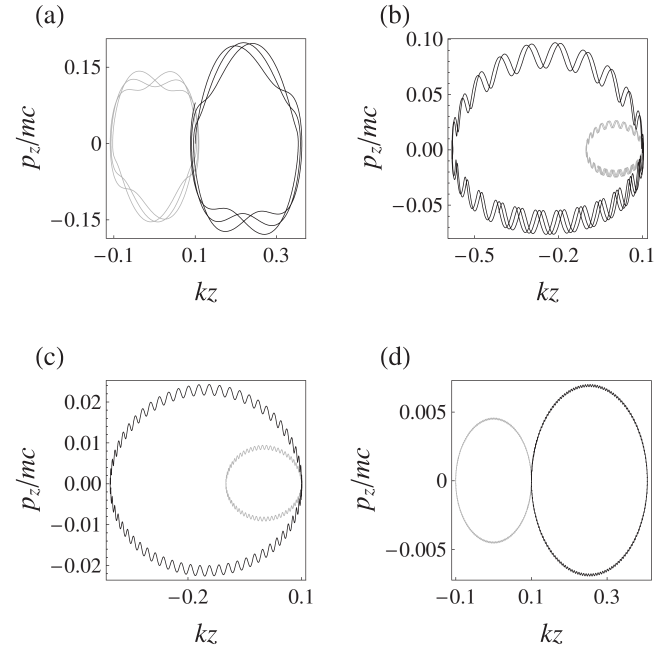

FIG. 6. Phase plots (z,p_z) showing bounce oscillations along a magnetic field due to inhomogeneity of E or B. An additional perturbation force $F_{\parallel}$ shifts the equilibrium point in the direction determined by the sign of $F_{\parallel}/m_{\parallel}$ (not just $F_{\parallel}$), with stable bouncing to persist for either sign of $m_{\parallel}$. (a) and (b) $E(z)$ has a minimum at $z=0$; B is uniform. Bounce oscillations are stable for the branches 1 (a) and 2 (b) yet unstable for branch 3 (particles seek high E; not shown); $F_{\parallel}=0$ (gray) and $F_{\parallel}=10^{-2}mc\omega$ (black). (c) and (d) E is uniform. (c) $B(z)$ has a maximum at $z=0$. Bounce oscillations are stable for branch 2 (shown) yet unstable for branches 1 and 3 (particles seek low B; not shown); $F_{\parallel}=0$ (gray) and $F_{\parallel}=10^{-3}mc\omega$ (black). (d) $B(z)$ has a minimum at $z=0$. Bounce oscillations are stable for branches 1 (not shown) and 3 (shown) yet unstable for branch 2 (particles seek high B; not shown); same $F_{\parallel}$ as in (c). All figures: particles initially have $kz=0.1$, $p_z=0$, $s=0$; z and p_z are measured in units k^{-1} and mc, respectively. At the extrema: $a_0\approx 5\sqrt{2}$ and $\sigma_0\approx 8.3$; $m_{\parallel 1,3}>0$, $m_{\parallel 2}<0$.

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APPENDIX A: GUIDING CENTER LAGRANGIAN

Consider a dynamical system, which exhibits slow translational motion in some guiding center variables $(\mathbf{Q},\mathbf{P})$ superimposed on fast oscillations in angle-action variables $(\boldsymbol{\theta},\mathbf{J})$. In the adiabatic regime, $\mathbf{J}$ is conserved, so the system action S can be put in the form

$$S=\mathbf{J}\cdot\Delta\boldsymbol{\theta}+\int\mathbf{P}\cdot d\mathbf{Q}-\int H\,dt, \tag{A1}$$

where $\Delta\boldsymbol{\theta}$ is the increment of $\boldsymbol{\theta}$ along a given trajectory, $H(\mathbf{Q},\mathbf{P};\mathbf{J})$ is the Hamiltonian, and t is the time. Suppose that we are only interested in guiding center trajectories, that is, those in the $(\mathbf{Q},\mathbf{P})$ space. In this case, we can neglect the first term in Eq. (A1), so as to come up with a reduced variational principle $\delta\mathcal{S}=0$, where

$$\mathcal{S}=\int(\mathbf{P}\cdot\dot{\mathbf{Q}}-H)\,dt \tag{A2}$$

is the new action to be varied with respect to $\mathbf{Q}$ and $\mathbf{P}$ only (cf., e.g., Ref. [[90], Sec. 44]).

Using $S=\int L\,dt$, where L is the Lagrangian, Eq. (A2) can be written as [31]

$$\mathcal{S}=\int(L-\mathbf{J}\cdot\dot{\boldsymbol{\theta}})\,dt. \tag{A3}$$

By definition, the integrand here must be expressed in terms of the guiding center variables only. Hence $\dot{\boldsymbol{\theta}}=\boldsymbol{\nu}(\mathbf{Q},\mathbf{P})$ (a parametric dependence on $\mathbf{J}$ is implied hereafter), so $\mathbf{J}\cdot\dot{\boldsymbol{\theta}}\,dt$ is *not* an exact differential, and the first term in Eq. (A3) is transformed as follows: By definition, L is a periodic function of $\boldsymbol{\theta}$, except that it may contain nonperiodic terms that are full-time derivatives. Since omitting the latter does not affect the motion equations, below we assume that $L=\langle L\rangle+L_{\sim}$, where the angle brackets stand for time averaging, $\langle L_{\sim}\rangle=0$, and $\langle L\rangle$ is a function of $(\mathbf{Q},\mathbf{P})$, or $(\mathbf{Q},\dot{\mathbf{Q}})$ only. On time scales of interest, that is, $\Delta t\gg\nu_i^{-1}$ and $\Delta t\gg|\nu_i-\nu_j|^{-1}$ (ν_i being any of the oscillation frequencies), the oscillatory term in Eq. (A3) vanishes; thus

$$\mathcal{S}=\int[\langle L\rangle-\mathbf{J}\cdot\boldsymbol{\nu}]\,dt. \tag{A4}$$

We can now introduce a guiding center Lagrangian $\mathcal{L}(\mathbf{Q},\dot{\mathbf{Q}})$ as $\mathcal{S}=\int\mathcal{L}\,dt$. Since the equality

$$\int\mathcal{L}\,dt=\int[\langle L\rangle-\mathbf{J}\cdot\boldsymbol{\nu}]\,dt \tag{A5}$$

must hold for any time interval, one has

$$\mathcal{L}=\langle L\rangle-\mathbf{J}\cdot\boldsymbol{\nu}, \tag{A6}$$

in agreement with Refs. [91,92]. (For a system exhibiting oscillations on multiple time scales, different $\mathcal{L}$'s can be introduced depending on how the time averaging is defined.) One can also show that Eq. (A6) conforms to the requirement of gauge invariance: Replacing L with $L+dG/dt$, where $G(\boldsymbol{\theta},\mathbf{Q},t)$ is an arbitrary function, will result in $\mathcal{L}\to\mathcal{L}+\Delta\mathcal{L}$, where

$$\Delta\mathcal{L}=\boldsymbol{\nu}\cdot\left\langle\frac{\partial G}{\partial\boldsymbol{\theta}}\right\rangle+\dot{\mathbf{Q}}\cdot\frac{\partial\bar{G}}{\partial\mathbf{Q}}+\frac{\partial\bar{G}}{\partial t}, \tag{A7}$$

and $\bar{G}(\mathbf{Q},t)=\langle G(\boldsymbol{\theta},\mathbf{Q},t)\rangle$. The first term is equal to zero due to G being periodic in $\boldsymbol{\theta}$. Thus one has $\Delta\mathcal{L}=d\bar{G}/dt$, i.e., $\Delta\mathcal{L}$ is, too, a full-time derivative, and therefore does not affect the average motion equations

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{Q}}}\right)=\frac{\partial\mathcal{L}}{\partial\mathbf{Q}}. \tag{A8}$$

For L being a periodic function of t (rather than, or in addition to, $\boldsymbol{\theta}$), the above procedure would map out the time variable, thus yielding an analog of the Maupertuis principle [[90], Sec. 44]. However, should t be kept (unlike $\boldsymbol{\theta}$) as the independent variable, the derivation of the guiding center Lagrangian is modified as follows. Consider the fast time $\tilde{t}$ and the slow time $\hat{t}$ separately. Then we obtain an extended system having the action

$$\hat{S}=\mathbf{J}\cdot\Delta\boldsymbol{\theta}-\tilde{H}\,\Delta\tilde{t}+\int\mathbf{P}\cdot d\mathbf{Q}-\int\hat{H}\,d\hat{t}, \tag{A9}$$

where the formally introduced momentum $-\tilde{H}$ conjugate to $\tilde{t}$ is to remain constant in the adiabatic regime. The super-Hamiltonian $\hat{H}$ must generate the same canonical equations as those of the original system; it must also provide that $d\tilde{t}/d\hat{t}=1$, as follows from the definition of $\tilde{t}$. These conditions are satisfied if one takes $\hat{H}=H+\tilde{H}$, so the super-Lagrangian

$$\hat{L}=\mathbf{P}\cdot\dot{\mathbf{Q}}+\mathbf{J}\cdot\dot{\boldsymbol{\theta}}-\tilde{H}\dot{\tilde{t}}-\hat{H} \tag{A10}$$

equals L. Then the guiding center Lagrangian reads

$$\hat{\mathcal{L}}=\langle L\rangle-\mathbf{J}\cdot\boldsymbol{\nu}+\tilde{H}, \tag{A11}$$

which is equivalent to Eq. (A6), since constant $\tilde{H}$ can be omitted.

Results similar to those in this appendix were obtained earlier for particle motion in a dc magnetic field [3,35,41], oscillations in nonrelativistic high-frequency waves [8,93], and laser-driven relativistic electron dynamics in vacuum [16,24,25]. In the main text, we make use of the general form of the theorem (A6), which contains the earlier results as particular cases. This generality allows us to formulate a fundamental concept of the effective mass for an oscillating particle without making preliminary assumptions on the nature of the oscillations.

APPENDIX B: PONDEROMOTIVE FORCES

1. NONRELATIVISTIC WAVE FIELDS

Let us apply the effective mass formalism to derive ponderomotive forces, starting with those in the nonrelativistic regime [7,10–12,94]. Consider a particle oscillating in a high-frequency wave $\mathbf{E}=-\boldsymbol{\nabla}\varphi$, where

$$\varphi=\varphi_0(\mathbf{r},t)\cos(\omega t-\mathbf{k}\cdot\mathbf{r}). \tag{B1}$$

We will assume that $k\bar{v}\ll\omega$; we will also assume that the envelope $\varphi_0(\mathbf{r},t)$ varies little on the time scale ω^{-1} and has a spatial scale ℓ large compared to the amplitude of the particle oscillations ($\ell\gg eE/m\omega^2$) and the guiding center displacement on the oscillation period ($\ell\gg\bar{v}/\omega$) [7]. Then $\mathcal{L}'=-mc^2+\langle L_{\mathrm{osc}}\rangle$, where

$$L_{\rm osc} \approx \frac{m\dot{\mathbf{r}}_{\rm osc}^2}{2} - e\mathbf{r}_{\rm osc}\cdot\mathbf{E}(\overline{\mathbf{r}},t) \qquad (B2)$$

is obtained using $\varphi(\mathbf{r}) \approx \varphi(\overline{\mathbf{r}}) - \mathbf{r}_{\rm osc}\cdot\mathbf{E}$, with the quiver displacement $\mathbf{r}_{\rm osc} = -e\mathbf{E}/m\omega'^2$, and the Doppler-shifted frequency $\omega' = \omega - \mathbf{k}\cdot\overline{\mathbf{v}}$. Then $\langle L_{\rm osc}\rangle = -\Phi$, where

$$\Phi = \frac{e^2E_0^2}{4m(\omega - \mathbf{k}\cdot\overline{\mathbf{v}})^2} \qquad (B3)$$

is known as the ponderomotive potential [7,10–12,94], E_0 being the field amplitude; thus

$$m_{\rm eff} = m + \Phi/c^2. \qquad (B4)$$

Omitting an insignificant constant, the guiding-center Lagrangian reads $\mathcal{L} = \frac{1}{2}m\overline{v}^2 - \Phi$. Hence the Hamiltonian takes the well-known form

$$\mathcal{H} = \frac{1}{2m}\overline{p}^2 + \Phi, \qquad (B5)$$

Φ playing a role of an effective potential, as expected.

Suppose now that, under the same conditions, an additional dc magnetic field $\mathbf{B}$ is imposed. Assuming that $\mathbf{B}$ is smooth, one has $\mathcal{L}' = -mc^2 - \mu B + \langle L_{\rm osc}\rangle$, where

$$\mu = \frac{m}{2B}(\mathbf{v}_\perp - \mathbf{v}_{\rm osc})^2 \qquad (B6)$$

is the new adiabatic invariant proportional to the action of the particle Larmor rotation at frequency $\Omega_0 = eB/mc$, $\mathbf{v}_{\rm osc}$ is the induced oscillatory velocity linear in E [11,93,95,96], and $\langle L_{\rm osc}\rangle$ is quadratic in E. Suppose $\mathbf{B} \approx \hat{\mathbf{z}}B(z)$ [Eq. (21)] and take

$$\mathbf{E} = \mathrm{Re}(E_0^{(+)}\hat{\tau}^{(+)} + E_0^{(-)}\hat{\tau}^{(-)} + E_0^{(\parallel)}\hat{\tau}^{(\parallel)})e^{-i\omega t + ikz}, \qquad (B7)$$

where $E_0^{(j)}$ are smooth envelopes, and $\hat{\tau}^{(j)}$ are unit vectors denoting circular polarization in the plane transverse to the dc magnetic field and linear polarization parallel to $\mathbf{B}$:

$$\hat{\tau}^{(\pm)} = (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})/\sqrt{2}, \quad \hat{\tau}_0^{(\parallel)} = \hat{\mathbf{z}}. \qquad (B8)$$

In this case, $\langle L_{\rm osc}\rangle = -\Phi_B$ [93], where

$$\Phi_B = \frac{e^2}{4m\omega'^2}\left\{\frac{|E_0^{(+)}|^2}{1+\Omega_0/\omega'} + \frac{|E_0^{(-)}|^2}{1-\Omega_0/\omega'} + |E_0^{(\parallel)}|^2\right\} \qquad (B9)$$

matches the known ponderomotive potential in a dc magnetic field [10,11]. Thus $m_{\rm eff} = m + (\mu B + \Phi_B)/c^2$, and

$$\mathcal{H} = \frac{1}{2m}\overline{p}^2 + \mu B + \Phi_B, \qquad (B10)$$

in agreement with the earlier results [93].

2. RELATIVISTIC LASER WAVE IN VACUUM

Now consider relativistic electron motion in a vacuum laser field $\mathbf{A} = \mathbf{A}_0(\mathbf{r},t)\cos\xi$ of arbitrary polarization, assuming that the vector-potential envelope $\mathbf{A}_0(\mathbf{r},t)$ has a scale that is large compared to the wavelength, $\xi = \omega(t - \mathbf{n}\cdot\mathbf{r}/c)$ is the phase, and $\hat{\mathbf{n}} = \mathbf{k}/k$ is a unit vector, say, in the $\hat{\mathbf{z}}$ direction. Using Eqs. (23)–(26) and conservation of the transverse canonical momentum $\mathbf{p}'_\perp = -(e/c)\mathbf{A}'$, one has

$$\mathcal{L}' = -mc^2\frac{1 + \langle a'^2\rangle_\xi}{\langle\gamma'\rangle_\xi}, \qquad (B11)$$

where $\mathbf{a}' = e\mathbf{A}'/mc^2$, and $\langle a'^2\rangle_\xi = \langle a^2\rangle_\xi$ is an invariant. Using Eq. (29), one also has, without solving the motion equations, that

$$\langle\gamma'\rangle_\xi = \sqrt{1 + \langle p'^2_\perp\rangle_\xi/(mc)^2} = \sqrt{1 + \langle a^2\rangle_\xi}, \qquad (B12)$$

and $\mathcal{L}' = -mc^2\langle\gamma'\rangle_\xi$. Thus $m_{\rm eff}$ equals [16–29]

$$m_{\rm eff} = m\sqrt{1 + \langle a^2\rangle_\xi}, \qquad (B13)$$

which is a Lorentz invariant independent of $\overline{\mathbf{v}}$. Correspondingly, the guiding center momentum reads $\overline{\mathbf{p}} = \overline{\gamma}m_{\rm eff}\overline{\mathbf{v}}$, and the well-known Hamiltonian is given by

$$\mathcal{H} = \sqrt{m_{\rm eff}^2c^4 + \overline{p}^2c^2}, \qquad (B14)$$

the ponderomotive force resulting from the $m_{\rm eff}$ dependence on $\overline{\mathbf{r}}$ and, possibly, slow dependence on t.

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