Simplified model of nonlinear Landau damping

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The nonlinear interaction of a plasma wave with resonant electrons results in a plateau in the electron distribution function close to the phase velocity of the plasma wave. As a result, Landau damping of the plasma wave vanishes and the resonant frequency of the plasma wave downshifts. However, this simple picture is invalid when the external driving force changes the plasma wave fast enough so that the plateau cannot be fully developed. A new model to describe amplification of the plasma wave including the saturation of Landau damping and the nonlinear frequency shift is proposed. The proposed model takes into account the change of the plasma wave amplitude and describes saturation of the Landau damping rate in terms of a single fluid equation, which simplifies the description of the inherently kinetic nature of Landau damping. A proposed fluid model, incorporating these simplifications, is verified numerically using a kinetic Vlasov code. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3160604]

I. INTRODUCTION

The problem of nonlinear Landau damping is a classical problem in plasma physics. Landau damping was first predicted analytically by Landau[1] and later observed many times in various physical systems both experimentally[2] and numerically.[3] Classical Landau damping occurs through the interaction of a plasma wave with warm plasma. Electrons, moving with velocities close to the phase velocity of the wave, strongly interact with the wave, since they travel for a long time seeing the same electric field of the wave. As a result, these electrons get accelerated or decelerated, depending on their original phase in the plasma wave. The average velocity of nearly resonant electrons approaches the phase velocity of the wave. If the original number of near-resonant electrons moving slower than the phase velocity is larger than the number of electrons moving faster than the phase velocity, the total energy of the resonant electrons increases. This extra energy comes from the energy of the plasma wave.

For about 40 years, nonlinear studies of Landau damping focused on different aspects of the phenomenon such as time-evolution of the plasma wave and nonlinear reduction of the Landau damping rate,[4–7] the nonlinear frequency shift of the plasma wave due to trapped particles,[8–10] long-time asymptotic evolution of nonlinear Landau damping,[11] and plasma echo.[12] Recently, the importance of nonlinear Landau damping has been explored with regard to backward Raman scattering[13] and backward Brillouin scattering[14] in laser-plasma interactions.

The applications of high-power laser systems include various parametric processes, which employ plasma waves. The use of plasma is partially motivated by the fact that plasma cannot be damaged by heating. Moreover, plasma waves can mediate parametric interactions with high efficiency. Raman scattering is one example of this parametric interaction. In this process, two electromagnetic (laser) waves propagate in plasma. If the difference between their frequencies is on the order of the plasma frequency $\omega_p = (4\pi e^2 n/m)^{1/2}$, a plasma wave can be resonantly excited by the ponderomotive potential of the electromagnetic waves. Since the amplitude of the ponderomotive force is proportional to its wavenumber, counterpropagating electromagnetic waves generate the plasma wave with the highest efficiency. The wavelength of the generated plasma wave is almost half the laser wavelength in this setup. The short wavelength of the plasma wave results in its small phase velocity, which can lead to a large number of the electrons, resonantly interacting with the wave. Therefore, a plasma wave with small wavelength is likely to experience noticeable Landau damping, which might affect the wave coupling and reduce the growth rate of the instability. On the other hand, the interaction of the plasma wave with near-resonant electrons can nonlinearly reduce the Landau damping rate and enhance the parametric coupling. Whether Raman scattering is a desirable or an undesirable effect, the accurate calculation of the influence of nonlinear Landau damping on the output signal will be required. Moreover, it would be advantageous if the main effects could be captured in a simple model.

In this paper we study nonlinear Landau damping of a plasma wave, which is nearly resonantly amplified by the external force. For example, the plasma wave can be driven parametrically. This study is relevant to a number of important applications. One application is the laser reflectivity in inertial confinement fusion experiments (see, for example, Ref. 15). Another is backward Raman amplifier in plasma,[16] where the mitigation Landau damping effects are particularly critical in extreme compression regimes[17] or the compression of extremely short wavelengths.[18] Yet another application is Raman current drive,[19] where, like for other methods of current drive, the current drive efficiency can be large if the plasma wave accelerates high energy electrons.[20] The purpose of this paper is to develop and to verify a simple model of nonlinear Landau damping, which can quantitatively describe the class of problems mentioned above. Also, the same approach can be applied for studying Landau damping of the ion-acoustic wave in the case of parametric wave coupling.

We consider the evolution of the plasma wave to be fast enough that issues of the time-asymptotic behavior of the plasma wave are already resolved.[11] Fast enough dynamics also implies that collisions do not affect the system unlike the opposite case scenario,[5,13] where steady-state solutions can be found. In our studies, we also assume that the amplitude of the plasma wave grows in time. The classical results for saturation of Landau damping in the presence of strong plasma wave of a constant amplitude[6,7] are thus not applicable. Moreover, the plasma wave starts to grow in the linear regime from a small amplitude, $\omega_B < \nu_0$, where $\omega_B = (e|E|k/m)^{1/2}$ is the bounce frequency and ν_0 is the linear Landau damping rate. By the time the system enters the parameters region of Mazitov and O'Neil,[6,7] $\omega_B \gg \nu_0$, the Landau damping rate is most likely to be significantly reduced, which does not allow one to use the approach described in those papers.

Landau damping, in principle, should be described in at least three-dimensional (3D) time-coordinate-velocity space. Analytical and even numerical studies of a fully kinetic problem can be challenging if many plasma wavelengths are taken into account. The development of a simplified model, which describes the interaction of the wave with near-resonant particles, will allow reducing the complexity of the system. In developing a simple model of nonlinear Landau damping, we focus on the evolution of the main plasma wave parameters, such as its amplitude and phase. In that way, we reduce the kinetic description of the nonlinear Landau damping to a set of fluid equations. In parametric interactions, the wavenumber of the plasma wave is typically well-defined by the resonance condition for the interacting waves. This simplifies the description, since we need to describe Landau damping of a single plasma wave mode only, unlike the classic quasilinear model[21] or fluid models for Landau damping of an arbitrary plasma wave.[22] The main finding here is the simplified set of equations, which is shown empirically to describe the essential physics.

The paper is organized as follows. In Sec. II we develop the simplified model for Landau damping and derive fluid equations for the Landau damping rate and the nonlinear frequency shift of the plasma wave. In Sec. III we verify the proposed model numerically and demonstrate that it has about 90% accuracy within the applicability limits. In Sec. IV we summarize the main results of the paper.

II. ANALYTICAL MODEL

In this section we develop a model describing nonlinear Landau damping. The derivations are not fully rigorous, but rather to some extent heuristic. They serve as an analytical motivation for the proposed model. After developing the model, we verify it computationally. We show that this simple heuristic model describes the kinetic solution well in cases of interest.

We study a plasma wave, which is amplified by a near-resonant external electric field. We assume that the plasma wave grows slowly enough compared to the characteristic time for the evolution of the electron distribution function, which is on the order of the inverse bounce frequency, $\partial_t \omega_B / \omega_B^2 \ll 1$. This regime was considered earlier and it was studied using a quasistatic adiabatic approach $\partial_t \omega_B / \omega_B^2 \to 0$.[9] The nonlinear quasistatic solution results in zero Landau damping rate and some nonlinear frequency shift of the plasma wave caused by the change of the electron distribution function. However, this solution does not describe the change of the plasma wave, since it is assumed to be a given function of time. While developing the model, which can self-consistently describe the evolution of the plasma wave driven by an external force, we will find the first nonvanishing term (on the order of $\partial_t \omega_B / \omega_B^2$) of Landau damping.

The change of the nonlinear frequency shift was obtained in earlier papers.[9,23] In those papers, the time-asymptotic nonlinear frequency shift was derived from the unsimplified Vlasov equation. That solution assumed fully saturated Landau damping rate. If the plasma wave is driven by external force, it evolves in the linear regime first and reaches the nonlinear stage when its amplitude becomes large enough. Therefore, the nonlinear quasistatic solution for the fully saturated Landau damping[9,23] is not applicable here. We will find the first nonvanishing term of the Landau damping rate on the order of $\partial_t \omega_B / \omega_B^2$. At the same time, we will use the time-asymptotic expression for the nonlinear frequency shift of the plasma wave, since it is the first nonvanishing in the case of $\partial_t \omega_B / \omega_B^2 \to 0$. We consider the plasma temperature to be large and the plasma wave amplitude to be small so that the fluid corrections to the nonlinear frequency shift can be neglected.[24]

We apply the quasilinear approach to describe the saturation of the Landau damping rate. To be precise, the quasilinear theory describes the case of a broadband plasma wave rather than a single wave. It can be applied if resonance domain in the velocity space caused by the broadband plasma wave is larger than the resonance domain of a single wave, $\delta v = v_{\rm ph} \Delta k / k \gg \omega_B / k$.[21] These parameters are the same for a single plasma wave. As a consequence, the amplitude of the plasma wave experiences oscillations on the bounce frequency time scale.[7] However, these oscillations can be not of interest in particular applications in which only the average amplitude of the plasma wave plays role. At the same time, the quasilinear theory correctly describes the energy transfer between the plasma waves and the average distribution function. In Appendix we show that the quasilinear theory can be a reasonable model, which correctly describes the interaction of a single plasma wave with most of the resonant particles. Thus, it stands to reason that the time-asymptotic quasilinear limit $(\partial_t \omega_B / \omega_B^2 \to 0)$ might be a good model even for a single wave in many situations of interest. This approach is similar to the approach considered in Ref. 25.

In quasilinear theory, the plasma wave is considered to be small enough so its change can be described in the wave envelope approximation,

$$E(z,t) = E(t) e^{i\Omega_0 t - ikz}, \quad (1)$$

$$\partial_t E - i\delta\Omega E + \nu E = \frac{E_0(t)}{i\partial_\Omega D}, \tag{2}$$

where $E(z,t)$ is the complex amplitude of the plasma wave and $E(t)$ is its envelope; $E_0(t)$ is the amplitude of the external electric field envelope and $\delta\Omega$ is the nonlinear frequency shift of the plasma wave in respect to the linear frequency Ω_0; ν is the Landau damping rate; k is the wavenumber of the plasma wave; $D(\Omega,k)$ is the plasma dielectric function defined by Eq. (3); e, m are electron charge and mass, respectively; t is time, z is longitudinal coordinate along the wavevector of the plasma wave.

The Landau damping rate and the frequency shift of the plasma wave correspond to the background distribution function $F_0(v,t)$, which satisfies the quasilinear equation

$$D(\Omega,k) = 1 + \frac{\omega_p^2}{k}\int \frac{\partial_v F_0}{\Omega - kv} dv = 0, \tag{3}$$

$$\partial_t F_0 = \mathrm{Re}\left\{ \frac{1}{2i} \left| \frac{eE}{m} \right|^2 \partial_v \frac{\partial_v F_0}{\Omega - kv} \right\}. \tag{4}$$

Equation (3) describes the change in both the Landau damping rate and the nonlinear frequency shift of the plasma wave. However, we apply the quasilinear theory to describe only the saturation of the Landau damping rate. Alternatively, we use the nonlinear quasistatic solution for the nonlinear frequency shift,[9,23] since it is lowest order solution at $\partial_t \omega_B / \omega_B^2 \to 0$.

First, we consider that the domain of the near-resonant particles has sharp boundaries in velocity space. The width of the resonant region is proportional to the bounce frequency,

$$\partial_t F_0(v,t) = 0, \quad |v - v_{\mathrm{ph}}| > \alpha\omega_B/k, \tag{5}$$

where α is a numerical factor, which we deduce later. The Taylor expansion of the distribution function is valid within the resonant domain, since the width of the resonant domain is much smaller than the characteristic scale of the distribution function, $\omega_B/k \ll F_0/\partial_V F_0$. The first two terms of the Taylor expansion are required to describe the effect of Landau damping $F_0(v,t) = F_0(v_{\mathrm{ph}},t) + (v - v_{\mathrm{ph}})\partial_v F_0(v_{\mathrm{ph}},t)$. Using the conservation laws (A10) and (A11), we find the equation describing the saturation of Landau damping,

$$\partial_t[(\nu_0 - \nu)\omega_B^3(t)] = \nu \frac{3\pi}{4\alpha^3}\omega_B^4(t), \tag{6}$$

where ν_0 is the linear Landau damping rate.

Now we deduce the empirical numerical factor α by comparing solutions of the exact nonlinear[26] and the simplified model Eq. (6). We consider the adiabatic amplification of the plasma wave until it reaches some amplitude E. The time-asymptotic stage results in saturated Landau damping and a steady-state plasma wave. The near-resonant particles are accelerated during the wave-particle interaction, which results in a plasma current (plasmon momentum transfers into electron momentum).

To obtain the numerical factor α, we now compare currents predicted by different theories. The change of the electron momentum predicted by the nonlinear kinetic solution[26] can be written as

$$\Delta P_{\mathrm{kin}} = \frac{64}{9\pi}\partial_v F(v_{\mathrm{ph}},0)\frac{\omega_B^3 m}{k^3}. \tag{7}$$

On the other hand, the simplified fluid theory deduced heuristically here predicts the change of the electron momentum as

$$\Delta P_{\mathrm{fluid}} = \frac{2\alpha^3}{3}\partial_v F(v_{\mathrm{ph}},0)\frac{\omega_B^3 m}{k^3}. \tag{8}$$

Comparing these two results, we conclude that $\alpha^3 = 32/3\pi$ in order to correctly describe the final stage of Landau damping. Then the self-consistent set of equations describing nonlinear Landau damping of a driven plasma wave:

$$\partial_t E - i\delta\Omega E + \nu E = \frac{E_0(t)}{i\partial_\Omega D}, \quad \omega_B = \left| \frac{eEk}{m} \right|^{1/2}, \tag{9}$$

$$\partial_t[(\nu_0 - \nu)\omega_B^3(t)] = \frac{9\pi^2}{128}\nu\omega_B^4(t), \tag{10}$$

$$\delta\Omega(t) = 1.09\omega_B(t)\frac{\omega_p^2}{k^3\partial_\Omega D}\partial_{vv}^2 F_0(v_{\mathrm{ph}},0). \tag{11}$$

This model describes both the nonlinear Landau damping rate and the nonlinear frequency shift of the externally driven plasma wave. These effects are described in terms of fluid equations rather than kinetic equations. Equation (10) describes the saturation of the Landau damping rate. This phenomenon is described for an arbitrary ratio of the bounce frequency and the Landau damping rate. Thus, Landau damping can be described appropriately both in the linear and the nonlinear regimes. Note from Eq. (10) that, for a constant amplitude of the plasma wave, the Landau damping rate decays exponentially. The timescale of this decay is on the order of the inverse bounce frequency, which is consistent with more rigorous derivations (e.g. Ref. 7).

The model (9)–(11) describes Landau damping for an externally driven plasma wave. It takes into account that the resonant domain in velocity space expands, while the amplitude of the plasma wave grows. This results in reduced saturation of Landau damping, since new particles become resonant with the growing plasma wave. To illustrate this, we consider a rapidly growing plasma wave. We consider this growth to be exponential-like. A time-asymptotic analysis of Eq. (10) for a given time dependence of the plasma wave amplitude results in the approximate solution for the Landau damping rate,[27]

$$\nu = \nu_0 \frac{128}{3\pi^2}\frac{1}{\int_0^t \omega_B(t)dt}. \tag{12}$$

Now the Landau damping rate decreases much slower than for a constant-amplitude plasma wave. The timescale of this saturation is the same as the timescale of the amplitude growth, $\partial_t \log \nu = \partial_t \log \omega_B$, rather than the inverse bounce frequency. Such a slow saturation of Landau damping may be important.

To establish the regime of validation of the model, recall that the simplified fluid model of nonlinear Landau damping (9)–(11) was derived using a number of assumptions. First, the oscillatory part of the distribution function should be small compared to the background distribution function, which limits the maximum allowed amplitude of the plasma wave,

$$\omega_B \ll \frac{k^2 v_T^2}{\Omega_0}. \tag{13}$$

The same condition states that the initial background distribution function $F_0(v,t=0)$ does not change a lot within the resonant domain, making valid its Taylor expansion within the resonant domain.

We also assumed that the amplitude of the plasma wave does not change rapidly compared to the bounce frequency timescale,

$$\partial_t \omega_B \ll \omega_B^2. \tag{14}$$

This condition allows one to use expression (11) for the nonlinear frequency shift of an adiabatically driven plasma wave. This condition also guaranties that the width of the resonant domain in velocity space is proportional to the bounce frequency rather than the bandwidth of the plasma wave envelope.

We assumed the amplification of the plasma wave,

$$\partial_t \omega_B > 0. \tag{15}$$

This condition states that the near-resonant electrons remain within the resonant domain, while the amplitude of the plasma wave changes in time. Otherwise, some energy of the damped plasma wave would be associated with the electrons outside of the resonant domain, and the total particle energy would not be proportional to the slope of the distribution function. In other words, the increase in the plasma wave amplitude results in the trapping of the electrons, which did not interact with the wave at early time. Therefore, extra cold electrons are trapped, which changes the slope of the distribution function in the resonance domain. This effect is absent for the reducing amplitude of the plasma wave. In that case the saturation of Landau damping is similar to the case of a constant plasma wave, $\partial_t \nu \propto \nu\omega_B$.

The condition for the electrons remaining trapped also implies that the change of the resonant velocity $v_{\mathrm{ph}}=(\Omega_0+\delta\Omega)/k$ does not result in the electron untrapping, $\delta\Omega < \alpha\omega_B$. This condition is satisfied automatically in Maxwellian plasma.

The simplified model describes the average evolution of the Landau damping rate and the nonlinear frequency shift. It does not describe oscillations of the wave envelope on the bounce frequency timescale. The plasma wave should be driven long enough in order to observe this averaging out,

$$\omega_B t \gg 1. \tag{16}$$

Finally, the plasma temperature should not be very high, so that the dispersion relation (3) allows a solution for the linear plasma wave mode,[28]

$$\frac{k v_T}{\omega_p} < 0.67. \tag{17}$$

III. NUMERICAL VERIFICATION OF SIMPLIFIED FLUID MODEL

In this section we compare the results of a numerical simulation of the Vlasov–Poisson set of Eqs. (A1) and (A2) with a solution of the simplified fluid models (9)–(11).

We consider a homogeneous plasma wave, which is driven with a given external force. We consider the driving force to be monochromatic, i.e., its amplitude and frequency are constant in time: $E_0(t)=E_0 \exp^{i(\Omega_S-\Omega_0)t}$, E_0,Ω_S=const. The plasma wave grows if the frequency of the source is close to the resonant frequency of the plasma wave, $\Omega_S \approx \mathrm{Re}\,\Omega_0$. In the beginning of amplification, the plasma wave grows in the linear regime of Landau damping. The amplification then enters the nonlinear stage of Landau damping, in which the Landau damping rate is significantly reduced. However, the growth of the plasma wave can still be saturated due to dephasing between the plasma wave and the driving force, since the nonlinear frequency shift increases with the plasma wave amplitude. The frequency of the driving force was chosen to be smaller than the linear frequency of the plasma wave in order to partially compensate the nonlinear frequency shift. As a result, the dephasing between the plasma wave and the external force remained small and the plasma wave amplitude was monotonically growing. Such a compensation mechanism of the dephasing is efficient at relatively small amplification time as seen below in Fig. 1. Further amplification of the plasma wave would eventually result in dephasing. The frequency chirp of the driving force would be required in order to compensate the nonlinear frequency shift of the plasma wave at longer amplification time.

The plasma wave can be described in terms of the amplitude and phase envelopes,

$$E(z,t) = \mathrm{Re}\, E(t) e^{i\phi(t)} e^{i\Omega_0 t - ikz} = \mathrm{Re}\big(E(t) e^{i\int \delta\Omega(t)dt} e^{i\Omega_0 t - ikz}\big). \tag{18}$$

This representation of the plasma wave is not unique, since some arbitrary fraction of the wave phase can be associated with the complex wave amplitude $E(t)$. This representation becomes unique if both E and ϕ are real functions.

The electric field amplitude oscillates at any given coordinate z_0,

$$E(z_0,t) = E(t)\mathrm{Re}(e^{i\Omega_0 t + i\phi(t) + i\phi_0}). \tag{19}$$

We can find the amplitude and the phase of the signal in its maxima. Then we can interpolate these data and restore the time dependence of the amplitude and the phase of the plasma wave if the envelopes change slowly within the plasma wave period. The frequency of the electric field is defined as the time derivative of its phase. To compare solutions of two different models, we define their amplitudes and phases consistently as described above.

The solution of the Vlasov–Poisson set of Eqs. (A1) and (A2) results in a time-dependent amplitude of the electric field. The frequencies of both the plasma wave mode and the external force affect the frequency of the generated electric field. In order to determine the frequency of the plasma wave, we assume that the plasma wave is the only mode, excited by the driving force. This assumption is in agreement with the detailed numerical studies.[29] Then the plasma wave frequency and the damping rate can be deduced from Eq. (9). If the solution of this equation is $E(t)e^{i\phi(t)}$, then we have from Eq. (9)

$$\nu(t) = \mathrm{Re}\left(\frac{E_0(t)}{iE(t)e^{i\phi}\partial_\Omega D}\right) - \frac{d_tE(t)}{E(t)}, \tag{20}$$

$$\delta\Omega(t) = d_t\phi + \mathrm{Re}\left(\frac{E_0(t)}{E(t)e^{i\phi}\partial_\Omega D}\right). \tag{21}$$

This technique allows one to find full time dependence of the plasma wave parameters such as its frequency and the damping rate and compare the semianalytical and Vlasov code results.

A numerical study of nonlinear saturation of Landau damping was performed with a standard Vlasov code, which uses a time split-scheme for solving Vlasov and Poisson equations. The common normalization, in which $e=m=1$, was used. The evolution of the plasma wave with a single wavelength $|k|=1$ was simulated. We performed runs exciting the plasma wave either with a single traveling wave or with a standing wave (two counter propagating waves). There was no significant difference between these two cases for small enough amplitudes of the plasma wave. The typical dimensionless plasma parameters were: $v_T=0.25-0.4$, $\omega_p=1$. The driving force E_0 was typically constant in time and its amplitude in different runs varied from 10^{-5} to 10^{-3}. The frequency of the driving force was constant and downshifted with respect to the linear frequency of the plasma wave, which avoids early dephasing between the plasma wave and the driving force. Typical numerical parameters of the simulations are time step $dt=0.1$, maximum calculation time $t_{\max}=1500$, maximum calculated velocity domain $|v|<2.5$, number of grid points in velocity space $N_V=1024$, and number of grid points in space $N_x=32$.

Typical results of the Vlasov simulations and their comparison with the solution of the analytical models (9)–(11) are presented in Fig. 1. Note that the time dependence of the wave amplitude and the damping rate of the plasma wave can be well approximated by the simplified models (9)–(11). The electric field, calculated using a Vlasov code, exhibits slow oscillations. The timescale of these oscillations is on the order of the inverse bounce frequency. Therefore, these oscillations can be interpreted as an influence of an individual particle motion in the potential of the electric field. Since our simplified approach does not take into account individual particle motion, it misses this feature. However, our simplified model correctly describes the average change of the electric field. Comparing the two solutions, we find that the nonlinear frequency shift of the plasma wave can be described by Eq. (11) (the second order corrections on the

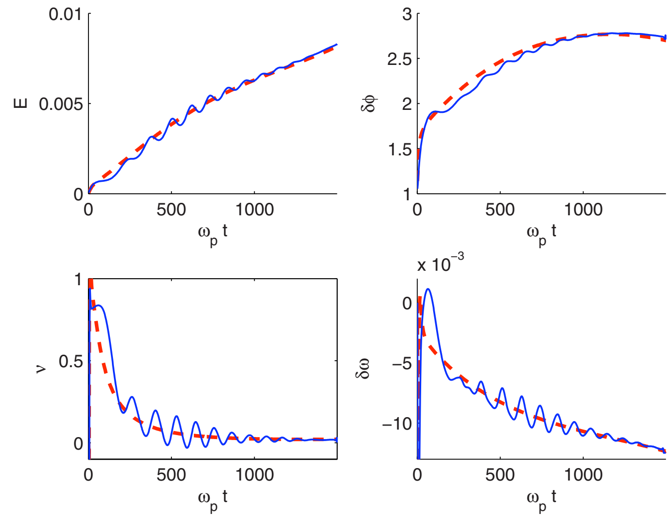

FIG. 1. (Color online) The results of numerical simulations using a Vlasov code (solid blue lines) and their approximations with the simplified model of nonlinear Landau damping (dashed red lines). Parameters of the simulation: $v_T=0.35$, $E_0=2\times10^{-4}$, $\Omega_S=1.1961$, and ($\Omega_0=1.2112$). The upper left plot shows the time dependence of the plasma wave amplitude. The upper right plot shows the phase mismatch between the plasma wave and the driving source. The lower plots show the Landau damping rate (lower left plot) and the nonlinear frequency shift (lower right plot) of the plasma wave derived from Eqs. (20) and (21).

plasma wave amplitude was used for a better fit[23]) with a numerical factor slightly different from the theoretical value of 1.09. In a numerical run presented in Fig. 1, this factor was 1.1; typically the difference varies within 10% range. This difference is probably caused in the initial stage of the amplification, when the plasma wave is small and does not grow adiabatically. Also, this difference can be partially caused by small corrections to the dispersion relation in the case of a driven plasma wave.[30] In the analysis presented in Figs. 1 and 2, we adjusted the theoretically predicted factor of 1.09 in Eq. (11) to fit the time-asymptotic dependence of the nonlinear frequency shift in the Vlasov simulations.

The accuracy of the simplified model was checked for different plasma temperatures and amplitudes of the driving force. The results are presented in Fig. 2. We introduce a norm of the function in order check if solution of the simplified model E_{fluid} is close to the solution of the Vlasov simulation E_{Vlasov},

$$\|E_{\mathrm{fluid}} - E_{\mathrm{Vlasov}}\| = \left[\frac{\int |E_{\mathrm{fluid}} - E_{\mathrm{Vlasov}}|^2 dt}{\int |E_{\mathrm{Vlasov}}|^2 dt}\right]^{1/2}. \tag{22}$$

Both the amplitudes and the phases of two solutions should be close to each other so that the norm is small. Using this norm, the numerical results presented in Fig. 2 show that the simplified fluid models (9)–(11) matches the Vlasov simulations to an accuracy of about 90%. The norm has the minima versus both the plasma temperature and the amplitude of the driving force, which defines the parameters region for the simplified fluid model. The boundaries of this parameter region are described in Sec. II and are consistent with the numerical results. The maximum amplitude of the driving force and the minimum thermal velocity are determined by condition (13). The growth of the norm at high plasma tem-

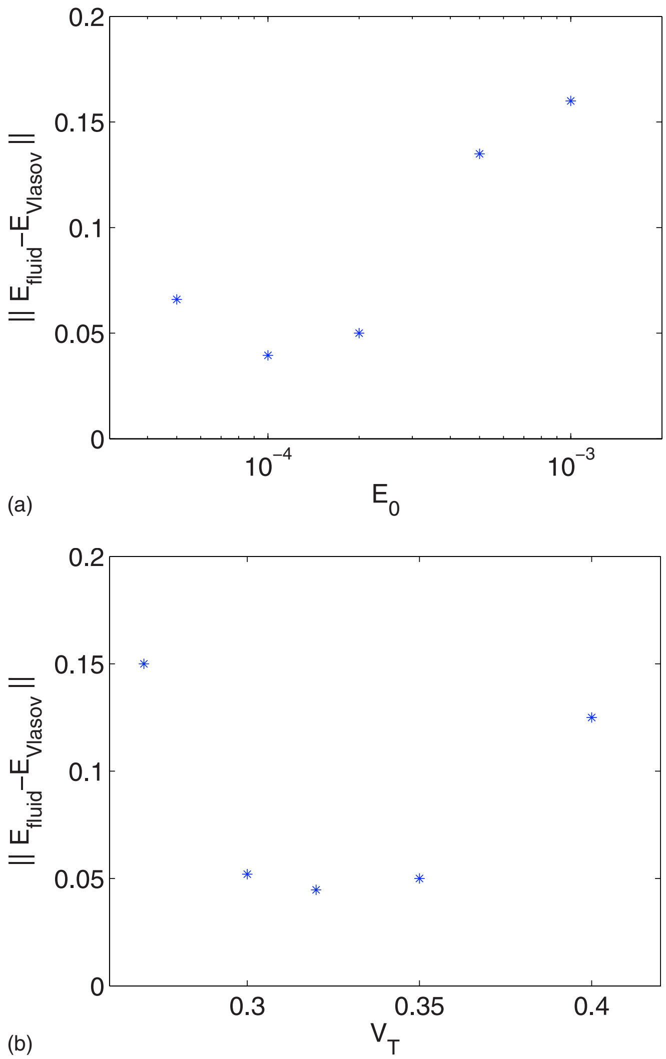

FIG. 2. (Color online) Accuracy of the simplified fluid model of Landau damping. The upper plot shows the difference between the solutions of the fluid and the Vlasov models vs the amplitude of the driving force E_0 at $v_T = 0.35$. The lower plot shows the difference between the solutions vs the plasma temperature at $E_0 = 2\times 10^{-4}$.

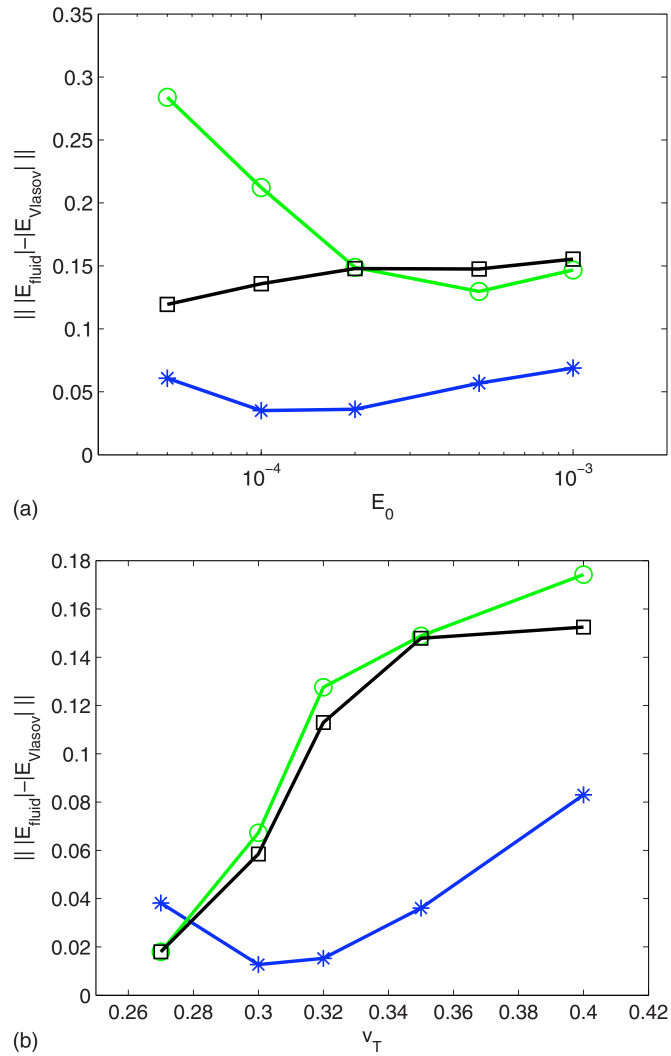

FIG. 3. (Color online) Accuracy of the simplified fluid models of Landau damping. The difference between the absolute amplitudes of the kinetic and fluid solutions are shown for the same parameters as in Fig. 2. Blue stars represent the simplified model presented in this paper. Green circles and black squares represent the solutions predicted by Lindberg *et al.* (Ref. 32) and Benisti and Gremille (Ref. 31) models, respectively.

perature is caused by approaching the limits (16) and (17). The relatively poor accuracy of the fluid model at small amplitude of the driving force can be explained by long time of the linear regime of Landau damping. During this stage, the nonlinear frequency shift does not reach its time-asymptotic value and, therefore, it cannot be described by Eq. (11). This results in a significant phase mismatch between the two solutions for the electric field.

We also compare the accuracy of our model to that of other similar models recently proposed.[31,32] The results are presented in Fig. 3. In order to adequately compare the accuracy of saturation of Landau damping, we use the same expression for the nonlinear frequency shift of the plasma wave as described in this paper. Moreover, the largest discrepancy between the exact kinetic and simplified fluid models in Fig. 2 comes from the phase mismatch of two solutions. We eliminate this effect by plotting the norm of the absolute, rather than complex, amplitude in Fig. 3. The results presented in Fig. 3 indicate that all three simplified fluid models provide reasonable agreement with the kinetic solution. However, the accuracy of our fluid models (9)–(11) is better than that of the models described in Refs. 31 and 32. At the same time, those fluid models have their own advantages.

The main advantage of the fluid model proposed by Benisti and Gremillet is the ability to explicitly express the nonlinear Landau damping rate in terms of the local plasma wave amplitude and its growth rate [Eq. (49) in Ref. 31]. Therefore, the growth of the plasma wave is described by the first-order ordinary differential equation (ODE), rather than the second-order set of ODEs (9)–(11) presented in this paper. The simplified fluid model for saturated Landau damping proposed by Lindberg *et al.*[32] uses the same approach as in Ref. 31. The authors use a reasonable but heuristic approach to describe the transition stage of saturation of Landau damping. Surprisingly, the accuracy of their model is almost the same as that of the more accurate Benisti and Gremillet's model in the regime of relatively short transition stage (relatively large amplitude of the external force in Fig. 3). The advantage of this model is its potential ability to correctly describe saturated Landau damping in the presence of strong plasma wave beyond the limitation (13). However, the description becomes kinetic in this regime since it requires evaluation of an integral in velocity space at each time step. We did not investigate this regime of model of Lindberg *et al.* and limited ourselves to the study of the "small-amplitude" regime [Eq. (16) in Ref. 32].

IV. CONCLUSIONS

We developed a simplified model for the nonlinear saturation of Landau damping. It self-consistently describes the growth of a plasma wave, driven by a near-resonant external force. The simplified model describes the saturation of Landau damping and the nonlinear frequency shift in terms of fluid equations. It does not describe a number of kinetic effects, such as the plasma echo and oscillation of the plasma wave amplitude on the bounce frequency timescale. However, it describes the saturation of Landau damping as well as the linear damping and the transition stages, when Landau damping is not fully saturated. The solution of the simplified model approximates well both the amplitude and the phase of the plasma wave.

This model was verified through numerical simulations using a Vlasov code. The introduced norm (22) allows comparing two solutions quantitatively, which showed about 90% accuracy of the simplified model within its applicability limits. The proposed model can be used for solving complicated systems, when nonlinear Landau damping of the plasma wave is present. It can be included in fluid codes instead of using full kinetic codes, for example, in describing Landau damping. This model can also be used for finding analytical solutions, since the equations has a much simpler form than does the original Vlasov–Poisson set of equations. For example, this model can be successfully applied to study nonlinear Landau damping in the backward Raman amplifier.[27] Also, using the technique employed here, it should be possible to generalize the model to describe nonlinear Landau damping of the ion-acoustic wave.

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APPENDIX: QUASILINEAR MODEL FOR A SINGLE PLASMA WAVE

The appendix argues that the use of the quasilinear limit even for a single wave may be appropriate in capturing most of the essential physics.

We start from the one-dimensional (1D) Vlasov–Poisson set of equations, describing the change of the distribution function and the amplitude of the electric field,

$$\partial_t F + v\partial_z F + \frac{e(E+E^0)}{m}\partial_v F = 0, \tag{A1}$$

$$\partial_z E = 4\pi e n \int F dv, \tag{A2}$$

where $E(z,t)$ is the amplitude of the plasma wave, $E^0(z,t)$ is the amplitude of the external electric field, $F(z,v,t)$ is the electron distribution function, e, m are electron charge and mass, respectively, n is the electron density, t is time, z is longitudinal coordinate along the wavevector of the plasma wave, and v is the electron velocity.

We linearize Eqs. (A1) and (A2) and seek the solution as a sum of slowly changing in time background distribution function $F_0(v,t)$ and fast oscillating distribution function $F_1(z,v,t)\propto e^{i\Omega t-ikz}$, which corresponds to the plasma wave,

$$F(z,v,t) = F_0(v,t) + \mathrm{Re}\{\tilde{F}(v,t)e^{i\Omega t-ikz}\}, \tag{A3}$$

$$|\tilde{F}| \ll |F_0|. \tag{A4}$$

Here Ω and k are the complex frequency and the wavenumber of the plasma wave, respectively. The characteristic timescale of F_0 and $\tilde{F}$ is on the order of the inverse bounce frequency, $\omega_B^{-1} \gg \Omega^{-1}$, which validates this decomposition.

For simplicity, we consider $\tilde{F}$ to be constant in time, thus, applying the quasistatic solution for the plasma wave. This solution is a reasonable approximation in the $\partial_t\omega_B/\omega_B^2 \to 0$ limit. The oscillating part of the distribution function $\tilde{F}=ie(E+E_0)/m\ \partial_v F_0/(\Omega-kv)$ becomes singular close to the phase velocity of the plasma wave $v_{\mathrm{ph}}=\mathrm{Re}\ \Omega/k$. This singularity results in Landau damping, like in the case of a linear plasma wave. At the same time, the oscillating part of the distribution function $\tilde{F}$ becomes larger than the background distribution function F_0 close to the resonant velocity, which violates assumption (A4). However, Eq. (A4) can still be valid for the most of the resonant particles $|v-v_{\mathrm{ph}}|\sim\omega_B/k$ if the amplitude of the plasma wave is small enough,

$$\omega_B \ll \frac{k^2 v_T^2}{\Omega_0}, \tag{A5}$$

where $v_T=\sqrt{T/m}$ is the thermal velocity of the original distribution function (here it is assumed to be Maxwellian) and Ω_0 is the frequency of a linear (small electric field) plasma wave. Note that this condition becomes milder when the plateau in the background distribution function is formed.

The linearized set of Vlasov–Poisson Eqs. (A1) and (A2) results in a solution for the amplitude of the plasma wave,

$$D(\Omega,k,t)E_{\Omega,k} = [D(\Omega,k,t)-1]E^0_{\Omega_k}, \tag{A6}$$

$$D = 1 + \frac{\omega_p^2}{k}\int \frac{\partial_v F_0}{\Omega - kv}dv, \tag{A7}$$

where D is the plasma dielectric function and $E_{\Omega,k}$ is a Fourier harmonic of the electric field, $E(z,t) = \int E_{\Omega,k}e^{i\Omega t - ikz}d\Omega dk$. The change of a quasi-monochromatic electric field $E(z,t)=E(t)e^{i\Omega_0 t - ikz}$ can be deduced from Eq. (A6). Here we consider the carrier frequency of the electric field Ω_0 to be equal to the resonant frequency of the linear plasma wave, $D(\Omega_0 + i\nu_0, k, t=0)=0$,

$$\partial_t E - i\delta\Omega E + \nu E = \frac{E_0(t)}{i\partial_\Omega D}. \tag{A8}$$

Here $\delta\Omega = \mathrm{Re}(\Omega - \Omega_0)$ is the frequency shift of the plasma wave due to the nonlinear change of the background distribution function, $D(\Omega,k,t)=0$; $\nu = \mathrm{Im}(\Omega) \approx -\pi(\omega_p/k)^2\partial_v F_0/\partial_\Omega D$ is the Landau damping rate of the plasma wave.

Averaging in time the Vlasov equation we find the equation describing the change of the background distribution function F_0. This equation has a quasilinear form,

$$\partial_t F_0 = \mathrm{Re}\left\{\frac{1}{2i}\left|\frac{eE}{m}\right|^2 \partial_v \frac{\partial_v F_0}{\Omega - kv}\right\}. \tag{A9}$$

Here we took into account that during the near-resonant amplification of the wave, its amplitude becomes much larger than the amplitude of the driving force, $E \gg E^0$. This equation describes the average change of the distribution function, since the steady-state solution for the oscillatory part of the distribution function $\tilde{F}$ was used. The solution for F_0 misses the oscillations at the bounce frequency timescale, since the solution for $\tilde{F}$ fails close to the phase velocity of the plasma wave. However, these oscillations will be averaged out if the interaction time is large compared to the inverse bounce frequency. These oscillations might not be important in some applications such as parametric interactions.

Even though Eq. (A9) is approximate, it still satisfies the basic conservation laws. Combining Eqs. (A8) and (A9) in the absence of the external force, $E_0=0$, one can demonstrate the conservation of the number of particles, the momentum, and the energy. Here the momentum and the energy of the plasma wave include both the electrostatic and particle parts,

$$\partial_t\left(n\int F_0 dv\right) = 0, \tag{A10}$$

$$\partial_t\left(nm\int vF_0 dv\right) = 2\nu\partial_\Omega(D\Omega)\frac{|E|^2}{16\pi}\frac{k}{\Omega}, \tag{A11}$$

$$\partial_t\left(\frac{nm}{2}\int v^2 F_0 dv\right) = 2\nu\partial_\Omega(D\Omega)\frac{|E|^2}{16\pi}. \tag{A12}$$

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