Metrics for comparing plasma mass filters

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High-throughput mass separation of nuclear waste may be useful for optimal storage, disposal, or environmental remediation. The most dangerous part of nuclear waste is the fission product, which produces most of the heat and medium-term radiation. Plasmas are well-suited to separating nuclear waste because they can separate many different species in a single step. A number of plasma devices have been designed for such mass separation, but there has been no standardized comparison between these devices. We define a standard metric, the separative power per unit volume, and derive it for three different plasma mass filters: the plasma centrifuge, Ohkawa filter, and the magnetic centrifugal mass filter. © *2011 American Institute of Physics*. [doi:10.1063/1.3646311]

I. INTRODUCTION

A number of plasma techniques have been proposed for separating particles based on mass.[1–4] Early techniques like the plasma centrifuge were primarily for isotope separation, but more recent designs have focused on separation of nuclear waste. In order to evaluate these devices, quantitative measures of comparison need to be developed.

The Archimedes Technology Group designed and built a plasma mass separator based on the design of Ohkawa.[3,5,6] It was thought that plasma mass separation would have many advantages in the separation of high level nuclear waste.[7] The filter is not sensitive to chemical properties of elements and only acts on the mass (actually charge-to-mass ratio) of ions. It can, therefore, replace many species extraction steps with one, provided desired elements are grouped together in mass. It does not introduce a working fluid or effluent stream that will increase the total waste mass.

The Ohkawa filter uses a rotating plasma in a magnetic field to create a radial confinement condition that separates heavy ions from light ones.[3] The separation is collisionless and requires a collector on the radial limiter for the heavy stream and at the ends for the light stream. While the Archimedes device was able to demonstrate separation, difficulty in creating the plasma and operating the end electrodes prevented the demonstration of adequate results to continue operation.

While the Ohkawa filter is a collisionless filter that depends on a radial confinement condition, the plasma centrifuge is a collisional mass separator that separates particles within a confined region.[1] The plasma centrifuge has been studied for decades. Early experiments were limited by the Alfven critical ionization velocity (CIV), while later ones had limited throughput because of pulsed operation.[2,8,9] These limits might be overcome by driving rotation with radio frequency waves.[10]

A more recent theoretical device, the magnetic centrifugal mass filter (MCMF), has been proposed for this type of problem.[4] The MCMF uses different magnetic and centrifugal confinement conditions on either end to produce separation based on mass. Because it is collisional, it may have higher throughput than the Ohkawa filter. In addition, it has reduced proliferation risk compared to other mass filters because the throughput becomes exponentially small for small mass differences.

In this paper, we will compare these three separation methods for separating nuclear waste and spent nuclear fuel. In Sec. II, we look for measures by which to compare separation technologies and in Sec. III, we define a "mass filter" as a type of mass separation device. In Secs. IV–VI, we derive expressions describing the ideal operation of each filter. Then in Sec. VII, we compare the filters.

II. NUCLEAR WASTE

In comparing mass separation schemes, one needs to define an objective function or "measure of goodness." In developing such a function, we need to address what we are separating and how effectively we need to separate it.

Isotopes in nuclear waste can be divided into three groups by mass: the lightest group (1–65 amu) is bulk mass that entered the waste stream through reprocessing or leaching, the intermediate group (80–160 amu) is the highly radioactive fission product, and the heavy group (225–250 amu) is the series of actinoids, moderately radioactive and potentially fissionable. The compositions of Hanford high level waste and spent nuclear fuel in terms of these categories are shown in Tables I and II. In both cases, the fission product produces more than 99% of the radioactivity despite making up a small fraction of the total mass.

At Hanford, the waste disposal requirements are set by an agreement between the DOE, the NRC, and Washington state.[13] The most common metal in the Hanford waste is ^{27}Al, and the elements responsible for most radioactivity are ^{90}Sr and ^{137}Cs. The requirement for disposal is that the ^{90}Sr activity be less than 20 $\mathrm{Ci/m^3}$, while on average, the waste contains 240 $\mathrm{Ci/m^3}$.[14] For ^{137}Cs, the feed has about 230 $\mathrm{Ci/m^3}$, and the limit is set by the Department of Energy at 0.105 $\mathrm{Ci/m^3}$ to reduce radiation exposure to personnel. These two separation problems are hard in different ways: ^{90}Sr has a smaller mass gap but ^{137}Cs has higher separation requirements. To choose the best mass filter scheme, our example will take the harder problem in both mass gap and reduction factor. We define the separation of high level

TABLE I. Inventory of all Hanford high level waste divided into three mass categories (Ref. 11).

	Bulk elements 1–65 amu	Fission product 80–160 amu	Actinoids 225–250 amu
Mass (kg)	1.49×10^8	1.05×10^6	5.82×10^5
	98.9%	0.7%	0.4%
Radioactivity	1.51×10^5	1.93×10^8	3.56×10^5
(Ci)	0.1%	99.7%	0.2%

waste as one with a mostly light feed with mass $m_\ell = 27$ amu for which we reduce the density of ^{90}Sr by a factor of 2300.

Spent nuclear fuel (SNF) is mostly ^{238}U, and the majority of radioactivity comes from ^{137}Cs (depending on the time after discharge).[15] The radioactivity in SNF is approximately 1.2×10^5 Ci/m^3, compared to a Nuclear Regulatory Commission limit for low level waste of 4600 Ci/m^3.[16,17] We, therefore, define the separation of spent nuclear fuel as removing 96% of the ^{137}Cs ions from a mostly heavy feed with $m_h = 238$ amu.

Another important measure for a separation device is throughput. For reference, we compare plasma devices to existing chemical separation plants. The Rokkasho Reprocessing Plant in Japan has a throughput of 800 MT/year and was built at a cost of \$20 billion. The throughput for the plasma centrifuge operating on a uranium feed is (from Eq. (12))

$$F \approx 200 \frac{n_0}{10^{14}/\mathrm{cm}^3} \left(\frac{a}{10\,\mathrm{cm}}\right)^2 \sqrt{\frac{T}{10\,\mathrm{eV}}}\,\mathrm{MT/year}, \quad (1)$$

where n_0 is the plasma density, a is the plasma radius, and T is the plasma temperature. Therefore, a 20 cm radius plasma with a density of 10^{14}/cm^3 and temperature of 10 eV would have a separative power of 800 MT/year, equal to the Rokkasho Plant. Larger and denser plasmas are easily conceivable.[5,9]

III. PLASMA MASS FILTERS

Plasma mass filters have a number of advantages over chemical separation processes. Because the particles are ionized, there is no dependence on the chemistry of the input stream. Many species are separated in one step rather than requiring a different process for each element. There is no working fluid introduced to increase the total waste mass. Plasma devices may also be much smaller physically than comparable chemical plants, reducing the cost of constructing and operating the plant.

TABLE II. Spent nuclear fuel produced per year from a 1 GW(electric) light water reactor, divided into three mass categories (Ref. 12).

	Bulk elements 1–65 amu	Fission product 80–160 amu	Actinoids 225–250 amu
Mass (kg)	…	9.2×10^2	2.6×10^4
		3.4%	96.6%
Radioactivity	…	2.2×10^7	4×10^4
(Ci)		99.8%	0.2%

While there is a significant energy cost to ionizing all particles, this amounts to only \$1/kg for uranium (assuming 100 eV per particle lost to ionization and \$0.10/kWh). These costs are small compared to the cost of chemical processing (at least \$1000/kg uranium (Ref. 18)) although a full cost comparison is beyond the scope of this work.

There is a clear tradeoff in separation problems between throughput and separation factor.[19] For example, two separation stages with a separation factor of α can be used in parallel to double the throughput, or they can be used in series to produce a separation factor like α^2. A useful metric in comparing separation methods is the separative power,[20]

$$SP = P(2x-1)\ln(x/(1-x)) + W(2y-1)\ln(y/(1-y)) - F(2z-1)\ln(z/(1-z)). \quad (2)$$

Here, F is the feed rate, P is the product (heads) flow rate, and W is the waste (tails) flow rate. The concentration of the enriched species in the product is x, in the waste is y, and in the feed is z. It is useful to define the fraction of each group that ends up in the product stream as

$$f_{\ell p} = \frac{xP}{zF}, \quad f_{hp} = \frac{(1-x)P}{(1-z)F}. \quad (3)$$

Likewise, we define the waste fractions as $f_{\ell w} = 1 - f_{\ell p}$ and $f_{hw} = 1 - f_{hp}$. These designations assume that the light element is enriched in the product and the heavy element is enriched in the waste. Substituting in to the separative power using $f_{\ell p}$ and f_{hp},

$$SP = F\big(f_{\ell p} z - f_{hp}(1-z)\big)\ln\frac{f_{\ell p}}{f_{hp}} + F\big((1-f_{\ell p})z - (1-f_{hp})(1-z)\big)\ln\frac{f_{\ell w}}{1-f_{hp}}. \quad (4)$$

We define a *mass filter* as a type of separation device where the majority of light particles exit through the product and the majority of heavy particles exit through the waste. That is, $f_{\ell w} \ll 1$ and $f_{hp} \ll 1$. In this case, the separative power may be rewritten as

$$U \approx F\big[(f_{hp}(1-z) - z)\ln f_{hp} + (f_{\ell w} z + z - 1)\ln f_{\ell w}\big]. \quad (5)$$

In the case that $f_{\ell w} = f_{hp} = \epsilon \ll 1$,

$$U \approx -F \ln \epsilon. \quad (6)$$

IV. PLASMA CENTRIFUGE

A plasma centrifuge uses the centrifugal force in a rotating plasma to control the radial distribution of ion species. The centrifugal force produces an azimuthal drift that is larger for heavy ion species than light ion species. Collisional drag between the two types of ions leads to an inward drift of light particles and an outward drift of heavy particles. Equilibrium in a plasma rotating at frequency Ω is reached when the density ratio satisfies[20]

$$n_2(r)/n_1(r) = n_2(0)/n_1(0)\exp\big(\Delta m \Omega^2 r^2/2T\big). \quad (7)$$

Unlike in a gas centrifuge, an arbitrary total density profile $n(r)$ can be produced because of radial confinement produced by the magnetic field.

The countercurrent plasma centrifuge amplifies the radial separation many times by creating a specific flow pattern through the device. The plasma near the core, which is enriched in the light species, travels in one direction, while the outside plasma, depleted in the light species, travels in the other direction. This effectively creates a cascade of centrifuges connected in series. The maximum number of times the radial separation can be magnified in length L is approximately L/λ_i, where λ_i is the ion mean free path.

Because of the form of Eq. (7), it is more natural to use $\beta = x(1-z)/z(1-x)$ and $\gamma = z(1-y)/y(1-z)$ in the equation for separative power rather than $f_{\ell w}$ and f_{hp}. We find the full separative power is

$$\begin{aligned} SP = F[&\log(\beta\gamma - 1) - z\log(\beta - 1)] \\ &- F(1-z)\log(\gamma - 1) \\ &+ F(1-2z)\log(z + \gamma - z\gamma) \\ &+ F(2z-1)\log(1 + z(\beta - 1)). \end{aligned} \tag{8}$$

In the limit of large separation factors, $\beta \gg 1$ and $\gamma \gg 1$, using the overall separation factor $\alpha = \beta\gamma$,

$$SP \approx F\ln(\alpha). \tag{9}$$

As a result, the separation factor for a countercurrent plasma centrifuge of length L is

$$SP \approx F\frac{L}{\lambda_i}\frac{\Delta m \Omega^2 a^2}{2T}. \tag{10}$$

We can find an upper limit for the feed rate by integrating across the exit surface, assuming a very small mirror ratio and Maxwellian ions,

$$\begin{aligned} F_i &= 2A\int_0^\infty dv_\parallel \int_0^\infty 2\pi v_\perp dv_\perp\, v_\parallel n_{0i}\frac{\exp\left(-m_i(v_\parallel^2 + v^2)/2T\right)}{(2\pi T/m_i)^{3/2}} \\ &= n_{0i}A\sqrt{2\pi}v_{ti}. \end{aligned} \tag{11}$$

By particle balance, $F_\ell = zF$ and $F_h = (1-z)F$, so with $n_0 = n_{0\ell} + n_{0h}$,

$$F = \frac{n_0 A\sqrt{2\pi}v_{th}}{1 - z + z\sqrt{m_\ell/m_h}}. \tag{12}$$

The total separative power is, therefore,

$$SP = \frac{n_0 A\sqrt{2\pi}v_{th}}{1 - z + z\sqrt{m_\ell/m_h}}\frac{L}{\lambda_i}\frac{\Delta m \Omega_E^2 a^2}{2T}. \tag{13}$$

V. OHKAWA FILTER

The Ohkawa filter is fundamentally different from the plasma centrifuge.[3,21] The separation method is collisionless, rather than collisional. In addition, while plasma centrifuges can rotate in either direction at any speed, the Ohkawa filter, at least as originally envisioned, has a fixed rotation speed and direction for a certain magnetic field and mass cutoff. Because the Ohakwa filter uses a radial confinement condition for separation, the heavy stream must be collected on the radial wall of the separation device, while the light stream is collected at the axial ends.

The Ohkawa filter is based on the balance of centrifugal and magnetic forces in a rotating plasma. Because heavier particles experience a larger centrifugal force, it is possible to choose system parameters so that the heavy particles are not confined in the system but light particles are. That is, the electric and magnetic fields are chosen so that for heavy particles, at any rotation speed the outward forces (centrifugal and electrostatic) exceed the inward force (magnetic). We can find the Ohkawa mass cutoff condition by solving the force balance equation for the rotation speed Ω,

$$\begin{aligned} 0 &= m\Omega^2 r + eEr - e\Omega rB, \\ \Omega &= \frac{\Omega_i}{2}\left(1 - \sqrt{1 + 4\Omega_E/\Omega_i}\right), \end{aligned} \tag{14}$$

where $\Omega_E = -E/rB$ is the $E \times B$ rotation frequency. One can see that there is no physical solution to Eq. (14) if $\Omega_E < -\Omega_i/4$.

We can define the cutoff mass through $\Omega_E = -\Omega_{ic}/4$,

$$m_c = \frac{erB^2}{4E}. \tag{15}$$

Typically, a parabolic potential is used so that E/r is constant.[3] Therefore, particles with a mass above m_c are not confined in the filter ($\Omega_E < -\Omega_i/4$), while particles with a mass below m_c are confined ($-\Omega_i/4 < \Omega_E$). The unconfined heavy particles circle the device axis and exit radially, while the confined particles remain tied to a field line and exit along the axis.

In the collisionless limit, the exit streams are mixed due to the thermal distribution of injected ions. Some light particles have enough perpendicular energy to exit radially and some heavy particles have enough parallel energy to exit along the axis.

A. Light particles

We will calculate the fraction of light particles that exit radially by first estimating the outer radius of an ion created at the center of the device with radial speed v_{r0} and then integrating over a Maxwellian distribution to find the total number of particles that intersect the outer wall.

Because of the rotation, light particles undergo cyclotron orbits much larger than they would in a stationary plasma, especially as their mass approaches the cutoff mass. The effective cyclotron frequency in the frame rotating at frequency Ω is[22]

$$\begin{aligned} \Omega_i' &= \Omega_i + 2\Omega, \\ &= \Omega_i\sqrt{1 + 4\frac{\Omega_E}{\Omega_i}}. \end{aligned} \tag{16}$$

If all particles are injected on the axis, the maximum radius of a particle with radial velocity v_r is $r_{max} = 2v_r/\Omega'_i$. The density of light particles with a maximum at a given radius can be found by integrating over the Maxwellian,

$$n_\ell(r) = \int d^3v\,\delta\left(r - \frac{2v}{\Omega'_{i\ell}}\right) f_{M\ell}(v) = n_{0\ell}\sqrt{\frac{2}{\pi}}\left(\frac{\Omega'_{i\ell} r}{2v_{t\ell}}\right)^3 \exp\left(-\left(\Omega'_{i\ell} r/2\right)^2/2v_{t\ell}^2\right). \tag{17}$$

We can use Eq. (16), the cutoff condition $\Omega_{ic}/4 = -\Omega_E$, and the mass difference $\Delta m_\ell = m_c - m_\ell$ to write

$$\Omega'_{i\ell} = -4\Omega_E \frac{\sqrt{m_c \Delta m_\ell}}{m_\ell}. \tag{18}$$

Using this in Eq. (17),

$$n_\ell(r) = n_{\ell 0}\sqrt{\frac{2}{\pi}}\left(\frac{4\sqrt{m_c\Delta m_\ell}\,\Omega_E^2 r^2}{m_\ell v_{t\ell}}\right)^{3/2} \exp\left(-\frac{2\Delta m_\ell \Omega_E^2 r^2}{T}\frac{m_c}{m_\ell}\right). \tag{19}$$

Any light particle with a maximum radius $r > a$, where a is the device radius, is lost to the outside wall with the heavy particles. Integrating Eq. (19) over radius, we find for large values of the exponent (small values of $f_{\ell w}$),

$$f_{\ell w} \approx \frac{4\left(2\Omega_E^2 a^2 m_c \Delta m\right)^{3/2}}{3\sqrt{\pi} v_{th}^3 m_i^3} \exp\left[-\frac{2m_c \Delta m_\ell \Omega_E^2 a^2}{m_\ell T}\right]. \tag{20}$$

B. Heavy particles

On the other hand, heavy particles enter the light stream if they exit in a parallel direction before they reach the outer wall. We can determine the time heavy particles take to exit radially from the conservation of angular momentum and energy. The canonical angular momentum is

$$P_\theta = mr\left(v_\theta + \frac{eA_\theta}{mc}\right). \tag{21}$$

For a constant magnetic field, the vector potential $A_\theta = \frac{1}{2}B_z r$, so if the particle starts on axis $v_\theta = -erB_z/2mc = -\Omega_{ih} r/2$.

The potential profile needed to produce the $E \times B$ rotation frequency Ω_E is $\Phi = -\Omega_E B_z c r^2/2$. Using the relation $eB_z/m_c c = -4\Omega_E$ and the previous value for v_θ, we write the energy conservation equation,

$$\frac{2m_c^2}{m_h}\Omega_E^2 r^2 + \frac{1}{2} m_h v_r^2 = 2m_c \Omega_E^2 r^2. \tag{22}$$

From this, we find $v_r^2(r) = v_r^2(0) + 4\Omega_E^2 r^2 m_c \Delta m_h/m_h^2$, with $\Delta m_h = m_h - m_c$. Assuming $v_r(a) \gg v_r(0)$,

$$t_{exit} \approx \frac{\log\left[-4a\Omega_E\sqrt{m_c\Delta m_h}/m_h v_r(0)\right]}{-2\Omega_E\sqrt{m_c\Delta m_h}/m_h}. \tag{23}$$

We can, therefore, find the fraction of heavy particles that exit along the field line before they are removed. To simplify the integration over phase space, we assume that $v_{r0} = v_{th}$. Then all heavy ions with $|v_{||}| > L/2t_{exit}$ enter the light stream. It is useful to define the scaled length $L^\star = L/\log\left(-4a\Omega_E\sqrt{m_c\Delta m_h}/m_h v_{th}\right)$ so that

$$f_{hp} = \frac{v_{th} m_h}{-L^\star \Omega_E \sqrt{\pi m_c \Delta m_h}} \exp\left[-\frac{m_c \Delta m_h}{m_i^2}\frac{\Omega_E^2 L^{\star 2}}{v_{th}^2}\right]. \tag{24}$$

C. Separative power

Combining Eqs. (5), (20), and (24),

$$SP \approx Fz\left[\frac{m_c \Delta m_h}{m_h^2}\frac{\Omega_E^2 L^{\star 2}}{v_{th}^2} - \ln\left(-\frac{v_{th}}{L^\star \Omega_E\sqrt{\pi}\sqrt{m_c\Delta m}/m_h}\right)\right] + F(1-z)\left[\frac{m_c\Delta m_\ell}{m_\ell}\frac{2\Omega_E^2 a^2}{T} - \ln\left(\frac{4\left(2\Omega_E^2 a^2 m_c \Delta m\right)^{3/2}}{3\sqrt{\pi} v_{t\ell}^3 m_\ell^3}\right)\right]. \tag{25}$$

Because we have already assumed that the terms that were in the exponentials were large, we can discard the log terms compared to the below equation

$$SP \approx F\frac{m_c}{T}\left(z\frac{\Delta m_h}{m_h}\Omega_E^2 L^{\star 2} + (1-z)\frac{\Delta m_\ell}{m_\ell} 2\Omega_E^2 a^2\right). \tag{26}$$

If the flow rate of light particles is limited by the density exiting through the ends, from Eq. (11),

$$F_\ell \approx n_{0\ell} A\sqrt{2\pi} v_{t\ell}, \tag{27}$$

while heavy particles exit in a time given by Eq. (23),

$$F_h = \frac{n_{0h} AL}{t_{exit}} = n_{0h} A\left(-2\Omega_E L^\star\sqrt{m_c\Delta m_h}/m_h\right). \tag{28}$$

The total flow rate is, therefore, using $F_\ell = zF$ and $F_h = (1-z)F$,

$$F = \frac{n_0 A v_{t\ell}}{(1-z) v_{t\ell} m_h/\left(-2\Omega_E L^\star\sqrt{m_c\Delta m_h}\right) + z/\sqrt{2\pi}}. \tag{29}$$

The density is limited by the collisionless condition $\Omega_i \tau_i \gg 1$,[3]

$$n_0 \ll \frac{3BT^{3/2}}{4\sqrt{\pi m_h}\lambda c e^3}, \tag{30}$$

$$n_0 \ll 2 \times 10^{15}\frac{(B/\mathrm{Tesla})(T/\mathrm{eV})^{3/2}}{\lambda Z^3 \mu^{1/2}}\ \mathrm{cm}^{-3}. \tag{31}$$

Therefore,

$$SP \approx \frac{n_0 A m_c v_{t\ell}}{T}\frac{z\dfrac{\Delta m_h}{m_h}\Omega_E^2 L^{\star 2} + (1-z)\dfrac{\Delta m_\ell}{m_\ell} 2\Omega_E^2 a^2}{z/\sqrt{2\pi} + (1-z)\dfrac{v_{t\ell} m_h}{-2\Omega_E L^\star\sqrt{m_c\Delta m_h}}}. \tag{32}$$

VI. MAGNETIC CENTRIFUGAL MASS FILTER

The MCMF is a collisional rotating plasma device that produces two well separated output streams.[4] The geometry is similar to the asymmetric centrifugal trap (ACT) proposed by Volosov in 1997 as an aneutronic fusion device.[23,24] However, unlike the ACT which is a hot collisionless confinement device that separates particles of different energy, the MCMF is a cold high-throughput collisional filter that separates particles of different mass.

A key element of the MCMF is that, in a rotating system, either an increase in the magnetic field or a decrease in radius confines the plasma. The confinement condition for a particle at the midplane with parallel energy $W_{||0}$, perpendicular energy $W_{\perp 0}$, and rotation energy $W_{E0} = m\Omega_E^2 r^2/2$ in a trap with magnetic mirror ratio $R_m = B_m/B_0$ and radial mirror ratio $R_r = r_0^2/r_m^2$ is

$$W_{||0} < W_{\perp 0}(R_m - 1) + W_{E0}\left(1 - R_r^{-1}\right). \tag{33}$$

For $R_m \ll 1$ and $R_r \gg 1$, the confinement condition becomes $W_{\perp 0} + W_{||0} < W_{E0}$. The confinement depends only on the midplane energy, not on pitch angle, and on W_{E0} which varies according to mass (not charge-to-mass ratio). If the plasma is collisional, both heavy and light ions have the same average kinetic energy, but the barrier is much higher for heavy particles. Therefore, more light particles exit through the boundary.

The boundary on the other side uses $R_r < 1$, which accelerates heavy particles toward the exit. Heavy and light particles are confined by a magnetic mirror, $R_m > 1$. Because the radial acceleration is stronger for heavy particles, more heavy particles will exit through this boundary.

In deriving the separative power, we assume that the midplane plasma is collisional so that both species have the same temperature and the particle distribution function is Maxwellian. We will first find the throughput at the light and heavy boundaries and then combine them to find the overall separation factor.

A. Light boundary

The light boundary is a simple energy threshold in the limit $R_m \ll 1$ (see Eq. (33)). If we define the cutoff velocity on the light side as $v_{c\ell} = v_{E0}\sqrt{1 - R_r^{-1}}$, the flow rate of species i in the product is

$$P_i = A\int_{v > v_{c\ell}} d^3v\, v_{\|} f_{mi}(\mathbf{v}), \tag{34}$$

$$P_i = n_{0i}A\int_{v_{c\ell}}^{\infty} dv\, \frac{\pi^2 v^3}{2}\frac{\exp(-v^2/2v_{ti}^2)}{\left(2\pi v_{ti}^2\right)^{3/2}}, \tag{35}$$

$$P_i = n_{0i}Ae^{-v_{c\ell}^2/2v_{ti}^2}\frac{\sqrt{\pi/2}}{2}v_{ti}\left(\frac{v_{c\ell}^2}{2v_{ti}^2} + 1\right). \tag{36}$$

We cannot determine the fraction of heavy particles in the light stream, f_{hp}, until we know the flow rate through the heavy boundary.

B. Heavy boundary

The heavy boundary is like a regular mirror loss cone shifted upward. From Eq. (33), we find that the critical velocity for particles to be confined on the heavy side is $v_{ch} = v_{E0}\sqrt{\left(R_r^{-1} - 1\right)/(R_m - 1)}$.

To integrate, we divide particles flowing through the heavy boundary into two regions in phase space: the totally unconfined region of particles with $v < v_{ch}$ and the particles in the loss cone with $v > v_{ch}$ but $W_{\perp 0}(R_m - 1) < W_{||0}$,

$$W_i \approx A\left[\int_{v < v_{ch}} d^3v\, v_{\|} f_{mi}(v) + \delta_h\int_{v > v_{ch}} d^3v\, v_{\|} f_{mi}(v)\right], \tag{37}$$

where the fraction of fast particles in the loss cone,

$$\delta_h \approx 1 - \frac{R_m - 1}{\sqrt{(R_m - 1)^2 + 1}} \approx \frac{1}{2R_m^2}. \tag{38}$$

Equation (37) can be rewritten as

$$W_i \approx n_0 A\left[(1 - \delta_m)\int_0^{v_{ch}} d^3v\, v_{\|} f_{mi}(v) + \delta_m\frac{\sqrt{\pi/2}}{2}v_{ti}\right], \tag{39}$$

where we have integrated the second term over all phase space. Doing the first integral, using the approximate form of Eq. (38),

$$W_i \approx n_{0i}A\frac{\sqrt{\pi/2}}{2}v_{ti}\left[1 - \left(1 - \frac{1}{2R_m^2}\right)e^{-v_{ch}^2/2v_{ti}^2}\left(\frac{v_{ch}^2}{2v_{ti}^2} + 1\right)\right]. \tag{40}$$

For $2R_m^2 \gg 1$ and $v_{ch}^2 \ll 2v_{ti}^2$, which we want to choose for a large separation factor,

$$W_i \approx n_{0i}A\frac{\sqrt{\pi/2}}{2}v_{ti}\left(\frac{1}{2R_m^2} + \frac{v_{ch}^4}{8v_{ti}^4}\right). \tag{41}$$

Combined with Eq. (36), we can now solve for the fraction of heavy particles in the light stream and light particles in the heavy stream, which leads to an expression for the separative power.

C. Separative power

To find the separative power, we first find f_{hp} and $f_{\ell w}$ and then substitute into Eq. (5). The fraction of heavy ions in the product is

$$f_{hp} = \frac{e^{-v_{c\ell}^2/2v_{th}^2}\left(\frac{v_{c\ell}^2}{2v_{th}^2} + 1\right)}{\frac{1}{2R_m^2} + \frac{v_{ch}^4}{8v_{th}^4} + e^{-v_{c\ell}^2/2v_{th}^2}\left(\frac{v_{c\ell}^2}{2v_{th}^2} + 1\right)}. \tag{42}$$

Because we are assuming $f_{hp} \ll 1$, we may rewrite this as

$$f_{hp} \approx e^{-v_{c\ell}^2/2v_{th}^2}\left(\frac{v_{c\ell}^2}{2v_{th}^2} + 1\right)\Bigg/\left(\frac{1}{2R_m^2} + \frac{v_{ch}^4}{8v_{th}^4}\right). \tag{43}$$

For the light ions in the waste stream,

$$f_{\ell w} = \frac{\frac{1}{2R_m^2} + \frac{v_{ch}^4}{8v_{th}^4}}{\frac{1}{2R_m^2} + \frac{v_{ch}^4}{8v_{th}^4} + e^{-v_{c\ell}^2/2v_{th}^2}\left(\frac{v_{c\ell}^2}{2v_{th}^2}+1\right)}, \tag{44}$$

$$f_{\ell w} \approx \left[\frac{1}{2R_m^2} + \frac{v_{ch}^4}{8v_{t\ell}^4}\right] e^{v_{c\ell}^2/2v_{t\ell}^2}\left(\frac{v_{c\ell}^2}{2v_{t\ell}^2}+1\right)^{-1}. \tag{45}$$

There is a potential issue in choosing a value for $v_{c\ell}$ or v_{ch} because if the flow through one end of the filter is choked off, everything may just flow through the other end, violating our assumption that both f_{hp} and $f_{\ell w}$ are small. Therefore, while not strictly necessary for the design of an MCMF, we choose our parameters to filter equally on both sides of the MCMF, $f_{hp} = f_{\ell w} = \epsilon$. Multiplying Eqs. (43) and (45), we can define the constant of order unity (maximum at m_h/m_ℓ and minimum at m_ℓ^2/m_h^2),

$$c_m = \frac{4T^2 + R_m^2 m_\ell^2 v_{ch}^4}{4T^2 + R_m^2 m_h^2 v_{ch}^4}\frac{m_h v_{c\ell}^2 + 2T}{m_\ell v_{c\ell}^2 + 2T}, \tag{46}$$

to write

$$\epsilon^2 = c_m e^{-\Delta m v_{c\ell}^2/2T}. \tag{47}$$

Using Eq. (6), we find the separative power,

$$SP = F\left(\frac{\Delta m v_{c\ell}^2}{2T} - \ln c_m\right). \tag{48}$$

We can simplify terms in the flow rate using Eq. (47). Setting Eq. (43) equal to ϵ, we find

$$\frac{1}{2R_m^2} + \frac{v_{ch}^4}{8v_{th}^4} = \frac{e^{-\Delta m v_{c\ell}^2/4T}}{\sqrt{c_m}} e^{-m_\ell v_{c\ell}^2/2T}\left(\frac{v_{c\ell}^2}{2v_{th}^2}+1\right) \tag{49}$$

so that Eq. (41) can be written, for heavy particles,

$$W_h = n_{0h}A\frac{\sqrt{\pi/2}}{2}v_{th}e^{-m_\ell v_{c\ell}^2/2T}\frac{e^{-\Delta m v_{c\ell}^2/4T}}{\sqrt{c_m}}\left(\frac{v_{c\ell}^2}{2v_{th}^2}+1\right). \tag{50}$$

Because of our assumption that $\epsilon \ll 1$, we can approximate the feed rate by adding together the light particles in the product (Eq. (36)) and heavy particles in the waste (Eq. (50)). By conservation of each species, $zF = P_\ell$ and $(1-z)F = W_h$,

$$zF = n_{0\ell}Ae^{-v_{c\ell}^2/2v_{t\ell}^2}\frac{\sqrt{\pi/2}}{2}v_{t\ell}\left(\frac{v_{c\ell}^2}{2v_{t\ell}^2}+1\right), \tag{51}$$

$$(1-z)F = n_{0h}A\frac{\sqrt{\pi/2}}{2\sqrt{c_m}}v_{th}e^{-(m_\ell+\Delta m/2)v_{c\ell}^2/2T}\left(\frac{v_{c\ell}^2}{2v_{th}^2}+1\right). \tag{52}$$

For $m_\ell u_{c\ell}^2 \gg 2T$, defining $n_0 = n_{0h} + n_{0\ell}$,

$$F = \frac{n_0 A v_{th} e^{-m_\ell v_{c\ell}^2/2T}\sqrt{\pi/2}}{(1-z)\sqrt{c_m}e^{\Delta m v_{c\ell}^2/4T} + z\sqrt{m_h/m_\ell}}\frac{m_h v_{c\ell}^2}{4T}. \tag{53}$$

This gives us the final form for the separative power, dropping the small log term in Eq. (48) and using $v_{c\ell} \approx \Omega_E a$,

$$SP = \frac{n_0 A v_{th} e^{-m_\ell \Omega_E^2 a^2/2T}\sqrt{\pi/2}}{(1-z)\sqrt{c_m}e^{\Delta m \Omega_E^2 a^2/4T} + z\sqrt{m_h/m_\ell}}\frac{m_h \Delta m \Omega_E^4 a^4}{8T^2}. \tag{54}$$

VII. COMPARISON

Surprisingly, despite fundamentally different separation mechanisms, each method studied has approximately the same separation factor as a function of rotation speed, $\alpha \approx e^{\Delta m \Omega^2 a^2/2T}$. Because of varying throughput levels, however, there is a substantial difference in separative power. To compare between devices that may have a different size, we use the separative power per unit volume. We also consider the energy required per particle processed.

A. Separative power per volume

Plasma volume is important as a proxy for overall cost for a design. This is because the volume must be surrounded by magnetic field coils, which are a major expense in magnetic confinement devices.

Because of our collisional assumption, we use the ion mean free path λ_i to estimate the length of the MCMF. Because λ_i also appears in the denominator of the centrifuge separative power, in determining the relative separative power, there is a λ_i/L term in the Ohkawa separative power per unit volume. This can be rewritten using $\Omega_{cc} = -4\Omega_E$ to be $\lambda_i/L = v_{th}\tau_i/L = -v_{t\ell}\Omega_{c\ell}\tau_{i\ell}m_\ell/4m_c\Omega_E L$.

TABLE III. Relative separative power per unit volume, assuming equal density and temperature.

	Majority light	Majority heavy
Centrifuge	$\sqrt{\frac{m_h}{m_\ell}}\frac{\Delta m\Omega^2 a^2}{2T}$	$\frac{\Delta m\Omega^2 a^2}{2T}$
Ohkawa	$\Omega_{c\ell}\tau_{i\ell}\frac{-\Delta m_h\Omega_E L/2m_h v_{th}}{\log^2\left(-4a\Omega_E\sqrt{m_c\Delta m_h}/m_h v_{th}\right)}$	$\Omega_{c\ell}\tau_{i\ell}\frac{2\Delta m_\ell\Omega_E^2 a^2}{T}\frac{\sqrt{m_c\Delta m_h}/\sqrt{2\pi m_h m_\ell}}{\log\left(-4a\Omega_E\sqrt{m_c\Delta m_h}/m_h v_{th}\right)}$
MCMF	$\sqrt{\frac{m_\ell}{m_h}}e^{-m_\ell\Omega_E^2 a^2/2T}\frac{m_h\Delta m\Omega_E^4 a^4}{8T^2}$	$\frac{1}{\sqrt{c_m}}e^{-(m_\ell + \Delta m/2)\Omega_E^2 a^2/2T}\frac{m_h\Delta m\Omega_E^4 a^4}{8T^2}$

Each device has a different throughput depending on the concentration of the light product in the feed. If the feed is mostly heavy particles, the Ohkawa filter has higher throughput because the heavy particles are removed throughout the volume (radially) and exit quickly. The MCMF, however, has a somewhat less throughput under a majority of heavy feeds because heavy particles are better confined overall.

The relative separative power over volume is shown in Table III. To get an idea for practical values of the separative power, we consider the problems described in Sec. II: high-level waste that is a mostly light feed with $m_h/m_\ell = 3$ and spent nuclear fuel that is a mostly heavy feed with $m_h/m_\ell = 2$. For the Ohkawa filter, we assume $L = 2a$ and $\Delta m_h = \Delta\ m_\ell = \Delta\ m/2$ (which ensure that $f_{\ell w}$, $f_{hp} \ll 1$), and $\Omega_{c\ell}\tau_{i\ell} = 10$ (to ensure $\Omega_{c\ell}\tau_{i\ell} \gg 1$).

The relative separative power per unit volume versus the log of the separation factor $\Delta m\Omega^2 a^2/2T$ (assuming equal density and temperature) is shown in Figs. 1 and 2. For majority light particles, the plasma centrifuge has the highest separative power, while for majority heavy particles, the Ohkawa filter seems most efficient. However, we note that the density in the Ohkawa filter is limited by Eq. (30). Curves for the Ohkawa filter with reduced density (factors of 10 and 100) are shown by dotted lines.

The disadvantage of the MCMF is clear in these diagrams as the separative power is much lower than competing technologies at equal density. It is the reduction in throughput that produces the stronger proliferation resistance of the MCMF. It is worth noting that if throughput is limited by the source rate (rather than density), this analysis is no longer relevant and the MCMF could have the same separative power for nuclear waste as other methods.

Unlike the centrifuge and Ohkawa filter, the separative power of the MCMF decreases exponentially at large values of $\Delta m\Omega^2 a^2/2T$. This means that the MCMF may not be practical for some separation problems requiring very high purity levels.

B. Energy use

Although the Ohkawa filter excels at separating a mostly heavy input stream, it does so at the expense of greater energy usage. Each heavy particle moves from the center of the device to the radial edge, moving from the positive to negative electrode. This requires energy $W = 2m_c\Omega_E^2 a^2$, about 500 eV in a practical device.[7]

This is much higher than the required energy for other plasma separation methods. Because particles exit along the field lines, the rotation and kinetic energies can be recovered by shaped end electrodes.[25] The only remaining energy losses are due to ionization, radiation, etc.

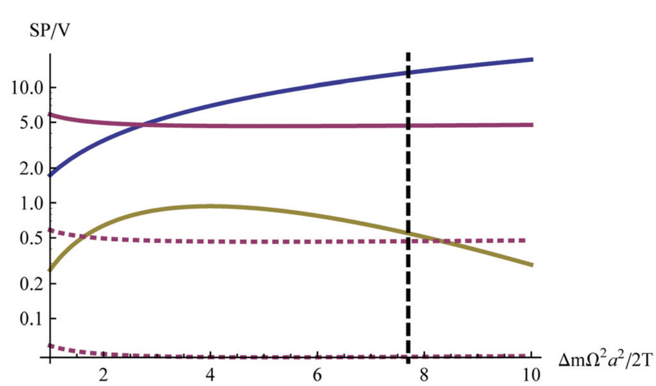

FIG. 1. (Color) Relative separative power per unit volume versus rotation energy for plasma centrifuge (blue), Ohkawa filter (magenta), and MCMF (yellow), for high level nuclear waste separation (mostly light input stream and $m_h/m_\ell = 3$). All devices have the same density and temperature. Relative power for the Ohkawa filter with reduced density (10 times and 100 times) shown by dotted lines. Vertical dashed line shows required separation level.

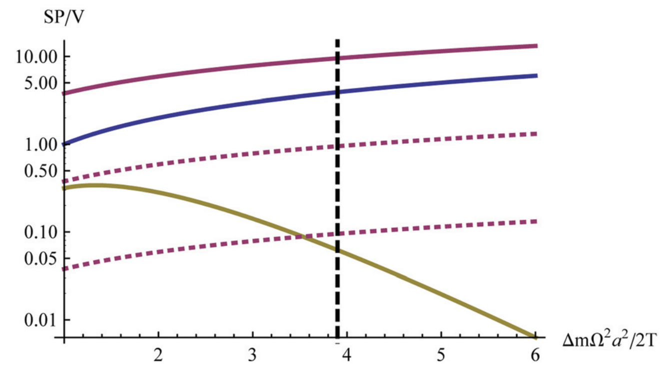

FIG. 2. (Color) Same as Fig. 1, but for spent nuclear fuel, a mostly heavy input stream with $m_h/m_\ell = 2$.

VIII. CONCLUSION

We have described the problem of nuclear waste separation and defined a measure of comparison for plasma separation technologies, the separative power per unit volume. The measure was calculated for three devices: the plasma centrifuge, Ohkawa filter, and MCMF.

The plasma centrifuge was found to have a consistently high separative power per unit volume. According to this metric, the plasma centrifuge would be the best method separating high level nuclear waste. It would also be the best method for separating spent nuclear fuel if densities 2-3 times higher than the collisionless limit (Eq. (30)) are used.

The Ohkawa filter was found to be comparable in separative power per unit volume to the plasma centrifuge, given equal density and temperature. It has a slight edge for mostly heavy feeds because it can produce a higher throughput of heavy particles. However, this is paid for by higher energy cost. The Ohkawa filter is limited overall by the collisionless density requirement.

The MCMF has a lower separative power per unit volume than other devices at the same density and temperature. However, if the density can exceed the Ohkawa filter because of the collisionless limit, it might achieve better separative power over volume. For nuclear waste, the density would have to be 10 times higher and for spent nuclear fuel, it would need to be 100 times higher.

This analysis provides a baseline for comparing the fundamental operation of each separation technology. However, there are many other factors that will influence which technique should be chosen. There are practical concerns that may limit how close to the ideal each technology can operate. There are also political considerations such as nuclear non-proliferation.

All of these devices have the potential to efficiently and effectively separate high level nuclear waste or spent nuclear fuel. It is possible that the costs for plasma mass separation are dramatically less than chemical separation. This could produce significant savings of time and money on the Hanford project. It could also make nuclear fuel reprocessing practical, reducing the demand for geological storage and making spent fuel safer for the future.

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