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# A Hamiltonian model of dissipative wave–particle interactions and the negative-mass effect

A.I. Zhmoginov *, I.Y. Dodin, N.J. Fisch

*Princeton University, Princeton, NJ 08544, United States*

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ABSTRACT

The effect of radiation friction is included in the Hamiltonian treatment of wave–particle interactions with autoresonant phase-locking, yielding a generalized canonical approach to the problem of dissipative dynamics near a nonlinear resonance. As an example, the negative-mass effect exhibited by a charged particle in a pump wave and a static magnetic field is studied in the presence of the friction force due to cyclotron radiation. Particles with negative parallel masses $m_\parallel$ are shown to transfer their kinetic energy to the pump wave, thus amplifying it. Counterintuitively, such particles also undergo stable dynamics, decreasing their transverse energy monotonically due to cyclotron cooling, whereas some of those with positive $m_\parallel$ undergo cyclotron heating instead, extracting energy from the pump wave.



## 1. Introduction

Wave–particle resonant interactions are conveniently approached within a Hamiltonian theory [1,2], which however renders it difficult to account for dissipative forces. In this respect, of special interest are interactions that lie beyond the traditional model of nonlinear resonance [3–5], particularly those where ponderomotive forces are essentially velocity-dependent [6]. An example here is the particle motion in a homogeneous magnetic field $\boldsymbol{B}_0$ and a co-propagating circularly-polarized electromagnetic wave. Due to autoresonant phase-locking [4,7] at the cyclotron resonance, the wave and the magnetic field effectively modify the inertia of the particle oscillation center (OC) with respect to low-frequency (or static) forces applied along $\boldsymbol{B}_0$. Hence, the OC mass $m_\parallel$ along $\boldsymbol{B}_0$ may be seen as negative, thereby yielding what is called the negative-mass effect, or NME [8–11].

Due to particle oscillations in a wave, dissipation is always associated with ponderomotive interactions, particularly in the form of the radiation friction, which can transfer energy from the particle "internal" (e.g., cyclotron) motion to the outgoing waves [12,13]. Hence, it can influence possible practical applications [9] of the NME and related effects, anticipated by analogy with other systems where negative mass of charge carriers is realized [10]. Therefore, a new formalism is necessary that would unite the existing Hamiltonian theory of autoresonant effects like NME [10] with the description of essentially non-Hamiltonian forces like the radiation friction.

The purpose of this Letter is, thus, twofold. First, we propose how dissipative perturbations can be included in the canonical formalism for a broad class of Hamiltonian systems similar to those yielding the NME, and also how the unperturbed phase space structure determines the effect of weak friction on the particle average dynamics at autoresonance. In application to wave–particle interactions, this theory, albeit quasi-Hamiltonian, allows one to describe slow dissipative dynamics of both the wave and the particles; hence, scenarios are found in which the wave is amplified or particles gain energy despite the fact that the system *total* energy decays. Second, we reconsider the NME, as produced by the charged particle interaction with a pump wave in a static magnetic field, and study how this particular effect is altered by the presence of the radiation friction. We show that particles with negative parallel masses $m_\parallel$ transfer their kinetic energy to the pump wave, thus amplifying it. (An analogy here is how negative-energy waves are amplified in the presence of dissipation; see, e.g., Refs. [14,15].) Counterintuitively, such particles also undergo stable dynamics, decreasing their transverse energy monotonically due to cyclotron cooling, whereas some of those with positive $m_\parallel$ undergo cyclotron heating instead, extracting energy from the pump wave.

The Letter is organized as follows. In Section 2, we describe the general formalism showing how the effect of weak dissipation can be understood by studying the unperturbed phase space of a Hamiltonian system; we also explain how this formalism predicts the direction of the energy flow in dissipative wave–particle interactions. In Section 3, we study the effect of radiation friction on the particle dynamics driven by a pump wave in a static magnetic

* Corresponding author.
*E-mail addresses:* azhmogin@princeton.edu (A.I. Zhmoginov), idodin@princeton.edu (I.Y. Dodin), fisch@princeton.edu (N.J. Fisch).

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field and, particularly, how the NME persists through the friction and how the trajectories of particles with positive $m_\parallel$ can become unstable.

## 2. General formalism

In this section, we study the dynamics of a particle, treated as a generalized nonlinear dynamical system, under the action of a weak resonant wave in the presence of an even weaker dissipative force. First, in Section 2.1, we consider the direct effect of the wave on the particle. Then, in Section 2.2, we also show how the evolution of the wave itself can be predicted from its effect on the particle.

### 2.1. Driven system

Consider a small perturbation to a generalized dynamical system governed by a Hamiltonian $H$ [16],

$$H = H_0(\boldsymbol{I}) + \varepsilon H_1(\boldsymbol{I})\cos(\boldsymbol{\ell}\cdot\boldsymbol{\phi} - \omega_0 t). \tag{1}$$

Here $(\boldsymbol{I},\boldsymbol{\phi})$ are the action-angle variables of the unperturbed Hamiltonian $H_0(\boldsymbol{I})$, $\varpi_i = \partial H_0/\partial I_i$ are the unperturbed frequencies, $\varepsilon \ll 1$ is a small parameter, $\omega_0$ is some constant frequency, $\boldsymbol{\ell} = (\ell_1,\ldots,\ell_n)$ is a constant $n$-dimensional integer vector, and $n \equiv \dim\boldsymbol{I} = \dim\boldsymbol{\phi}$. Without loss of generality, assume non-zero $\ell_n$. Using a generating function

$$\Phi(\boldsymbol{J},\boldsymbol{\phi},t) = J_1\phi_1 + \cdots + J_{n-1}\phi_{n-1} + J_n(\boldsymbol{\ell}\cdot\boldsymbol{\phi} - \omega_0 t), \tag{2}$$

perform a canonical transformation to the new variables $(\boldsymbol{J},\boldsymbol{\theta})$; hence

$$\theta_{i<n} = \phi_i, \qquad \theta_n = \boldsymbol{\ell}\cdot\boldsymbol{\phi} - \omega_0 t, \tag{3}$$

$$J_{i<n} = I_i - \ell_i I_n/\ell_n, \qquad J_n = I_n/\ell_n. \tag{4}$$

The new Hamiltonian $\mathcal{H} \equiv H + \partial\Phi/\partial t$ depends on only one canonical coordinate $\theta_n$:

$$\mathcal{H} = H_0(\boldsymbol{J}) - \omega_0 J_n + \varepsilon H_1(\boldsymbol{J})\cos\theta_n. \tag{5}$$

Therefore, all $J_i$ with $i<n$ are constants, and Eq. (5) can be treated as a Hamiltonian of the motion in $(J_n,\theta_n)$ plane. The stationary points of this two-dimensional system are located near the resonance surface $\boldsymbol{\ell}\cdot\boldsymbol{\varpi} = \omega_0$ and are said to form *stationary surfaces* $J_n^0(J_1,\ldots,J_{n-1})$ in the $n$-dimensional $\boldsymbol{J}$ space (see also Ref. [17]).

Let us also introduce *stable surfaces* $J_n^*(J_1,\ldots,J_{n-1})$ formed by the stable stationary points of the system. The particle trajectories in the vicinity of a stable stationary point $(J_n^*,\theta_n^*)$ are periodic orbits with some characteristic period $T$. For a stable stationary point with $\sin\theta_n^* = 0$ [10], one has a real

$$T \approx 2\pi\left(-\varepsilon H_1 \frac{\partial\Omega}{\partial J_n}\cos\theta_n^*\right)^{-1/2}, \tag{6}$$

where $\Omega = \partial H_0/\partial J_n - \omega_0 + \varepsilon(\partial H_1/\partial J_n)\cos\theta_n^*$, and all functions of $\boldsymbol{J}$ are evaluated at $(J_{i<n}, J_n^*)$.

Consider a perturbation to the system (5) by a weak dissipative force $\boldsymbol{\xi}(\boldsymbol{J},\boldsymbol{\phi}) = (\boldsymbol{G},\boldsymbol{K})$, i.e.,

$$\dot{J}_i = \varepsilon\delta_{in}H_1(\boldsymbol{J})\sin\theta_n + G_i, \tag{7}$$

$$\dot{\theta}_i = \omega_i(\boldsymbol{J}) - \delta_{in}\omega_0 + \varepsilon\frac{\partial H_1}{\partial J_i}\cos\theta_n + K_i, \tag{8}$$

where $\delta_{ij}$ is the Kronecker symbol, and $\omega_i = \partial H_0/\partial J_i$. Assume initial conditions such that the system is close to an unperturbed stable surface $S$, that $\boldsymbol{\xi}(\boldsymbol{J},\boldsymbol{\phi})$ is not resonant to the oscillations at frequencies $\varpi_i$, and that $T$ is much larger than all $2\pi/\varpi_i$ and the corresponding beat frequencies. Then, for $i<n$, one can average Eqs. (7) and (8) over the fast oscillations in $\boldsymbol{\phi}$ to get

$$\dot{J}_{i<n} = \langle G_i\rangle, \qquad \dot{\theta}_{i<n} = \bar{\omega}_i, \tag{9}$$

where $\bar{\omega}_i = \omega_i + \varepsilon(\partial H_1/\partial J_i)\cos\theta_n + \langle K_i\rangle$, and $\langle\ldots\rangle$ denotes averaging over the phases $\phi_i$. Therefore, the effect of friction on the degrees of freedom corresponding to $i<n$ consists in adding a slow drift in $J_i$ and slightly modifying the frequencies to $\bar{\omega}_i$.

Now let us consider the remaining degree of freedom, $(J_n,\theta_n)$. Recalling that the dynamics in these variables is slow compared to the dynamics in $\boldsymbol{\phi}$, average Eqs. (7) and (8) for $i=n$ over the fast oscillations in $\boldsymbol{\phi}$:

$$\dot{J}_n = \varepsilon H_1(\boldsymbol{J})\sin\theta_n + \langle G_n\rangle, \tag{10}$$

$$\dot{\theta}_n = \omega_n(\boldsymbol{J}) - \omega_0 + \varepsilon\frac{\partial H_1}{\partial J_n}\cos\theta_n + \langle K_n\rangle. \tag{11}$$

Unlike for $i<n$ [Eqs. (9)], the average friction force can now be compensated by Hamiltonian forces. This means that the dissipation does not destroy the equilibrium in $(J_n,\theta_n)$ plane but rather shifts it to a new location $(\bar{J}_n^*,\bar{\theta}_n^*)$ given by

$$\varepsilon H_1\big(J_{i<n},\bar{J}_n^*\big)\sin\bar{\theta}_n^* = -\langle G_n\rangle, \tag{12}$$

$$\omega_n\big(J_{i<n},\bar{J}_n^*\big) - \omega_0 + \varepsilon\frac{\partial H_1}{\partial J_n}\cos\bar{\theta}_n^* = -\langle K_n\rangle, \tag{13}$$

which drifts slowly due to $J_{i<n}$ following Eqs. (9). Hence, we consider this (quasi-) equilibrium as a *local* stationary point of the system. The surface formed by the local stationary points $\bar{J}_n^*(J_1,\ldots,J_{n-1})$ will be further called a *local stationary surface* $\bar{S}$.

From expanding Eqs. (10) and (11) in the vicinity of the new stable stationary point $(\bar{J}_n^*,\bar{\theta}_n^*)$, it follows that the effect of friction is determined by derivatives of $\langle G_n\rangle$ (and $\langle K_n\rangle$) rather than $\langle G_i\rangle$ itself [unlike in Eqs. (9)]. To see this, consider the normalized phase space area that the particle orbit encircles in $(J_n,\theta_n)$ plane:

$$\Lambda = \frac{1}{2\pi}\oint J_n\,d\theta_n. \tag{14}$$

Without friction $\Lambda$ would be an adiabatic invariant [18], i.e., $\dot{\Lambda}\approx 0$, as shown in Ref. [2]. The conservation of $\Lambda$ causes the autoresonant phase-locking effect [4,7] when the system state “sticks” to the local stationary surface $\bar{S}$. With the friction, however, an argument similar to that in Ref. [2] yields

$$2\pi\dot{\Lambda} \approx \oint_l \langle G_n\rangle\,d\theta_n - \oint_l \langle K_n\rangle\,dJ_n, \tag{15}$$

where $l$ is the periodic orbit of the Hamiltonian system (5). Using the Stokes theorem, Eq. (15) can then be transformed to

$$2\pi\dot{\Lambda} \approx \int_M \left(\frac{\partial\langle G_n\rangle}{\partial J_n} + \frac{\partial\langle K_n\rangle}{\partial\theta_n}\right)dJ_n\,d\theta_n, \tag{16}$$

where $M$ is the oriented area of the phase space $(J_n,\theta_n)$ encircled by $l$. (The area sign is positive for clockwise-rotating orbits and negative otherwise.) Assuming that the system oscillates in a small vicinity of the stable point $(\bar{J}_n^*,\bar{\theta}_n^*)$, and neglecting the variations of $\partial\langle G_n\rangle/\partial J_n$ on this scale, one can approximate:

$$\frac{\dot{\Lambda}}{\Lambda} \approx \frac{\partial\langle G_n\rangle}{\partial J_n}, \tag{17}$$

where we used $\Lambda = (2\pi)^{-1}\int_M dJ_n\,d\theta_n$. This shows that the phase space within the $(J_n,\theta_n)$ orbit grows or shrinks depending on the sign of $\partial\langle G_n\rangle/\partial J_n$. If $\partial\langle G_n\rangle/\partial J_n > 0$, a particle perturbation from the stable stationary surface grows exponentially, thus indicating *dissipation-induced instability* as introduced in the general theory of dynamical systems [19,20].

The subset of the local stationary surface $\bar{S}$ close to the stationary surface $S$ of the original Hamiltonian system, for which $\partial\langle G_n\rangle/\partial J_n < 0$, is an *attractor* of the system with friction. Specifically, trajectories are pulled toward this surface due to friction, and further motion along $\bar{S}$ is determined by the properties of the dissipation function $\boldsymbol{\xi}$. Notice also that the attracting part of $\bar{S}$ is generally characterized by a specific value of $\theta_n^*$, therefore indicating the *phase bunching* occurring in the system. At intersections of $\bar{S}$ with $\langle G_{i<n}\rangle = 0$ (which is, generally, a set of one-dimensional curves in $n$-dimensional space $\boldsymbol{J}$), the attractor has stationary points. The stability of these points in the $J_{i<n}$ subspace can be determined by calculating the eigenvalues of the matrix $D_{ij} \equiv \partial\langle G_i\rangle/\partial J_j$. Specifically, if there is at least one eigenvalue of $D_{ij}$ with a positive real part, the corresponding stationary point is unstable.

### 2.2. Wave–particle system

Suppose now that the generalized dynamical system that we introduced above actually describes the interaction between a particle and a wave, with $\omega_0$ being the wave frequency; hence, one may ask what happens to the wave action $I_w$ as the friction force $\boldsymbol{\xi}$ is applied to a particle. Despite the total action in the particle-field system decays (by definition of $\boldsymbol{\xi}$), $I_w$ may, in fact, grow, meaning that the wave is amplified through dissipation. Below, we derive the general conditions under which such a dissipative amplification of the wave is possible.

Consider system where the wave is treated as a single independent degree of freedom. The corresponding Hamiltonian then reads as

$$H' = H_0(\boldsymbol{I}) + \varepsilon H_1(\boldsymbol{I}, \mathcal{E})\cos(\boldsymbol{\ell}\cdot\boldsymbol{\phi} - \psi) + \omega_0 I_w, \tag{18}$$

where $\psi$ is the wave canonical phase, and $I_w$ is the action variable conjugate to $\psi$ [21]. Instead of the generating function (2), take

$$\Phi = J_1\phi_1 + \cdots + J_{n-1}\phi_{n-1} + J_n(\boldsymbol{\ell}\cdot\boldsymbol{\phi} - \psi) + \mathcal{I}\psi, \tag{19}$$

so $\mathcal{I} = I_w + J_n$ is the new action representing the total number of quanta in the two resonant degrees of freedom, $\theta_n$ and $\psi$. Then, $(J_i, \theta_i)$ are the new actions and angles correspondingly, defined, as before, through Eqs. (3) and (4), and

$$\theta_n = \boldsymbol{\ell}\cdot\boldsymbol{\phi} - \psi. \tag{20}$$

Since the new Hamiltonian, $\mathcal{H}' = H'$, or

$$\mathcal{H}' = H_0(\boldsymbol{J}) + \varepsilon H_1(\boldsymbol{J}, \mathcal{I})\cos\theta_n + \omega_0(\mathcal{I} - J_n), \tag{21}$$

does not depend on $\psi$ or $\theta_{i<n}$, one concludes that $J_{i<n}$ and $\mathcal{I}$ are constants of motion.

In the presence of a dissipative force $\boldsymbol{\xi}$ on the particle, one then obtains, like in Section 2.1:

$$\langle\dot{\mathcal{H}}'\rangle = \sum_{i<n}\left\langle\frac{\partial\mathcal{H}'}{\partial J_i}G_i\right\rangle + \left\langle\frac{\partial\mathcal{H}'}{\partial J_n}G_n\right\rangle + \left\langle\frac{\partial\mathcal{H}'}{\partial\theta_n}K_n\right\rangle + \left\langle\frac{\partial\mathcal{H}'}{\partial\mathcal{I}}G_{\mathcal{I}}\right\rangle, \tag{22}$$

where the effective dissipative force $G_{\mathcal{I}}$ corresponding to $\mathcal{I}$ [in the sense of Eq. (7)] satisfies $G_{\mathcal{I}} = G_n$. (Remember that, without interacting with the particle, the wave is assumed undamped.) Close to $\bar{S}$, the second and the third terms are negligible; then, $\langle\dot{H}'\rangle \approx \boldsymbol{\Omega}\cdot\langle\boldsymbol{G}\rangle$, where $\boldsymbol{\Omega} = (\omega_1, \ldots, \omega_{n-1}, \omega_0)$. Therefore, the force $\boldsymbol{\xi}$ dissipates energy if [22]

$$\boldsymbol{\Omega}\cdot\langle\boldsymbol{G}\rangle < 0. \tag{23}$$

Even when the latter is satisfied, though, the wave energy does not necessarily decay. Indeed, consider the evolution of the wave action $I_w$:

$$\dot{I}_w = \langle G_n\rangle - \dot{J}_n. \tag{24}$$

Assuming, as before, that the system operates near $\bar{S}$, one obtains:

$$\dot{I}_w = \langle G_n\rangle - \sum_{i<n}\frac{\partial\bar{J}_n^*}{\partial J_i}\langle G_i\rangle. \tag{25}$$

The same can be written as

$$\dot{I}_w = \sum_{i\le n}\langle G_i\rangle\frac{\partial R}{\partial J_i}, \tag{26}$$

where $R(J_1, \ldots, J_n) = J_n - \bar{J}_n^*(J_1, \ldots, J_{n-1})$. Hence, $\dot{I}_w$ equals the derivative of $R$ along $\langle\boldsymbol{G}\rangle$ in $\boldsymbol{J}$ space:

$$\dot{I}_w = \langle\boldsymbol{G}\rangle\cdot\nabla_{\boldsymbol{J}}R. \tag{27}$$

On the other hand, since $R$ is constant (zero) on the stationary surface, the gradient of $R$ in $\boldsymbol{J}$ space, $\nabla_{\boldsymbol{J}}R$, is orthogonal to $\bar{S}$. Particularly, since $\partial R/\partial J_n > 0$, the vector $\nabla_{\boldsymbol{J}}R$ points toward larger $J_n$. Thus, the dissipation causes the wave energy to decrease ($\dot{I}_w < 0$) only if $\langle\boldsymbol{G}\rangle$ at $\bar{S}$ points toward the lower one of the two halves of the $J_n$ space separated by $\bar{S}$.

In principle, though, $\langle\boldsymbol{G}\rangle$ can also point in the opposite direction, causing wave amplification; that is, $I_w$ increases in this case through dissipation, apparently, at the expense of the particle internal energy. (Notice that this effect is different from the conventional dissipation-induced instabilities [23–26]; in particular, the wave energy growth may not be exponential.) In Section 3, we illustrate how this effect is possible in a specific physical system.

## 3. Example: wave-driven particle in a magnetic field

### 3.1. Basic equations

Assume a homogeneous magnetic field of the form $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$, governed by the vector potential $\boldsymbol{A}_0 = -\hat{\boldsymbol{x}}By$, and a wave field with circular polarization, governed by $\boldsymbol{A}_w = (mc^2/q)(a_0/\sqrt{2})\times(\hat{\boldsymbol{x}}\cos\xi - \hat{\boldsymbol{y}}\sin\xi)$, where $m$ and $q$ are the particle mass and charge, $c$ is the speed of light, $a_0$ is the normalized wave amplitude, $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ are unit vectors directed along $x$, $y$, and $z$ correspondingly, and $\xi = \omega_0 t - kz$. The particle Hamiltonian reads as

$$H = \sqrt{m^2c^4 + c^2(\boldsymbol{P} - q\boldsymbol{A}/c)^2}, \tag{28}$$

where $\boldsymbol{A} = \boldsymbol{A}_0 + \boldsymbol{A}_w$, and $\boldsymbol{P}$ is the particle canonical momentum. After a series of canonical transformations, Eq. (28) can be cast [10] into the form of Eq. (5):

$$\mathcal{H} \approx H_0 - \omega J_2 - \frac{\varepsilon\sqrt{J_2}}{H_0}\cos\theta_2, \tag{29}$$

with

$$H_0 = c\left[m^2c^2 + 2m\Omega_0 J_2 + k^2(J_1 + J_2)^2\right]^{1/2}, \tag{30}$$

$$J_1 = p_\parallel/k - \tilde{\mu}, \qquad \theta_1 = kz, \tag{31}$$

$$J_2 = \tilde{\mu}, \qquad \theta_2 = \tilde{\theta} - \omega t + kz. \tag{32}$$

Here $p_\parallel \equiv P_z$ is the component of the particle kinetic momentum parallel to $\boldsymbol{B}_0$, $\varepsilon = mc^3\sqrt{m\Omega_0}a_0$ is the normalized (yet dimensional) amplitude playing a role of the small parameter, $\Omega_0 = qB_0/mc$ is the nonrelativistic Larmor frequency, and $\tilde{\mu}$ is the canonical momentum, which is related to the particle magnetic moment $\mu \equiv p_\perp^2/(2m\Omega_0)$ (here $\boldsymbol{p}_\perp$ is the kinetic momentum transverse to $\boldsymbol{B}_0$) as

$$\mu = \tilde{\mu} + \frac{mc^2 a_0^2}{4\Omega_0} - a_0 c\sqrt{\frac{m\tilde{\mu}}{\Omega_0}}\cos(\tilde{\theta} - \omega t + kz). \tag{33}$$

### 3.2. Radiation friction

The radiation reaction four-force on a particle reads as [27]:

$$g^i = \frac{2q^3}{3mc^3}\frac{\partial F^{ik}}{\partial x^l}u_k u^l - \frac{2q^4}{3m^2c^5}F^{il}F_{kl}u^k + \frac{2q^4}{3m^2c^5}\left(F_{kl}u^l\right)\left(F^{km}u_m\right)u^i, \tag{34}$$

where $F_{ik}$ is the electromagnetic four-tensor, and $u^i$ is the particle four-velocity. Since the dominant motion in our case is assumed [10] to be the cyclotron motion (rather than the wave-driven oscillations), keep only the terms due to the static field $B_0$; then,

$$g^x = -\frac{2q^4}{3m^2c^5}B_0^2\left[\left(u^x\right)^2 + \left(u^y\right)^2 + 1\right]u^x, \tag{35}$$

$$g^y = -\frac{2q^4}{3m^2c^5}B_0^2\left[\left(u^x\right)^2 + \left(u^y\right)^2 + 1\right]u^y, \tag{36}$$

$$g^z = -\frac{2q^4}{3m^2c^5}B_0^2\left[\left(u^x\right)^2 + \left(u^y\right)^2\right]u^z. \tag{37}$$

Using an approximate relation $\tilde{\mu} \approx (p_x^2 + p_y^2)(2m\Omega_0)^{-1}$ and Eqs. (35) and (36), one obtains

$$\langle G_2\rangle \approx -\frac{\kappa J_2}{\tilde{\gamma}}\left(2J_2 + \frac{mc^2}{\Omega_0}\right), \tag{38}$$

where $\kappa = (4q^2\Omega_0^3)(3m^2c^5)^{-1}$, and $\tilde{\gamma} = H_0/(mc^2)$. To find $\langle G_1\rangle$, recall that $J_1 = p_z/k - J_2$, and thus $G_1 = \dot{p}_z/k - G_2$, where $\dot{p}_z$ can be taken from Eq. (37):

$$\dot{p}_z \approx \frac{cg^z}{\tilde{\gamma}} = -\frac{4q^2\Omega_0^3}{3m^2c^5\tilde{\gamma}}k(J_1 + J_2)J_2. \tag{39}$$

Therefore,

$$\langle G_1\rangle = -\frac{\kappa J_2}{\tilde{\gamma}}\left(J_1 - J_2 - \frac{mc^2}{\Omega_0}\right). \tag{40}$$

### 3.3. Stationary points and dissipative dynamics

Consider now how the dissipative force $\boldsymbol{G}$ affects the average dynamics, taking into account that the particle remains attached to the stationary curve $\bar{S}$, or $\bar{J}_2^*(J_1)$, that we found in Ref. [10] (Fig. 1). Since

$$\partial\langle G_2\rangle/\partial J_2 < 0, \tag{41}$$

the dissipation makes this curve an attractor (as discussed in Section 2), in contrast to some other dynamical systems, in which the radiation friction can lead to an instability [13]. The stationary points $\mathcal{A}^{(j)} \equiv (J_1^{(j)}, J_2^{(j)})$ on this attractor are found from its intersection with the curve $\langle G_1\rangle = 0$, which, according to Eq. (40), represents a straight line in $(J_1, J_2)$ space, given by

$$J_1 - J_2 = mc^2/\Omega_0. \tag{42}$$

Depending on the wave refraction index, $n_0 \equiv kc/\omega_0$, there can be one or two stable stationary points, as seen in Fig. 1 [10]. Using Eq. (42) together with the equation for $S$ [10], one can find $\mathcal{A}^{(j)}$ explicitly to the leading order in $\varepsilon$, at least for $\omega_0$ close to $\Omega_0$ and

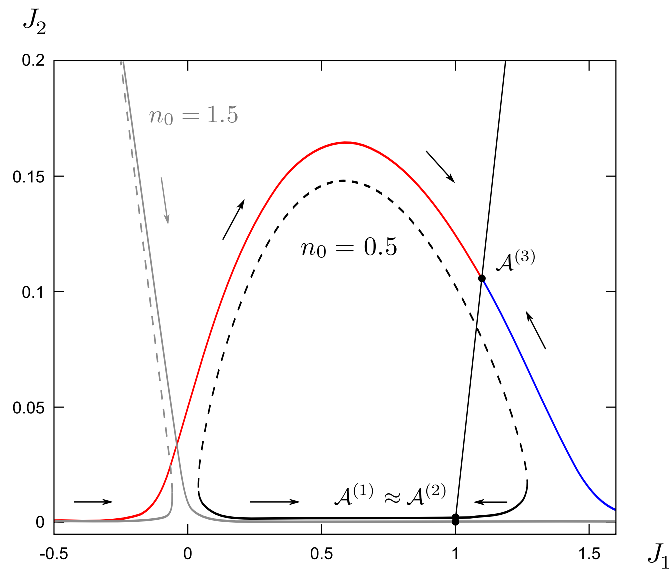


**Fig. 1.** (Color online.) Two stationary curves plotted for $n_0 = 0.5$ and $n_0 = 1.5$ for $\varepsilon = 0.01$, $\omega = 0.98\Omega_0$ (in units $m = q = c = 1$). The stable parts of the curves are solid and the unstable parts are dashed. The curve $G_1(J_1, J_2) = 0$, which is a straight line here, intersects the stationary curves for $n_0 = 1.5$ and $n_0 = 0.5$ in one and two points correspondingly. The direction of the dissipation-driven drift along the stationary curves is shown with arrows.

$|n_0 - 1| \ll 1$. Specifically, at $n_0 > 1$, there is only one stationary point, $\mathcal{A}^{(1)} = (J_1^{(1)}, J_2^{(1)})$, with

$$J_1^{(1)} \approx mc^2/\Omega_0, \tag{43}$$

$$J_2^{(1)} \approx \frac{\varepsilon^2(3 + 2\sqrt{2})}{8m^2c^4\Omega_0^2}, \tag{44}$$

whereas at $n_0 < 1$, there exist two such points, $\mathcal{A}^{(2)} \approx \mathcal{A}^{(1)}$ and $\mathcal{A}^{(3)} = (J_1^{(3)}, J_2^{(3)})$, with:

$$J_1^{(3)} \approx \frac{mc^2\Omega_0}{4\omega^2(1 - n_0)}, \tag{45}$$

$$J_2^{(3)} \approx \frac{mc^2}{\Omega_0}\left[\frac{\Omega_0^2}{4\omega^2(1 - n_0)} - 1\right]. \tag{46}$$

Since

$$\partial\langle G_1\rangle/\partial J_1 < 0, \tag{47}$$

*each* of the stationary points $\mathcal{A}^{(j)}$ ($j = 1, 2, 3$) is stable, i.e., attracts trajectories lying in its vicinity (Fig. 2). Hence, the particle response to the radiation friction in the system considered here can be summarized as follows: First, the particle is picked up by the wave and is accelerated by the light pressure. Yet, the altered longitudinal velocity affects the detuning from the cyclotron resonance and, thus, also the energy of wave-driven oscillations. On the other hand, the latter energy can either decrease or increase, depending on the initial conditions, because the stationary curve $J_2^*(J_1)$ is a multi-valued function (Fig. 1). Specifically, if a particle is originally closer to the branch connected to $\mathcal{A}^{(1)}$ (at $n_0 > 1$) or $\mathcal{A}^{(2)}$ (at $n_0 < 1$), then it will be further attracted to this branch and follow it toward the equilibrium, *losing* the transverse energy. On the other hand, if a particle is instead closer to the branch connected to $\mathcal{A}^{(3)}$ (this is only possible at $n_0 < 1$), it will follow the branch toward higher $J_2$, thereby *increasing* the transverse energy (Figs. 1 and 2).

The time scale for these processes can be estimated as follows. To the stationary curve, a particle is attracted on the time scale

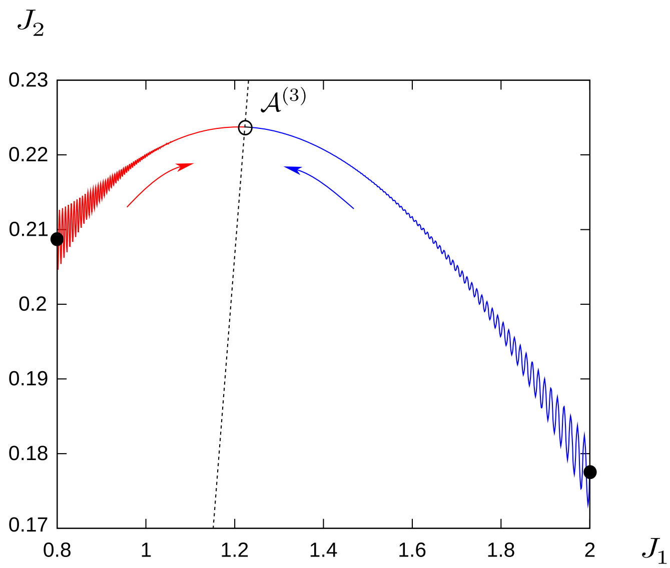


**Fig. 2.** (Color online.) Two trajectories of a particle in $(J_1, J_2)$ space obtained by numerical integration of the motion equations corresponding to the Hamiltonian $H$ [Eq. (28)] and added radiation friction $g$ [Eqs. (35)–(37)]. In this example, $n_0 = 0.5$, $\omega = 0.97\Omega_0$, $\varepsilon = 0.01$, and $m = c = q = 1$. Two initial particle states are shown with solid disks at $J_1 = 0.8$ and $J_1 = 2.0$. Both particle trajectories asymptotically converge (see the arrows) to the intersection (circle) of the stationary curve and the curve given by $G_1 = 0$ (dashed), yielding a stationary point $\mathcal{A}^{(3)}$ [cf. Fig. 1].

$\delta t_\Lambda$ derived from Eq. (17), with $\langle G_2\rangle \approx -\kappa J_2 mc^2/\Omega_0$ [Eq. (38)], where we assumed, for simplicity, that the particle motion is nonrelativistic. Then,

$$\delta t_\Lambda \sim \frac{\Omega_0}{mc^2\kappa} \approx 2.6\,\text{s} \times \frac{M^3}{Z^4 B_0[\text{T}]^2}, \tag{48}$$

where $M$ is the particle mass in the units of the electron mass, $Z$ is the particle charge in the units of the electron charge, and $B_0[\text{T}]$ is the static magnetic field measured in Teslas. For the motion along the curve, the time scale $\delta t$ is found from:

$$\frac{\dot{J}_2}{J_2} = \frac{d\bar{J}_2^*}{dJ_1}\frac{\dot{J}_1}{J_2} = -\frac{\kappa}{\tilde{\gamma}}\frac{d\bar{J}_2^*}{dJ_1}\left(J_1 - J_2 - \frac{mc^2}{\Omega_0}\right) \approx \frac{d\bar{J}_2^*}{dJ_1}\frac{mc^2\kappa}{\Omega_0\tilde{\gamma}}. \tag{49}$$

When $|d\bar{J}_2^*/dJ_1| \sim 1$ (on the negative-mass branch), one has $\delta t \sim \delta t_\Lambda$; however, on the low-energy, quasi-flat branch $\bar{J}_2^*(J_1)$, $\delta t$ can be much larger.

### 3.4. Negative-mass effect with radiation friction

Now let us study whether the wave–particle interaction leads to wave damping or amplification. As shown in Section 2.2, the wave–particle interaction in the presence of friction can lead to wave damping ($\dot{I}_w < 0$) or wave amplification ($\dot{I}_w > 0$). Assuming that the particle is nonrelativistic (or weakly relativistic), one can rewrite the condition for amplification, $\dot{I}_w > 0$ using Eq. (25) as $d\bar{J}_2^*/dJ_1 < \langle G_2\rangle/\langle G_1\rangle$. Using Eqs. (38) and (40), this also rewrites approximately as

$$\frac{d\bar{J}_2^*}{dJ_1} < \frac{2J_2 + mc^2/\Omega_0}{J_1 - J_2 - mc^2/\Omega_0}. \tag{50}$$

Interestingly, Eq. (50) coincides with the necessary and sufficient condition for the particle "effective parallel mass" $m_\parallel$ to be negative; cf. Eq. (56) in Ref. [10]. Hence, we can summarize the results of this section also as follows. All particles with negative parallel mass exhibit slow drift along the stable stationary curve, driven by the light pressure and accompanied by the wave amplification. The corresponding dynamics turns out to be "stable", in respect that the particle transverse energy is monotonically decreasing due to cyclotron cooling. On the other hand, some of particles with positive parallel mass exhibit "unstable" dynamics, i.e., while the particles are accelerated by the light pressure, their transverse energy also grows due to cyclotron heating. The characteristic time of the particle drift along the stable stationary surface $\bar{S}$ is given by Eq. (49). At $B \sim 1$ T, it is of order of seconds for electrons and $10^9$ s for ions. Hence, we can conclude that the NME predicted in Refs. [8] and [10] persists through dissipation and thereby represents a robust physical effect, potentially observable in experiment.

## 4. Conclusions

In this Letter, we showed that the effect of radiation friction can be included in the Hamiltonian treatment of wave–particle interactions with autoresonant phase-locking, yielding a generalized canonical approach to the problem of dissipative dynamics near a nonlinear resonance. As an example, the negative-mass effect exhibited by a charged particle in a pump wave and a static magnetic field is studied in the presence of the friction force due to cyclotron radiation. Particles with negative parallel masses $m_\parallel$ are shown to transfer their kinetic energy to the pump wave, thus amplifying it. Counterintuitively, such particles also undergo stable dynamics, decreasing their transverse energy monotonically due to cyclotron cooling, whereas some of those with positive $m_\parallel$ undergo cyclotron heating instead, extracting energy from the pump wave.

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