Amended conjecture on an upper bound to time-dependent space-charge limited current

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Notwithstanding the recent conjecture that the upper bound on the time-averaged current across a space-charge-limited diode is equal to the steady state Child-Langmuir limit (J_{CL}), Zhu and Ang used a one-dimensional (1D) particle in cell (PIC) code to show that in the regime where space charge effects limit the current to only a few electrons at a time, the time-averaged current can exceed J_{CL} by up to 13% [Y. Zhu and L. K. Ang, Appl. Phys. Lett. **98**, 051502 (2011)]. These results are, in fact, verified using our own 1D PIC code. However, the increase in the current is due to special boundary conditions that pertain in this regime and not to the time dependence of the current. To rule out discreteness effects, the conjecture on the upper bound may be reformulated to include only the case when the electric field at the cathode does not fall below zero. © *2012 American Institute of Physics*. [doi:10.1063/1.3671961]

The Child-Langmuir Law[1] gives the space-charged limited current in the classical problem of a 1-D diode

$$J_{CL} = \frac{4}{9}\epsilon_0\sqrt{\frac{2q}{m}}\frac{V^{3/2}}{d^2}. \tag{1}$$

Many interesting generalizations of this effect have been considered, particularly with respect to geometry,[2–4] non-zero injection velocities,[5] and relativistic[6,7] and quantum effects.[8,9] Time-dependent problems have also been studied, for example short current pulses[10,11] and time-varying voltage drops to control startup transients.[12,13]

In a previous paper,[14] we considered a diode consisting of a cathode at $x = 0$ that emits cold electrons with zero initial velocity, an anode at $x = d$, and a voltage difference V between them. In steady state, the maximum current density that can pass through the diode is given by the Child-Langmuir limit, Eq. (1), and occurs when space charge in the diode causes the electric field at the cathode to fall to zero. We considered the case where the supply of electrons at the cathode, modelled as a fluid and in the one-dimensional limit, could be controlled as a function of time so that the current emission could be selectively repressed before the space charge limit was reached. The boundary conditions for this situation are

$$qE(x = 0) \geq 0. \tag{2}$$

We were interested in whether this extra flexibility in the problem allowed for the time-averaged current to exceed the steady state limit. We did provide an upper bound, but not as low an upper bound as the steady state limit, J_{CL}. However, simulations with a 1D particle in cell (PIC) code led us to conjecture that the time-averaged current (for $t \to \infty$) could not exceed J_{CL}.

Zhu and Ang[15] considered space charge emission in the "coulomb blockade regime" which is similar to the configuration described above. The difference is that the voltage across the diode is so low that the space charge limit is reached when several electrons at a time pass through the diode. In both cases, the electrons are emitted with zero initial velocity and the voltage across the diode is constant in time. Because the charge can only be emitted in discrete quantities equal to e, the charge of one electron, the current emitted from the cathode is inherently time-dependent. They defined the threshold voltage, $V_{th} = ed/2\epsilon_0 A$, where A is the area of the diode, as the voltage below which even a single electron cannot be injected into the diode because of space charge limitation. Using a 1D code, they found that for diode voltages in the range of $1 \leq V/V_{th} \leq 4$, the time-averaged current can exceed J_{CL} by up to 13%. We note that in order to realistically model single electrons, three-dimensional effects should be taken into account, so that the electric field of the charges and image charges fall off as $(x - x_0)^{-2}$ instead of being constant in space. Neither Zhu and Ang's model nor our model includes this effect. However, three-dimensional simulations by Pedersen *et al.*[16] showed that a regime similar to the Coulomb blockade regime exists when many electrons are emitted in sheets from a two-dimensional cathode at low voltage. In this regime, electrons are emitted in bursts of tens to hundreds of electrons at a time which occur at a regular frequency even though all applied boundary conditions are held constant. In view of this result, it is of particular interest as well to investigate space charge effects in a simplified one-dimensional model.

We have verified the Zhu and Ang results using our own 1-D PIC code. We also determined that the mechanism for the increase above J_{CL} is not the time dependence of the current injection, but rather it is special boundary conditions that apply in the few electron regime. In this regime, the space-charge electric field of a single electron charge sheet, $e/2\epsilon_0 A$, is not small compared to the total electric field in the diode. In both 1D models, electrons are treated as charge sheets that are vanishingly thin in the x-dimension so there is a discontinuity in the electric field at the location of each electron, x_i. The field just behind an electron, $E(x_i - \delta) = \bar{E}(x_i) - e/2\epsilon_0 A$, is different from the field just in front of that electron, $E(x_i + \delta) = \bar{E}(x_i) + e/2\epsilon_0 A$, because of the reversal in sign of the space charge electric field of the electron. The force acting on each electron is proportional to the average of the electric field just in front and just behind that electron, $\bar{E}(x_i)$. Because of these discontinuities, the electric field at the cathode is substantially different from the electric field acting on an electron that has just been emitted. For an electron to be emitted, the electric field at the cathode must be smaller than zero (push electrons toward the anode). When this electron is emitted, the electric field at the cathode can rise above zero (push electrons toward the cathode) even though the force on the electron is still towards the anode. Thus, the boundary condition at the cathode is

$$qE(x=0) \geq \frac{-q^2}{2\epsilon_0 A}, \tag{3}$$

where $q=-e$ is the electron charge. These physically correct boundary conditions allow more charge into the diode than would be possible, if the charge were a continuous fluid. The breakdown of the traditional fluid boundary conditions in this regime explains why the average current density can increase over J_{CL}.

It is of great interest to determine whether the increase over J_{CL} is peculiar to the Coulomb blockade regime or whether it can be applied more generally. If the increase was due to the time dependence of the current, this might allow for J_{CL} to be exceeded in the many electron regime by, for example, using a photodiode to supply discrete bunches of electrons to mimic the time dependence of the Coulomb blockade regime. However, if, as we claim, the mechanism is the modified boundary conditions, then the increase over J_{CL} will vanish in the many electron limit. To demonstrate that the increase over J_{CL} is due to the boundary conditions, we performed two sets of simulations using different boundary conditions. In both cases, the time-dependence of the current injection was determined by the discrete nature of the charge sheets. The charge sheets can only be emitted with surface charge densities that are multiples of a minimum value, $e/A = 2\epsilon_0 V_{th}/d$, which represents the discrete charge of an electron in our 1D model. The charge sheets are emitted as soon as it is possible to do so without causing the electric field at the cathode to increase above the maximum value allowed by the boundary conditions. The difference between the two sets of simulations is that one set was run using "coulomb blockade boundary conditions" given by Eq. (3), while the other set was run using "traditional boundary conditions" given by Eq. (2). The traditional boundary conditions are not a correct description of the Coulomb blockade regime; however, they would apply if the same time dependent current injection of a diode in the Coulomb blockade regime was somehow mimicked in a diode with many electrons. The results of the simulation are shown in Fig. 1. The traditional boundary conditions double the threshold voltage for emission so that nothing can be emitted unless $V/V_{th} > 2$. We see that when traditional boundary conditions are used, the average current remains below J_{CL}. Accordingly, we propose the amended conjecture that J_{CL} cannot be exceeded on average (for $t \to \infty$) even for time-varying fields so long as $qE(x=0) \geq 0$. This conjecture as amended still enjoys numerical support, yet only a considerably larger upper bound has to date been analytically demonstrated.

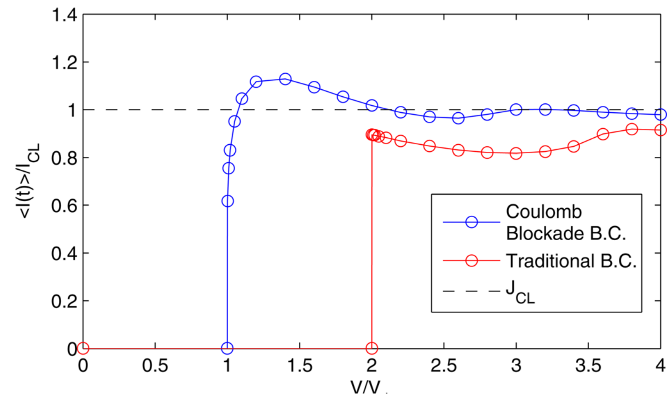

FIG. 1. (Color online) Time averaged current for simulations using coulomb blockade boundary conditions and traditional boundary conditions compared to J_{CL}, the steady state limit.

Another interesting feature of Fig. 1 is that the average current varies non-monotonically as V/V_{th} is varied for both boundary conditions. What appear as "oscillations" may occur because the timing of the emission of each charge sheet varies relative to when other charge sheets (that were previously emitted) exit the diode. The transit time of a charge sheet is most strongly affected by the acceleration it experiences soon after its emission because the velocity gained early on will carry the sheet forward for the duration of its transit, and faster transit times will result in larger average currents. The force on every charge sheet at the instant of emission is equal to zero, but the rate at which the force increases depends in part on when other charge sheets exit the diode, providing a sharp increase in the field. Thus, the relative timing of charge emission can shift in and out of its optimal value as V/V_{th} is varied.

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