Negative-Mass Instability in Nonlinear Plasma Waves

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The negative-mass instability, previously found in ion traps, appears as a distinct regime of the sideband instability in nonlinear plasma waves with trapped particles. As the bounce frequency of these particles decreases with the bounce action, bunching can occur if the action distribution is inverted in trapping islands. In contrast to existing theories that also infer instabilities from the anharmonicity of bounce oscillations, spatial periodicity of the islands turns out to be unimportant, and the particle distribution can be unstable even if it is flat at the resonance. An analytical model is proposed that describes both single traps and periodic nonlinear waves and concisely generalizes the conventional description of the sideband instability in plasma waves. The theoretical results are supported by particle-in-cell simulations carried out for a regime accentuating the negative-mass instability.

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Introduction.—It is well known that bounce oscillations of particles autoresonantly trapped in a wave can couple to wave sidebands, rendering them unstable [1–3]. The sideband instability (SI) was extensively studied in the past [4–8], more recently in application to free electron lasers [9] and storage rings [10], and now is attracting renewed attention [11,12] in the context of intense laser-plasma interactions (LPI) and the associated trapped-particle modulational instability (TPMI) [13], which is the SI's geometrical-optics limit [14]. Yet little effort was paid to unifying SI theories that appeared after the original Kruer-Dawson-Sudan work [1], further termed KDS. As a consequence, their results are often neglected today, and that, in turn, leads to misapplications [15]. Thus, even though quantitative predictions may be better left to simulations in any case, a transparent theory is needed, particularly as a practical tool for interpreting LPI-related numerical data, that would both comprehensively capture and elucidate the SI paradigmatic physics.

Here we propose such a theory for Bernstein-Greene-Kruskal (BGK) waves [16], paradigmatic in the LPI context, in one-dimensional (1D) electron collisionless plasma. We identify a new mechanism of the SI, which is additional to that implied in KDS. We call this mechanism a negative-mass instability (NMI), because of its resemblance to the NMI in accelerators and storage rings [17]. The NMI is not limited to periodic waves and thus can be treated also as an extension of the bunching instability recently found in ion traps [18]. Our analytical model describes both single traps and periodic nonlinear waves and represents a transparent generalization of KDS, reproducing the latter as a limit. In contrast to existing theories that also relate the SI to anharmonicity of bounce oscillations, in the NMI theory, the particle distribution can be unstable even if it is flat at the resonance. Below, we present our theory in detail and support it with results of particle-in-cell (PIC) simulations for a regime accentuating the NMI effect.

Physical mechanism.—For transparency, we will limit our consideration to waves that are initially phase mixed, so the initial trapped population can be characterized by the distribution $F(J)$ of the particle bounce actions, J. [As proved fruitful by Refs. [14,19–21], finding $F(J)$ is treated as an independent problem, not to be addressed here, but see Refs. [19,21].] Even in this case, sidebands are subjected to a whole zoo of instabilities, for which SI serves merely as an umbrella term. A variety of regimes is exhibited already by the KDS model [3], which assumes $F(J) = \delta(J)$. These instabilities feed on the free energy stored in the trapped-particle motion at the wave phase velocity, $\bar{u}$, much like the usual bump-on-tail instability. On the other hand, the distribution *inside* trapping islands can also become unstable by itself, if $F(J)$ is inverted (which occurs naturally [22]) and if the bounce frequency, Ω, is a decreasing function of J (which is typical). The specific mechanism is as follows.

Consider a pair of particles bouncing in the wave potential, i.e., rotating in phase space around a local equilibrium. Through Coulomb repulsion (strictly speaking, via collective fields), the leading particle increases its energy; then it moves to an outer phase orbit and slows down its phase space rotation (as $\Omega' < 0$), whereas the trailing particle moves to a lower orbit and speeds up, correspondingly. This way, mutually repelling particles can undergo phase bunching, or condensation, as if they had negative masses [23]. The condensation may or may not eventually saturate in the form of a stable "macroparticle" [24]; yet, its very formation, which one may expect to be a generic feature of ring-shaped distributions, constitutes a fundamental instability in itself, missed in KDS. We adopt the term NMI to refer to this distinct regime of the SI, by analogy with the well-known bunching mechanism for particles rotating in accelerators and storage rings [17]. A subtle difference, however, is that in our case particles rotate in *phase* space, so what determines the instability is the canonical frequency Ω rather than a physical angular velocity.

Clearly, the bunching mechanism applies to single traps too, thus bridging the SI with a similar instability found in ion traps [18]. To our knowledge, a Vlasov theory of the ion-trap NMI does not exist, so it is worth considering the two instabilities together, drawing on their similarity. As will be seen, this approach is different from both "parametric" theories [1–5], which deduce the SI from mode coupling, and "quasilinear" theories [6], which treat sidebands as independent harmonics and infer the SI as their inverse Landau damping on a nonlinearly perturbed slow part of the velocity distribution.

Single trap.—Let us first consider a single trap absent bulk plasma. Suppose a given 1D static potential well, $U(x)$, which causes an individual particle to oscillate at some nonlinear frequency, $\dot{\theta} = \Omega(J)$; here $(J, \theta) \equiv \Gamma$ are the corresponding action-angle variables. When multiple particles are placed in the same well, their oscillations become perturbed by the collective electrostatic potential, ϕ. For clarity, we will consider only perturbations along the x axis, so the transverse dynamics is decoupled and need not be considered below. The individual-particle Hamiltonian can then be written as $\mathcal{H} = \int_0^J \Omega(J')dJ' + e\phi(x,t)$, where e is the charge, and x is now understood as a function of the phase-space coordinates, $x = X(\Gamma)$. Absent collisions, the particle distribution $f(\Gamma, t)$ is governed by the Vlasov equation,

$$\frac{\partial f}{\partial t} + \left(\Omega + e\frac{\partial \phi}{\partial J}\right)\frac{\partial f}{\partial \theta} - e\frac{\partial \phi}{\partial \theta}\frac{\partial f}{\partial J} = 0, \qquad (1)$$

somewhat unusual as the electrostatic force is now formally a function of both coordinate and momentum. The corresponding Poisson's equation then can be written as

$$\frac{\partial^2 \phi}{\partial x^2} = -4\pi e N t \int_\Gamma d\Gamma\, \delta(x - X(\Gamma))\, f(\Gamma), \qquad (2)$$

where $\mathcal{N}_t$ is the average number of particles per unit surface transverse to the x axis, $\int_\Gamma d\Gamma f(\Gamma) = 1$, $\int_\Gamma d\Gamma \doteq \int_0^\infty dJ \int_{-\pi}^{+\pi} d\theta$, and the symbol $\doteq$ denotes definitions.

Let us search for a solution in the form $f = \bar{f}(J) + \mathrm{Re}\tilde{f}(\Gamma)e^{-i\omega t}$, where $\tilde{f} = \sum_m \tilde{f}_m(J)e^{im\theta} \ll \bar{f}$, and $\phi = \bar{\phi}(x) + \mathrm{Re}\tilde{\phi}(X(\Gamma))e^{-i\omega t}$. The static part of the potential, $\bar{\phi}$, is determined by $\bar{f}$ and can be eliminated by redefining $U(x)$. Correspondingly, $\tilde{\phi}$ is determined by $\tilde{f}$, which contains no net charge, $\int_\Gamma d\Gamma \tilde{f}(\Gamma, t) = 0$; thus the quiver field, $\tilde{E} \doteq -\partial\tilde{\phi}/\partial x$, vanishes at $x \to \pm\infty$. We will assume that $U(x)$ is even and the phase is defined such that the sign of $\cos\theta$ matches that of X. This gives $\int_0^\infty \delta(x - X(\Gamma))dx = H(\cos\theta)$, where H is the Heaviside step function; hence, integrating Eq. (2) over x from zero to infinity yields $\mathcal{E} = -4\pi e\mathcal{N}_t \int_0^\infty dJ \int_{-\pi/2}^{+\pi/2} d\theta \tilde{f}(\Gamma)$, where $\mathcal{E} \doteq \tilde{E}(x = 0)$. From the linearized Eq. (1), we have $\tilde{f}_m = -em\tilde{\phi}_m\bar{f}'/(\omega - m\Omega)$, and thus

$$\mathcal{E} = 8\pi e^2\mathcal{N}_t \sum_m \sin\left(\frac{\pi m}{2}\right)\int_0^\infty \frac{\tilde{\phi}_m(J)\bar{f}'(J)}{\omega - m\Omega(J)}dJ. \qquad (3)$$

The integration contour can be taken along the real axis in J space if $\gamma \doteq \mathrm{Im}\omega > 0$ but otherwise must be understood as a Landau contour in the J complex plane.

To simplify the right-hand side in Eq. (3), let us use $\tilde{\phi}_m(J) = \int_{-\pi}^{+\pi} \tilde{\phi}(X(\Gamma))e^{-im\theta}d\theta/(2\pi)$ and replace the potential with its Taylor series, $\tilde{\phi}(x) = \sum_{\ell=0}^\infty q_\ell x^\ell$; in particular, $q_1 = -\mathcal{E}$. We will assume that oscillations are close to linear, with some constant frequency Ω_0; hence $X(\Gamma) \approx A\cos\theta$, where $A = [2J/(M\Omega_0)]^{1/2}$, and M is the particle mass. Then, $\tilde{\phi}_m = -(\delta_{m,1} + \delta_{m,-1})A\mathcal{E}/2$, and Eq. (3) yields the following dispersion relation:

$$1 + 2\omega_t^2\sqrt{J_0}\int_0^\infty \frac{F'(J)\sqrt{J}}{\omega^2 - \Omega^2(J)}dJ = 0. \qquad (4)$$

Here we replaced $\Omega(J)$ with Ω_0 in the numerator and defined $F(J) \doteq 2\pi\bar{f}(J)$, so $\int_0^\infty F(J)dJ = 1$. Also, x_0 is the maximum amplitude of the unperturbed oscillations, so $J_0 \doteq M\Omega_0 x_0^2/2$ is their maximum action [i.e., $\bar{f}(J > J_0) = 0$]; $\bar{n}_t \doteq \mathcal{N}_t/(2x_0)$ is the trapped-particle average density; $\bar{\omega}_t \doteq (4\pi\bar{n}_t e^2/M)^{1/2}$ is the characteristic plasma frequency, and we introduced $\omega_t \doteq \bar{\omega}_t(2/\pi)^{1/2}$.

Suppose a ring distribution, $F(J) = \delta(J - J_0)$. For harmonic bounce oscillations $[\Omega(J) = \Omega_0]$, Eq. (4) yields $\omega^2 = \Omega_0^2 + \omega_t^2$; this is understood, because then ω_t happens to equal the local plasma frequency at $x = 0$, which must also be the eigenfrequency at vanishing Ω_0. Consider now nonzero $\alpha \doteq -\Omega'(J_0)J_0/\Omega_0$. In this case, Eq. (4) rewrites as $w^2 - \beta w + 4\alpha\beta = 0$, where $w \doteq (\omega/\Omega_0)^2 - 1$ and $\beta \doteq \omega_t^2/\Omega_0^2$. We will assume $\alpha \ll 1$ and $\beta \ll 1$, implying $\phi \ll U$ [3]; yet, for simplicity, we will also adopt $b \doteq \beta/(16\alpha) \ll 1$, so there are *very* few trapped particles. Then $w \approx \beta/2 \pm 2(-\alpha\beta)^{1/2}$, and thus $\omega \approx \Omega_0 + \delta\omega \pm \omega_t(-\alpha)^{1/2}$, where $\delta\omega = \omega_t^2/(4\Omega_0)$. Hence, having $\alpha > 0$, which corresponds to $\Omega' < 0$, leads to an instability with $\gamma \approx \omega_t\alpha^{1/2}$. This is the NMI cold limit.

Clearly, a nonzero width of the distribution, J_T, corresponding to the thermal spread of natural frequencies $\Delta\Omega \sim \Omega' J_T$, cannot affect the growth rate if $\Delta\Omega \ll \gamma$. (The effect on the real frequency shift is insignificant, as $\delta\omega \ll \gamma$). The latter rewrites as $J_T^2/J_0^2 \lesssim 16b$, so it can be satisfied even at $J_T \sim J_0$ due to the large numerical coefficient on the right-hand side. Thus, except at too small b, Eq. (4) yields $\gamma \sim \omega_t\alpha^{1/2}$ for almost any inverted $F(J)$, and it does not matter what part of the distribution realizes the exact resonance, $\Omega(J) = \Omega_0 + \delta\omega$. One can as well ascertain this numerically or by recalculating the dispersion relation for test cases such as a Gaussian or rectangular distribution, $F(J) = H(J_0 - J)H(J - J_0 + J_T)/J_T$.

Periodic BGK wave.—Now consider a sinusoidal BGK wave with amplitude $\bar{E}$, wave number $\bar{k} \equiv 2\pi/\bar{\lambda}$, and nonlinear frequency $\bar{\omega}(\bar{k}, \bar{E})$ [19,21]. In the frame traveling at (nonrelativistic) velocity $\bar{u} = \bar{\omega}/\bar{k}$, the medium is periodic and stationary, so any perturbation with a well-defined frequency ω' is a Bloch-Floquet wave. In particular, the charge density becomes $\tilde{\rho} = e^{-i\omega' t + ik'x'} \sum_\ell \tilde{\rho}_\ell e^{i\ell\bar{k}x'}$, where $x' = x - \bar{u}t$, ℓ hereupon spans from $-\infty$ to $+\infty$, and the constant k' is a quasi-wave-vector, which is restricted to the first Brillouin zone; i.e., $\kappa \doteq k'/\bar{k}$ satisfies $|\kappa| < 1/2$. In terms of x and t, this gives $\tilde{\rho} = \sum_\ell \tilde{\rho}_\ell e^{-i\omega_\ell t + ik_\ell x}$, where $\omega_\ell \doteq \omega + \ell\bar{\omega}$, $k_\ell \doteq k + \ell\bar{k}$, $\omega \doteq \omega' + k'\bar{u}$, and $k \doteq k'$. Gauss's law yields then $\tilde{E}_\ell = 4\pi\tilde{\rho}_\ell/(ik_\ell\epsilon_\ell)$, where $\epsilon_\ell \doteq \epsilon(\omega_\ell, k_\ell) = \epsilon(\omega' + (\ell + \kappa)\bar{\omega}, (\ell + \kappa)\bar{k})$. As in Ref. [1], we assume that the bulk plasma response is modeled with a linear dielectric function, say, $\epsilon(\omega, k) \approx 1 - \omega_p^2/(\omega^2 - 3k^2 v_T^2)$, where $\omega_p \doteq (4\pi n e^2/M)^{1/2}$, n is the bulk electron density, v_T is the electron thermal speed, and $\varkappa \doteq \bar{k}v_T/\omega_p \ll 1$.

Now use $\tilde{\rho}_\ell = L^{-1}\int dx' \tilde{\rho}(x', t) e^{i\omega' t - ik_\ell x'}$, where L is the plasma length, and substitute $\tilde{\rho} = \sum_j r_j e^{-i\omega' t}$, where $r_j = e\mathcal{N}_t[\int_\Gamma d\Gamma\, \delta(x' - j\bar{\lambda} - X(\Gamma))\tilde{f}(\Gamma)]_j$ are the contributions of individual islands, and $\tilde{f}$ can be taken from the single-trap problem. We will assume again that most significant are oscillations near the trapping-island center, so $k_\ell X(\Gamma) \ll 1$. (This overestimates the contribution of high-ℓ harmonics, but see below.) A straightforward calculation yields then $\tilde{\rho}_\ell = -(ik_\ell/4\pi)\bar{\omega}_t^2 \mathcal{J}\mathcal{A}$. Here $\bar{\omega}_t \doteq (4\pi\bar{n}_t e^2/M)^{1/2}$, like before, with the average trapped density being $\bar{n}_t = \mathcal{N}_t/\bar{\lambda}$; also, $\mathcal{J} \doteq \int_0^{J_*} dJ JF'(J)/[\omega'^2 - \Omega^2(J)]$, J_* is the separatrix action [so $F(J > J_*) = 0$], and $\mathcal{A} = (\bar{\lambda}/L)\sum_j \mathcal{E}_j e^{-ik_\ell j\bar{\lambda}}$, where $\mathcal{E}_j = \tilde{E}(x = j\bar{\lambda})$. Using Gauss's law, we hence arrive at $\tilde{E}_\ell = -\bar{\omega}_t^2\mathcal{J}\mathcal{A}/\epsilon_\ell$. On the other hand, $\mathcal{E}_j = \sum_{\ell'} \tilde{E}_{\ell'} e^{ik_{\ell'} j\bar{\lambda}}$ yields $\mathcal{A} = \sum_{\ell'}\tilde{E}_{\ell'}$, which does not depend on ℓ. Summing over all relevant ℓ then leads to

$$1 + \frac{\bar{\omega}_t^2}{\epsilon_{\text{eff}}}\int_0^{J_*} \frac{JF'(J)}{(\omega - k\bar{u})^2 - \Omega^2(J)}\, dJ = 0, \qquad (5)$$

where $1/\epsilon_{\text{eff}} \doteq \sum_\ell 1/\epsilon(\omega_\ell, k_\ell)$. As a side note, the amplitudes of individual harmonics are thereby locked ($\epsilon_\ell \tilde{E}_\ell = \text{const}$). This means, contrary to a popular misconception, that a sideband wave with well-defined ω and k has neither a single frequency nor a single wave vector, but rather consists of multiple harmonics with different (ω_ℓ, k_ℓ); in particular, all of them have identical growth rates, $\text{Im}\omega_\ell = \text{Im}\omega \equiv \gamma$, and similarly for $\text{Im}k_\ell$.

Absent plasma, the number of harmonics contributing to ϵ_{eff} is about $\bar{\lambda}/x_0$; then Eq. (4) is recovered, at least qualitatively. Plasma, in contrast, accentuates harmonics with small ϵ_ℓ, which is realized (assuming that nonlinear effects are weak, so $\omega' \ll \bar{\omega}$) at $(\ell + \kappa)\bar{\omega} \approx \pm\omega_p$. Since $\bar{\omega}$ is itself close to $\pm\omega_p$, that requires small κ and $\ell = \pm 1$. Following KDS, we retain only these resonant terms, so

$$1/\epsilon_{\text{eff}} = 1/\epsilon(\omega - \bar{\omega}, k - \bar{k}) + 1/\epsilon(\omega + \bar{\omega}, k + \bar{k}). \qquad (6)$$

If one redefines ω and k according to the (less symmetric) notation adopted in Ref. [1], the KDS result is hence reproduced from Eq. (5) as a limiting case corresponding to $\Omega'(J) = 0$, through integration by parts. [That being said, realistic $\Omega'(J)$ is nonnegligible even at zero J, so KDS is not the universal cold limit here; the shape of $F(J)$ matters even when its width is vanishingly small.]

Contrary to Ref. [2], the stationary-wave dispersion, $\bar{\omega}(\bar{k}, \bar{E})$, need not be derived separately, for it is already contained in the model as a limit. Since $\alpha \ll 1$ is assumed, substituting $\omega' = 0$ and $k' = 0$ into Eqs. (5) and (6) yields, within the accuracy that we have adopted,

$$\epsilon(\bar{\omega}, \bar{k}) + 2\bar{\omega}_t^2/\Omega_0^2 \approx 0, \qquad (7)$$

in agreement with the adiabatic theory [19]. This completes the set of equations [Eqs. (5)–(7)] generalizing the KDS model. The growth rate that flows from those is determined by both the KDS effect, yet quantitatively modified now because of non-constant $\Omega(J)$, and the NMI. As the NMI remains a *cold* instability, its rate is primarily determined by the integral principal value, so a wave can be unstable even if $F(J)$ is flat at the resonance. This, in particular, leads to a very different γ than the one that Ref. [5] attributes [25] to the resonance pole in an equation resembling Eq. (5). Below, we discuss results of numerical simulations that support our predictions.

Numerical results.—To illustrate the NMI and unambiguously validate the difference between our Eq. (5) and KDS (rather than to mimic a specific experiment), we performed 1D PIC simulations under conditions that controllably accentuate these effects. A self-consistent phase-mixed electron plasma wave was seeded, with ions modeled as a homogeneous background. We then emulated [26] plasma compression perpendicularly to the wave vector. During this compression, $\bar{n}_t/n \equiv \tau$, $\bar{k}$, and v_T remain fixed, but $\bar{\omega} \sim \omega_p(t)$ grows as $N^{1/2}$, where $N \doteq n(t)/n_0$. (The index 0 hereupon denotes initial values.) Electrons that were trapped initially are then accelerated such that their average velocity remains equal to $\bar{u}(t)$, so the trapping island detaches from the bulk distribution. The wave electrostatic energy density, $\mathcal{W}$, is adiabatically amplified through compression and then decays, as explained in Ref. [27]. Here we are interested in only the initial stage, when $\mathcal{W}$ grows; each trapped particle then preserves its J, but $J/J_*(t)$ decreases, so a deeply trapped ring-shaped distribution is formed.

Specifically, we start out with $\varkappa_0 \approx 0.26$, $\tau_0 \approx 6 \times 10^{-4}$, and $[\bar{n}_t m\bar{u}^2/(4\mathcal{W})]_0 \approx 0.29$. As we use periodic boundary conditions, only $k = 2\pi p/L$ is allowed,

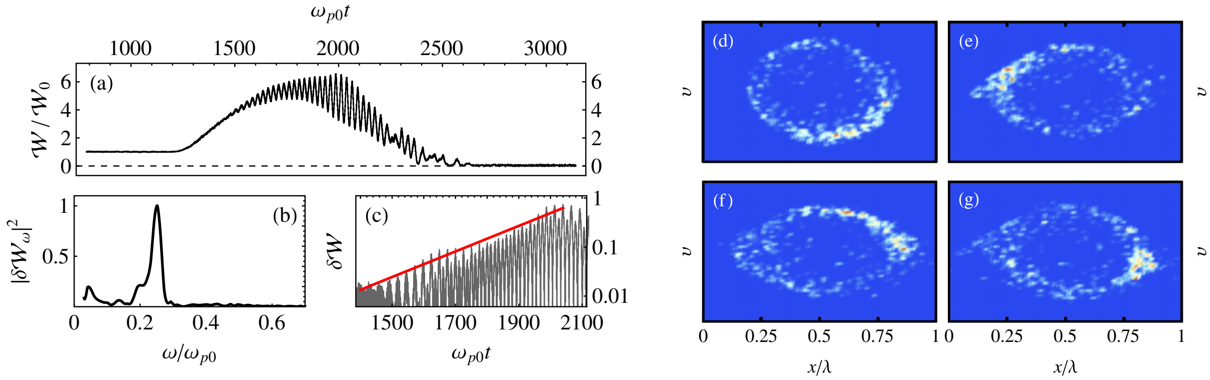

FIG. 1 (color online). (a) Electrostatic energy density, $\mathcal{W}(t)$, averaged over the simulation box. (b) Fourier spectrum of $\delta\mathcal{W}(t) \doteq \mathcal{W} - \langle\mathcal{W}\rangle$ on the time interval $1700 \leq \omega_{p0}t \leq 2400$; here $\langle\mathcal{W}\rangle$ is the moving time average. The peak is at the characteristic Ω, almost constant near the peak $\langle\mathcal{W}\rangle$. (c) Close up of (a) in logarithmic scale; straight line (red) is an exponential fitting corresponding to $\gamma = 0.006\omega_{p0}$. (d)–(g) Consecutive snapshots of the trapped-particle phase space illustrating bunch formation (the passing distribution, not shown, is beyond the figure limits). Arbitrary units; the index 0 denotes initial values.

where p is integer; then, $|\kappa| < 1/2$ leads to $|p| < \bar{k}L/(4\pi)$. We operate at the lowest spatial mode ($\bar{\lambda} = L$), so the inequality becomes $|p| < 1/2$, and, for integer p, this means $p = 0$; i.e., $\kappa = 0$. (Although the quasi-wave-vector is zero in this case, the perturbation, being a Bloch wave, yet contains spatial harmonics with wave vectors $\ell\bar{k}$). Under these conditions, the KDS model predicts zero γ (and so does Ref. [5]), but sideband amplification from noise is nevertheless observed [Fig. 1(a)]. Quantitative assessment of the instability is hindered by the fact that the plasma is nonstationary. Still, the inferred characteristic values, $\omega/\omega_{p0} \approx 0.25$ and $\gamma/\omega_{p0} \approx 0.006$ [Figs. 1(b) and 1(c)], agree with the theory (which predicts $\omega/\omega_{p0} \sim 0.2$ and $\gamma/\omega_{p0} \sim 0.01$ throughout the whole process), and phase bunching is observed indeed [Figs. 1(d)–1(g)]. We also theoretically calculated ω for realistic $F(J)$ at the specific t when $\mathcal{W}$ is at its maximum ($\mathcal{W}/\mathcal{W}_0 \approx 5$, $N \approx 5.4$). Both the real and imaginary parts of ω inferred from Eqs. (5) and (6) match the observed values within a few percent.

Discussion.—As the domain of certain quantitative validity for our theory is limited to particles trapped at orbits deeper than in the numerical example above, the high precision with which the theory matches our simulations may be, to some extent, accidental. What is actually important, however, is that we have been able to predict correctly the qualitative dynamics observed in *ab initio* simulations. This means that our theory adequately describes the leading-order thermal effects in the SI and thus can be used as an advancement of KDS, whether or not the NMI is present. This responds to the long-standing need for a transparent theory that would permit *analyzing* the effect of the trapped-particle distribution on the SI, at least partially, rather than solely relying on simulations or precariously ascribing to KDS the generality that the latter cannot possess in principle (see also Ref. [14]), as is often done in literature.

In particular, on the score of our theory being transparent, we have been able to identify a new mechanism of the SI, which is additional to that implied by KDS, and which we term NMI because of the resemblance to the NMI in accelerators and storage rings. This instability is caused by phase bunching of particles bouncing within trapping islands. Contrary to existing theories that also infer BGK-wave instability from the anharmonicity of bounce oscillations, spatial periodicity of the islands turns out to be unimportant for the NMI, and the particle distribution can be unstable even if it is flat at the resonance. An analytical model is proposed which describes both periodic nonlinear waves, thus generalizing KDS, and also single trapping islands, thus relating the SI to a similar instability recently found in ion traps. The theoretical results are supported by PIC simulations carried out for a regime accentuating the NMI effect.

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