PHYSICS OF PLASMAS **20**, 102105 (2013)

On plasma rotation induced by waves in tokamaks

Xiaoyin Guan,[1] I. Y. Dodin,[1] Hong Qin,[1,2] Jian Liu,[2] and N. J. Fisch[1]
[1]*Princeton Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08543, USA*
[2]*Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

(Received 28 June 2013; accepted 10 September 2013; published online 10 October 2013)

The momentum conservation for resonant wave-particle interactions, now proven rigorously and for general settings, is applied to explain in simple terms how tokamak plasma is spun up by the wave momentum perpendicular to the dc magnetic field. The perpendicular momentum is passed *through* resonant particles to the dc field and, giving rise to the radial electric field, is accumulated as a Poynting flux; the bulk plasma is then accelerated up to the electric drift velocity proportional to that flux, independently of collisions. The presence of this collisionless acceleration mechanism permits varying the ratio of the average kinetic momentum absorbed by the resonant-particle and bulk distributions depending on the orientation of the wave vector. Both toroidal and poloidal forces are calculated, and a fluid model is presented that yields the plasma velocity at equilibrium.
 [http://dx.doi.org/10.1063/1.4823713]

I. INTRODUCTION

It is well known that plasma rotation in toroidal geometry can improve confinement, so a question arises how to control this rotation externally. The traditional method relying on neutral-beam injection[1,2] will not suit next-generation tokamaks, such as ITER. Other methods of spinning the plasma need to be found then, and using radiofrequency waves is one of promising possibilities. For example, poloidal flow driven by ion Bernstein waves was observed on TFTR,[3] and toroidal rotation driven by fast magnetosonic waves was reported for JET.[4] More recently, toroidal and poloidal flows driven by mode-conversion heating in the ion-cyclotron radiofrequency (ICRF) range were also observed on Alcator C-Mod for D-^{3}He plasmas.[5,6]

A number of papers calculated these rotations in terms of the ponderomotive force that waves impose on plasma in equilibrium.[6–11] However, those studies do not quite elucidate *how*, and how rapidly, the wave momentum ends up in bulk particles. For instance, it is often assumed that the plasma acquires only the part of the wave momentum that is parallel to the magnetic field, as that one gets picked up by resonant particles that later pass the momentum to the bulk through collisions;[12] the perpendicular momentum is hence considered unimportant.[13–15] It is only recently that alternative interpretations were proposed, independently in Refs. 16 and 17, which suggest that the wave perpendicular momentum can also play a significant or even dominant role. Although leading to relatively concise expressions for plasma fluxes, these models build on the heavy machinery of (appropriately extended) quasilinear theory and are not entirely complete (see below). One may expect then that the momentum transfer between waves and bulk plasma can be given a simpler and further-reaching explanation.

It is the purpose of this paper to propose such an explanation as well as to extend and correct the existing understanding of how the wave perpendicular momentum influences plasma rotation. As in Refs. 16 and 17, we focus on the most controversial, *resonant* momentum transfer, as opposed to the effect of reactive or gradient ponderomotive forces.[18] A general theorem applies in this case that unambiguously relates the wave energy absorbed by a particle to the change of the particle canonical momentum, $\Delta\mathbf{P}$ (in contrast with the aforementioned theories, this theorem is also fully relativistic). The fraction of $\Delta\mathbf{P}$ that is due to the wave parallel momentum input drives a parallel current[32] and is transferred to the bulk plasma in a usual manner, i.e., on a relatively large time scale of hot-particle collisions. The remainder of $\Delta\mathbf{P}$, which is due to the wave perpendicular momentum, cannot be utilized as kinetic momentum and thus does not drive current. Instead, it is accumulated in the form of a Poynting flux, giving rise to the radial electric field that builds up due to charge separation across flux surfaces. (We are at odds with Ref. 17 regarding the direction, but not the magnitude, of the radial current.) The bulk plasma is then immediately accelerated up to the electric drift velocity, a mechanism which is entirely independent of collisions; i.e., in principle, collisions are not needed for a wave to spin up the bulk plasma.

We explicitly calculate the ratio of the associated collisional and collisionless forces on the plasma, which is the fundamental parameter that determines (i) how much current is produced per unit wave momentum[20] and (ii) how rapidly the wave momentum is transferred to the bulk. We find that it is possible, e.g., to accelerate resonant particles but not the bulk or, alternatively, the bulk but not resonant particles, depending on how the wave vector and the magnetic field are oriented with respect to each other. The direction and time history of the bulk acceleration can also be varied. We derive not only the toroidal acceleration (as in Refs. 16 and 17) but also the poloidal acceleration, which we identify as a characteristic feature of wave-plasma resonant interaction. We also present a fluid model, from which the effect of the parallel and perpendicular wave momenta can be estimated and which explains how kinetic models, such as ours and those in Refs. 16 and 17, relate to the earlier "ponderomotive" theories.[6–11]

II. CONSERVATION LAW

As rigorously proved in the Appendix, whenever a particle resonantly gains energy $\Delta\mathcal{E}$ from a wave, it also receives canonical momentum $\Delta\mathbf{P}$ such that its projection on every cyclic axis x^h is given by

$$\Delta P_h = k_h \Delta\mathcal{E}/\omega, \tag{1}$$

where k_h is the corresponding component of the wave vector, and ω is the wave frequency ("cyclic" means that the particle Lagrangian is independent of x^h). The effect of gradient ponderomotive forces, which can input a comparable amount of momentum,[19] is not included in Eq. (1) and will not be considered below, but could be introduced additively in a more general theory.

Let us apply the theorem (1) specifically to wave-particle interaction in static electric and magnetic fields, $\mathbf{E} = -\nabla\phi(\mathbf{X})$ and $\mathbf{B} = \nabla\times\mathbf{A}(\mathbf{X})$. The particle Lagrangian is then given by $\Lambda = \Lambda_0 + e\mathbf{V}\cdot\mathbf{A}/c - e\phi$; here $\Lambda_0 = \Lambda_0(\mathbf{V})$ is the free particle Lagrangian, $\mathbf{V}$ is the particle velocity, e is the particle charge, and c is the speed of light. For simplicity, assume cylindrical geometry (including slab geometry as a limit), so the field lines lie on surfaces of constant radius, r, and $\phi = \phi(r)$. Let us adopt the orthogonal coordinate system (r,η,b), such that $\mathbf{e}_b$ is along $\mathbf{B}$. Then, one can take $\mathbf{A} = \Psi(r)\mathbf{e}_\eta$, so $\mathbf{B} = \Psi'(r)\mathbf{e}_b$ (We assume orthonormal coordinates, so upper and lower indexes are hence interchangeable; in particular, covariant basis vectors, $\mathbf{e}_q$, are indistinguishable from contravariant basis vectors, $\mathbf{e}^q = \nabla x^q$,[21] where $q = r,\ \eta,\ b$.). Since η is a cyclic variable, the conjugate canonical momentum P_η satisfies Eq. (1); i.e., $k_\eta\Delta\mathcal{E}/\omega = P_{\eta\mathrm{f}} - P_{\eta\mathrm{i}}$, where the indexes f and i (hereupon replaced with a generic index t) denote the final and initial states. Assuming a particle is outside the wave in these states, one can take $P_{\eta\mathrm{t}} = p_{\eta\mathrm{t}} + e\Psi(r_\mathrm{t})/c$, where p_η is the kinetic momentum. However, $P_{\eta\mathrm{t}}$ is conserved outside the wave, so one can also replace it with the gyrophase-average, $P_{\eta\mathrm{t}} = \langle e\Psi(r_\mathrm{t})/c\rangle$, where we used $\langle p_{\eta\mathrm{t}}\rangle = 0$. For small enough Δr, Taylor expansion then yields $\Delta P_\eta = (e\Psi'/c)\Delta r_{\mathrm{gc}}$, where $\Delta r_{\mathrm{gc}} \doteq \langle\Delta r\rangle$ is the guiding-center displacement (the symbol $\doteq$ denotes definitions). This leads to $(eB/c)\Delta r_{\mathrm{gc}} = k_\eta\Delta\mathcal{E}/\omega$, or

$$\Delta r_{\mathrm{gc}} = ck_\perp\Delta\mathcal{E}/(eB\omega). \tag{2}$$

We switched to the conventional notation here, $k_\perp \equiv k_\eta$, but keep in mind that k_r does not contribute to this $k_\perp$, even when k_r is nonzero. Also note that Eq. (2) reproduces the key α-channeling equation,[22] which couples diffusion in energy space to diffusion in geometrical space. The difference is that our derivation is relativistic, accounts for static electric field (a part of $\Delta\mathcal{E}$ is actually spent on the work against $\mathbf{E}$), and does not assume the heating to be instantaneous, unlike in Ref. 22 (In Ref. 23, the latter assumption was relaxed too.).

Equation (2) describes an effect similar to what Ref. 17 identifies as an outward electron pinch,[24] except that we find it to be inward for $k_\perp > 0$ and $e < 0$. It is dominant over neoclassical pinches[26] for, say, Alcator C-Mod typical parameters.[17] Hence, we will not introduce a tokamak geometry separately; i.e., toroidal and cylindrical coordinates will not be distinguished. In particular, we will not consider trapped-particle effects, as the wave perpendicular momentum that we focus on here is absorbed through trapped and circulating particles in the same way. (It would, however, be interesting to revisit in the future neoclassical effects such the Ohkawa current drive[25] with the wave-induced electric field taken into account.)

Let us now adopt the standard cylindrical (or toroidal) coordinates (r,θ,φ) for the same system. The vector-potential components, $A_r = 0$ and $A_h = \Psi(r)(\mathbf{e}_\eta\cdot\mathbf{e}_h)$ (where $h = \theta, \varphi$), are independent of θ and φ, so Eq. (1) applies, yielding $\langle\Delta p_h\rangle + \langle\Delta\mathcal{P}_h\rangle = k_h\Delta\mathcal{E}/\omega$. Here $\langle\Delta p_h\rangle$ is the average kinetic momentum gained by a resonant particle (e.g., a nonrelativistic particle with mass m has $p_h = mV_h$), so $\langle\Delta\mathcal{P}_h\rangle = (eB/c)(\mathbf{e}_\eta\cdot\mathbf{e}_h)\,\Delta r_{\mathrm{gc}}$ must be the momentum transferred to the dc field *through* the particle. Due to Eq. (2), it equals $\langle\Delta\mathcal{P}_h\rangle = (\mathbf{e}_\eta\cdot\mathbf{e}_h)k_\perp\Delta\mathcal{E}/\omega$, where $\mathbf{e}_\eta\cdot\mathbf{e}_\theta = \mathbf{e}_\varphi\cdot\mathbf{e}_b = B_\varphi/B$ and $\mathbf{e}_\eta\cdot\mathbf{e}_\varphi = -\mathbf{e}_\theta\cdot\mathbf{e}_b = -B_\theta/B$.[31] Also notice that $k_h = \mathbf{k}_\parallel\cdot\mathbf{e}_h + \mathbf{k}_\perp\cdot\mathbf{e}_h$, where $\mathbf{k}_\parallel \doteq k_\parallel\mathbf{e}_b$ and $\mathbf{k}_\perp \doteq k_\perp\mathbf{e}_\eta$; thus (cf. Fig. 1)

$$k_\theta = \mathbf{k}_\parallel\cdot\mathbf{e}_\theta + k_\perp B_\varphi/B, \quad k_\varphi = \mathbf{k}_\parallel\cdot\mathbf{e}_\varphi - k_\perp B_\theta/B. \tag{3}$$

One can hence re-express the above results as follows:

$$\langle\Delta p_h\rangle = (\mathbf{k}_\parallel\cdot\mathbf{e}_h)\Delta\mathcal{E}/\omega, \quad \langle\Delta\mathcal{P}_h\rangle = (\mathbf{k}_\perp\cdot\mathbf{e}_h)\Delta\mathcal{E}/\omega. \tag{4}$$

III. WAVE-INDUCED FORCES

Adding plasma into the picture leads to deceleration of resonant particles and relaxation of the dc field, so both $\langle\Delta p_h\rangle$ and $\langle\Delta\mathcal{P}_h\rangle$ are transferred to the bulk *eventually*. Until then, however, the effects of these two momenta are quite different. The former is well-studied;[32] namely, the momentum $\langle\Delta p_h\rangle$ contributes to plasma current and, while decaying on the long time scale of the hot-particle collisions, creates a force directly applied to the bulk plasma

$$F_{\parallel h} = (\mathbf{k}_\parallel\cdot\mathbf{e}_h)\mathfrak{P}_{\mathrm{abs}}/\omega. \tag{5}$$

For the effect of $\langle\Delta p_h\rangle$ on trapped particles, see Ref. 30. Less understood is the action of the perpendicular momentum input, $\langle\Delta\mathcal{P}_h\rangle$. As we explained above, one can interpret it as the momentum that resonant particles pass to the dc field, so the rate of this transfer,

$$F_{\perp h} = (\mathbf{k}_\perp\cdot\mathbf{e}_h)\mathfrak{P}_{\mathrm{abs}}/\omega, \tag{6}$$

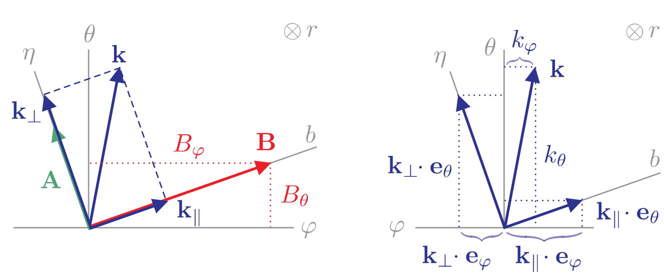

FIG. 1. Schematic of the assumed geometry.

can be understood as a wave-induced force on the field. In a tokamak, this is reflected in the change of the Poynting flux $\mathbf{S} \propto \mathbf{E} \times \mathbf{B}$ that is associated with the electric field $\mathbf{E} = E_r \mathbf{e}_r$ produced by the resonant-particle transport across flux surfaces [Eq. (2)]. The bulk plasma responds to $\Delta\mathbf{S}$ by adjusting its drift velocity, $\mathbf{U}$. For times smaller than those of collisions in the bulk, this leads simply to the change in $\mathbf{E} \times \mathbf{B}$ drift velocity, $\Delta\mathbf{U} \propto \Delta\mathbf{S}$, where we used $\Delta B/B \ll \Delta S_h/S_h$. Both electrons and ions pick up the same $\Delta\mathbf{U}$ within one gyroperiod (i.e., almost instantaneously), so no current is generated by $F_{\perp h}$ *per se*,[20] at least until the bulk pressure changes significantly. Also remarkably, collisions are not needed here to pass the wave momentum to the bulk; they can only moderate $\mathbf{U}$ by limiting E_r. Thus, absent collisions, bulk plasma would actually experience unlimited acceleration.

The difference between how $F_{\parallel h}$ and $F_{\perp h}$ act on the plasma renders their ratio, $\wp_h \doteq F_{\parallel h}/F_{\perp h}$, a fundamental parameter that determines (i) how much current is produced per unit wave momentum[20] and (ii) how rapidly the wave momentum is transferred to the bulk. This parameter happens to be merely a geometrical factor

$$\wp_h = (\mathbf{k}_\parallel \cdot \mathbf{e}_h)/(\mathbf{k}_\perp \cdot \mathbf{e}_h), \tag{7}$$

which depends on how $\mathbf{k}$ is oriented with respect to $\mathbf{B}$

$$\wp_\theta = k_\parallel B_\theta/(k_\perp B_\varphi), \quad \wp_\varphi = -k_\parallel B_\varphi/(k_\perp B_\theta). \tag{8}$$

Notice, in particular, that $\wp_h$ can have either sign, so plasma can be first accelerated by $F_{\perp h}$, which kicks in faster, and later decelerated by $F_{\parallel h}$; cf. Ref. 17.

For instance, lower-hybrid current drive (LHCD)[33] is performed at $k_\perp \gg k_\parallel$, so, together with $B_\theta \ll B_\varphi$, that gives $\wp_\varphi \sim 1$ (see below for estimates). This means that a significant portion of the wave toroidal momentum is transferred to plasma on the ion-gyroperiod time scale, because it is done by $F_{\perp\varphi}$, whereas the comparable effect of $F_{\parallel\varphi}$ is delayed by the collision time scale. (We discuss the momentum transfer only in the resonance region; it takes additional time to distribute this momentum within the whole tokamak volume.[32]) This is in agreement with Ref. 17. However, we also derive the *poloidal* momentum transfer here, and it is almost entirely due to $F_{\perp\theta}$, since $\wp_\theta \ll 1$. The appearance of this poloidal rotation (as well as the appearance of E_r) can be considered a characteristic feature of wave-plasma resonant interaction, at least in the limit of large aspect ratio.

IV. EQUILIBRIUM STATE

The total force on a plasma in equilibrium, $F_h = F_{\parallel h} + F_{\perp h}$, is determined by k_h

$$F_h = k_h \mathfrak{P}_{\text{abs}}/\omega. \tag{9}$$

It is seen then (contrary to the existing tradition of identifying k_φ with $k_\parallel$, as in Ref. 6) that, to maximize the toroidal force, one needs to maximize $\mathbf{k}_\parallel \cdot \mathbf{e}_\varphi - k_\perp B_\theta/B$ [Eq. (3)] rather than $k_\parallel$. Even the sign of F_φ is not necessarily the same as that of $k_\parallel$ and can be controlled by varying $k_\perp$. This also means that one can make k_φ zero at nonzero $k_\parallel$; then hot particles are resonantly accelerated toroidally by $F_{\parallel\varphi}$, but the bulk plasma is not, as $F_\varphi = 0$.

The plasma equilibrium under the action of F_h can be found from a simple hydrodynamic model, specifically, as follows. Let us treat the bulk plasma as a single fluid with velocity $\mathbf{u}$ and mass density ρ, acted upon by a force density f_h such that its integral over the plasma volume, $\Gamma = 2\pi^2 R a^2$, equals F_h (Here R and a are the tokamak major and minor radii.). Then one can adopt

$$\partial_t u_\theta = -\mu u_\theta + f_\theta/\rho, \quad \partial_t u_\varphi = \hat{\mathcal{D}} u_\varphi + f_\varphi/\rho, \tag{10}$$

cf. Refs. 8 and 34–36; here μ is the poloidal momentum damping rate, and the operator $\hat{\mathcal{D}}$ is defined as

$$\hat{\mathcal{D}} u_\varphi \doteq r^{-1}[\partial_r(r\chi\,\partial_r u_\varphi) + \partial_r(r \mathrm{v} u_\varphi)], \tag{11}$$

where χ is the toroidal momentum diffusivity, and v is the pinch velocity (The equation for u_θ ignores momentum transport because neoclassical damping is considered dominant, as usual.[8,34]). For negligible ∂_t, these yield $u_\theta = f_\theta/(\mu\rho)$ and $\hat{\mathcal{D}} u_\varphi = -f_\varphi/\rho$. Integration over the tokamak volume leads to $\Gamma \bar{u}_\theta = F_\theta/(\mu\rho)$ and $(2\Gamma/a^2)\int_0^a \hat{\mathcal{D}} u_\varphi r dr = -F_h/\rho$, where the bar denotes volume average. Using $\int_0^a \hat{\mathcal{D}} u_\varphi r dr = \chi a u'_\varphi(a)$ and estimating $a u'_\varphi(a)$ as $-\bar{u}_\varphi$, we get $\bar{u}_\theta = F_\theta/(\mu\rho\Gamma)$ and $\bar{u}_\varphi = a^2 F_h/(2\rho\chi\Gamma)$. Equation (9) gives then

$$\bar{u}_\theta = \frac{k_\theta \mathfrak{P}_{\text{abs}}}{2\pi^2 R a^2 \omega\mu\rho}, \quad \bar{u}_\varphi = \frac{k_\varphi \mathfrak{P}_{\text{abs}}}{4\pi^2 R \omega\rho\chi}. \tag{12}$$

These estimates agree with those yielded by "ponderomotive" theories such as Ref. 9, except that here we consider nonzero k_θ, and $\mathfrak{P}_{\text{abs}}$ accounts for the *total* power deposition through any resonances. We also can tell now that the relative contributions of the wave parallel and perpendicular momentum input to $\bar{u}_h$ are, correspondingly, $\wp_h/(\wp_h + 1) \ll 1$ and $1/(\wp_h + 1) \sim 1$. Finally, Eqs. (12) can as well be used to model the average rotation of essentially inhomogeneous plasma, i.e., account for reactive ponderomotive forces, as those do not affect the total momentum input in a tokamak equilibrium.

V. NUMERICAL ESTIMATES

To compare our predictions with experiment, consider parameters typical for LHCD on Alcator C-Mod. Specifically, assume that the wave power 1 MW is delivered at frequency of 4.6 GHz with parallel refractive index $k_\parallel c/\omega \approx 2$, so $k_\parallel \approx 1.93 \times 10^2\,\mathrm{m}^{-1}$. Also, $k_\perp/k_\parallel \sim (m_i/m_e)^{1/2} \approx 61$,[11] which gives $\wp_\theta \approx 1/600$ and $\wp_\varphi \approx -1/6$, assuming $B_\theta/B_\varphi \approx 0.1$. (The indexes i and e denote ions and electrons, correspondingly; m_i is twice the proton mass, assuming deuterium plasma.) Both toroidal and poloidal rotations are then mainly driven by the perpendicular momentum input, contrary to earlier theories.[15,37] We take the toroidal momentum diffusivity, χ, to be $0.15\,\mathrm{m}^2/\mathrm{s}$.[35,36] We also take $\mu \sim \nu_{ii} \sim 10^3\,\mathrm{s}^{-1}$, since the banana regime is realized [as $\nu_{ii} R q/[(a/R)^{3/2}(T_i/m_i)^{1/2}] \ll 1$, where ν_{ii} is the ion-ion collision frequency, and $q \sim 3$ is the safety factor], assuming the ion temperature $T_i \approx 1.5\,\mathrm{KeV}$.[38] The plasma mass density is $\rho \approx m_i n_i$, where

$n_i = 10^{20}\,\mathrm{m}^{-3}$. Together with $R = 0.67\,\mathrm{m}$ and $a = 0.21\,\mathrm{m}$, for these parameters Eqs. (12) yield $\bar{u}_\varphi \approx -30\,\mathrm{km/s}$ and $\bar{u}_\theta \approx 2\,\mathrm{km/s}$, where k_θ and k_φ are substituted from Eqs. (3). The value of $\bar{u}_\varphi$ is consistent with experiments.[39] The poloidal rotation, in contrast, has not been reported for LHCD, so our prediction for $\bar{u}_\theta$ is yet to be verified. On the other hand, our $\bar{u}_\theta$ is close to observations for plasma with mode-conversion ICRF heating.[6] (The direction of rotation is not necessarily the same as in the above calculation, because the experimental $k_\perp$ can have either sign.) However, keep in mind that our integral estimates do not resolve possible *redistribution* of the plasma intrinsic rotation through wave absorption, as is believed to occur, e.g., at electron cyclotron heating.[40–42] The momentum carried by the wave *per se* can be negligible in this case, so different estimates need to be applied that are not discussed here.

VI. DISCUSSION

In this paper, we show how tokamak plasma is spun up by the wave momentum perpendicular to the dc magnetic field. As in Refs. 16 and 17, we focus on the most controversial, *resonant* momentum transfer, as opposed to the effect of reactive, or gradient ponderomotive forces. (The latter, which we neglected here as in Refs. 16 and 17, could be introduced additively but remain outside the scope of our study.) As opposed to other recent theories that arrive at conclusions similar to ours through detailed (but nonrelativistic) quasilinear calculations, here we appeal merely to the momentum conservation theorem, which we prove rigorously for general resonant interactions (Appendix). This permits us to elucidate details of the underlying basic physics, including answering how much momentum gets deposited into resonant particles as opposed to the bulk. We find that it is possible, e.g., to accelerate resonant particles but not the bulk or, alternatively, the bulk but not resonant particles, because (and one may find this counterintuitive) collisions happen to be unnecessary for spinning up the bulk plasma. We derive not only the toroidal acceleration but also the poloidal acceleration, which we identify as a characteristic feature of wave-plasma resonant interaction. We then present a fluid model, from which the effect of the parallel and perpendicular wave momenta can be estimated and which explains how kinetic models, such as ours and those in Refs. 16 and 17, relate to the earlier "ponderomotive" theories.[6–11]

Per referee's suggestion, we also notice that our calculations point out the need for a fully consistent description of high-frequency wave-particle interactions (nonlinear and not limited to waves with fixed central ω and $\mathbf{k}$) that would manifestly conserve the total energy and momentum of the plasma. Most suitable for this purpose seems to be a field-theoretic variational formalism that would be similar in spirit to those recently reported, albeit in different contexts, in Refs. 27–29. Developing such a formalism for radiofrequency waves remains a matter of future research.

ACKNOWLEDGMENTS

The work was supported by the U.S. DOE through Contract No. DE-AC02-09CH11466, the ITER-China program through Grant No. 2010GB107001, and the National Natural Science Foundation of China through Grant No. NSFC-11075162. We thank J. E. Rice for providing experimental data from Alcator C-Mod.

APPENDIX: ENERGY-MOMENTUM CONSERVATION FOR GENERAL RESONANT INTERACTIONS

In this Appendix, we prove a general theorem describing energy-momentum conservation at wave-particle resonant interaction. In doing so, we rely only on the fundamental symmetry of such interactions, so our final result, Eq. (A1), is fully relativistic and applies to waves of arbitrary nature, including non-electromagnetic waves.

Let us start out with considering a nondissipative linear wave absent resonant particles. Such a wave can be assigned a Lagrangian density, $\mathfrak{L}$, expressed in terms of two independent real functions, namely, the (arbitrarily normalized) amplitude $\mathscr{A}(t,\mathbf{x})$ and phase $\xi(t,\mathbf{x})$; here t is time, $\mathbf{x} = (x^1,\ldots x^D)$ are general spatial coordinates, and D is the number of dimensions. In the geometrical-optics (GO) limit assumed below, one has $\mathfrak{L} = \mathscr{A}^2\mathfrak{D}(\omega,\mathbf{k};t,\mathbf{x})$, where $\omega \doteq -\partial_t\xi$ is the local frequency, and $\mathbf{k} \doteq \nabla\xi$ is the local wave vector;[18] the symbol $\doteq$ denotes definitions. (More generally, $\mathfrak{L}$ can also depend on derivatives of A, ω, and $\mathbf{k}$, but still not on ξ, which reflects conservation of the wave total action, or the number of quanta.) The densities of the wave canonical momentum and energy are then $\boldsymbol{\pi} = \mathbf{k}\mathfrak{L}_\omega$ and $w = \omega\mathfrak{L}_\omega$,[18] and subindexes hereupon denote partial derivatives except where explicitly stated otherwise.

Suppose now that a wave interacts with a resonant particle described by generalized coordinates $X^q(t)$ and velocities $V^q \doteq d_t X^q$; here $d_t \equiv d/dt$, and $q = 1,\ldots D$. We hence introduce a joint Lagrangian density, $\mathscr{L} = \delta(\mathbf{x},\mathbf{X})\,L + \mathfrak{L}$, where $\delta(\mathbf{x},\mathbf{X}) \doteq \delta(\mathbf{x}-\mathbf{X})/\sqrt{g(\mathbf{x})}$ is a generalized delta function,[43] g is the spatial metric determinant, $L = \Lambda + \lambda$ is the particle Lagrangian, $\Lambda = \Lambda(t,\mathbf{x},\mathbf{V})$ is that absent the wave, and $\lambda = \lambda(t,\mathbf{x},\mathbf{V};\xi,A,\omega,\mathbf{k})$ is the interaction Lagrangian. (To ensure that L satisfies GO requirements, one should, in fact, introduce many particles smoothly distributed along the wave, but this does not affect our conclusions.) The Euler-Lagrange equations (ELE) are then obtained from the least action principle, $\delta S = 0$, where $S \doteq \int L\,\mathrm{d}\mathbf{x}\,dt$, and $\mathrm{d}\mathbf{x} \doteq \sqrt{g(\mathbf{x})}\,dx^1\ldots dx^D$ is a volume element.

The first ELE, $\delta_{X^q} S = 0$, yields $d_t P_q = \Lambda_{X^q} + \partial_{X^q}\lambda$, where $P_q \doteq L_{V^q}$ is the particle canonical momentum, and the right-hand side is evaluated at $\mathbf{x} = \mathbf{X}(t)$. (We use ∂, and also ∇ below, to denote "full" derivatives in the sense that they treat each argument as a function of $(t,\mathbf{x})$; cf. Ref. 18.) Let us consider q such that Λ_{X^q} is zero or negligible. Under the same assumptions as in Ref. 17, we also assume that the main component of the wave-driven force is due to the resonance rather than due to gradients of A, ω, and $\mathbf{k}$. This means $\partial_{x^q}\lambda \approx k_q\lambda_\xi$, so $d_t P_q \approx k_q\lambda_\xi$. The second ELE, $\delta A S = 0$, gives the wave dispersion relation, $L_a = 0$; i.e., roughly, $\mathfrak{D}(\omega,\mathbf{k};t,\mathbf{x}) \approx 0$. (We will not need this equation below.) The third ELE, $\delta_\xi S = 0$, gives $\partial_t\mathfrak{L}_\omega - \nabla\cdot\mathfrak{L}_{\mathbf{k}} + \delta(\mathbf{x},\mathbf{X})\,\lambda_\xi = 0$; cf. Ref. 18. One may recognize this as Whitham's equation for the wave action density, $\mathfrak{L}_\omega$, whence the equation for

π_q is inferred readily.[44] Integrating that over the volume yields $d_t\Pi_q = \int \mathfrak{L}_\omega \dot{k}_q \,\mathrm{d}x - k_q\lambda_\xi$, where $\Pi_q \doteq \int \pi_q \,\mathrm{d}x$ is the wave total canonical momentum. The term $\dot{k}_q \equiv (\partial_t + \mathbf{v}_g \cdot \nabla)k_q$, where $\mathbf{v}_g \doteq -\mathfrak{L}_\mathbf{k}/\mathfrak{L}_\omega$ is the group velocity, describes the evolution of k_q due to both plasma inhomogeneity [i.e., nonzero $\omega_{x^q}(t, \mathbf{x}, \mathbf{k})$] and interaction with resonant particles. However, we have ruled out the former by neglecting the gradient force, and resonant particles cannot significantly affect k_q either, as it is determined primarily by the bulk plasma; thus, we drop $\dot{k}_q$. This leads to $d_t\Pi_q \approx -k_q\lambda_\xi$, so $d_t(P_q + \Pi_q) = 0$. For the canonical energy, $W \doteq \int w \,\mathrm{d}x$, one similarly finds that $d_tW = -\omega\lambda_\xi \approx \partial_t\lambda = -H_t = -d_tH$ (assuming a quasistationary state, so $\dot{\omega}$ is negligible, like $\dot{k}_q$), where $H = H(t, \mathbf{X}, \mathfrak{P})$ is the particle Hamiltonian; hence $d_t(H + W) = 0$. Assuming constant k_q and ω, one also has $d_t\Pi_q \approx (k_q/\omega)\, d_tW$. The increments of the particle momentum and energy are then expressed as

$$\Delta P_q = k_q \Delta\mathcal{E}/\omega, \quad \Delta H = \Delta\mathcal{E}, \tag{A1}$$

where $\Delta\mathcal{E} \doteq -\Delta W$ is the absorbed canonical energy. Notice that resonant interaction is a "package deal"; a particle simultaneously absorbs *all* components of the wave momentum, except those corresponding to zero k_q.

The above analysis explains how to interpret similar results that often appear in literature without a derivation.[45] For instance, it makes clear that Eq. (A1) describe the transfer of the wave *canonical*, rather than full, energy-momentum.[18] (The fact that there is more than one definition of the wave energy-momentum in plasma is often overlooked in applied literature.) Also note that our results apply in any geometry and are independent from specific motion equations; e.g., they are fully relativistic and are not limited to electromagnetic interactions. As a side corollary, the wave momentum need not be associated with a Poynting flux for Eq. (A1) to hold.

[1] S. A. Sabbagh, R. E. Bell, J. E. Menard, D. A. Gates, A. C. Sontag, J. M. Bialek, B. P. LeBlanc, F. M. Levinton, K. Tritz, and H. Yuh, Phys. Rev. Lett. **97**, 045004 (2006).
[2] Y. Liu, A. Bondeson, Y. Gribov, and A. Polevoi, Nucl. Fusion **44**, 232 (2004).
[3] B. P. LeBlanc, R. E. Bell, S. Bernabei, J. C. Hosea, R. Majeski, M. Ono, C. K. Phillips, J. H. Rogers, G. Schilling, C. H. Skinner, and J. R. Wilson, Phys. Rev. Lett. **82**, 331 (1999).
[4] L.-G. Eriksson, T. Johnson, T. Hellsten, C. Giroud, V. G. Kiptily, K. Kirov, J. Brzozowski, M. DeBaar, J. DeGrassie, M. Mantsinen, A. Meigs, J.-M. Noterdaeme, A. Staebler, D. Testa, A. Tuccillo, and K.-D. Zastrow, Phys. Rev. Lett. **92**, 235001 (2004).
[5] Y. Lin, J. E. Rice, S. J. Wukitch, M. J. Greenwald, A. E. Hubbard, A. Ince-Cushman, L. Lin, E. S. Marmar, M. Porkolab, M. L. Reinke, N. Tsujii, J. C. Wright, and Alcator C-Mod Team, Phys. Plasmas **16**, 056102 (2009).
[6] Y. Lin, J. E. Rice, S. J. Wukitch, M. J. Greenwald, A. E. Hubbard, A. Ince-Cushman, L. Lin, M. Porkolab, M. L. Reinke, and N. Tsujii, Phys. Rev. Lett. **101**, 235002 (2008).
[7] L. A. Berry, E. F. Jaeger, and D. B. Batchelor, Phys. Rev. Lett. **82**, 1871 (1999).
[8] J. R. Myra and D. A. D'Ippolito, Phys. Plasmas **7**, 3600 (2000).
[9] J. R. Myra and D. A. D'Ippolito, Phys. Plasmas **9**, 3867 (2002).
[10] J. R. Myra, L. A. Berry, D. A. D'Ippolito, and E. F. Jaeger, Phys. Plasmas **11**, 1786 (2004).
[11] Z. Gao, J. Chen, and N. J. Fisch, Phys. Rev. Lett. **110**, 235004 (2013).
[12] This applies to both circulating and trapped particles. The latter cannot increase their *average* parallel velocity (which remains zero), but local acceleration of trapped particles along magnetic field lines is possible and leads to a radiofrequency pinch.[30]
[13] T. Hellsten, Plasma Phys. Controlled Fusion **53**, 054007 (2011).
[14] S. Wang, Phys. Plasmas **18**, 102502 (2011).
[15] J. P. Lee, J. C. Wright, P. T. Bonoli, R. R. Parker, P. J. Catto, Y. A. Podpaly, J. E. Rice, and M. L. Reinke, in Proceedings of the 19th Topical Conference on Radio Frequency Power in Plasmas (2011), Vol. 1406, p. 459.
[16] X. Guan, H. Qin, J. Liu, and N. J. Fisch, Phys. Plasmas **20**, 022502 (2013).
[17] J. Lee, F. I. Parra, R. R. Parker, and P. T. Bonoli, Plasma Phys. Controlled Fusion **54**, 125005 (2012).
[18] I. Y. Dodin and N. J. Fisch, Phys. Rev. A **86**, 053834 (2012).
[19] Absent dissipation, gradient ponderomotive forces deposit the wave momentum $\Delta(\mathbf{k}W/\omega)$, where W is the wave energy.[18] Thus, under the typical conditions $\Delta|\mathbf{k}/\omega| \lesssim |\mathbf{k}/\omega|$ and $|\Delta W| \sim \Delta\mathcal{E}$, one can expect their effect to be comparable to resonant absorption.
[20] The current drive effect caused by the transfer of the wave *energy* to resonant particle[32] is not discussed here.
[21] R. B. White, *The Theory of Toroidally Confined Plasmas*, 2nd ed. (Imperial College Press, Singapore, 2006), Sec. 1.3.
[22] N. J. Fisch and J. M. Rax, Phys. Rev. Lett. **69**, 612 (1992).
[23] N. J. Fisch, Phys. Plasmas **2**, 2375 (1995).
[24] See Eq. (27) in Ref. 17, with the latter term dropped for an equilibrium at large-aspect ratio. The right-hand side there is proportional to our $F_{\perp\varphi} = -(B_\theta/B)k_\perp \mathfrak{P}_{\mathrm{abs}}/\omega$ [Eq. (6)], and the left-hand side is proportional to B_θ, yielding a similar radial velocity. For notation, see below.
[25] T. Ohkawa, General Atomic Company Report No. GA-A13847, 1976.
[26] X. Guan, H. Qin, and N. J. Fisch, Phys. Plasmas **17**, 092502 (2010).
[27] A. J. Brizard and N. Tronko, Phys. Plasmas **18**, 082307 (2011).
[28] B. Scott and J. Smirnov, Phys. Plasmas **17**, 112302 (2010).
[29] I. Y. Dodin, PPPL Report 4856, "On variational methods in the physics of plasma waves" [Fusion Sci. Technol. (to be published)].
[30] N. J. Fisch and C. F. F. Karney, Phys. Fluids **24**, 27 (1981).
[31] For orthonormal coordinates that we use, one can employ $\epsilon_{\alpha\beta\gamma}\mathbf{e}_\gamma = \mathbf{e}_\alpha \times \mathbf{e}_\beta$, where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita permutation symbol.[21] Alternatively, see Fig. 1.
[32] N. J. Fisch, Rev. Mod. Phys. **59**, 175 (1987).
[33] N. J. Fisch, Phys. Rev. Lett. **41**, 873 (1978).
[34] K. Ida, Plasma Phys. Controlled Fusion **40**, 1429 (1998).
[35] J. E. Rice, W. D. Lee, E. S. Marmar, P. T. Bonoli, R. S. Granetz, M. J. Greenwald, A. E. Hubbard, I. H. Hutchinson, J. H. Irby, Y. Lin, D. Mossessian, J. A. Snipes, S. M. Wolfe, and S. J. Wukitch, Nucl. Fusion **44**, 379 (2004).
[36] J. E. Rice, W. D. Lee, E. S. Marmar, N. P. Basse, P. T. Bonoli, M. J. Greenwald, A. E. Hubbard, J. W. Hughes, I. H. Hutchinson, A. Ince-Cushman, J. H. Irby, Y. Lin, D. Mossessian, J. A. Snipes, S. M. Wolfe, S. J. Wukitch, and K. Zhurovich, Phys. Plasmas **11**, 2427 (2004).
[37] L. G. Eriksson, T. Hellsten, M. F. F. Nave, J. Brzozowski, K. Holmström, T. Johnson, J. Ongena, K.-D. Zastrow, and JET-EFDA Contributors, Plasma Phys. Controlled Fusion **51**, 044008 (2009).
[38] K. C. Shaing and E. C. Crume, Phys. Rev. Lett. **63**, 2369 (1989).
[39] J. E. Rice, A. C. Ince-Cushman, P. T. Bonoli, M. J. Greenwald, J. W. Hughes, R. R. Parker, M. L. Reinke, G. M. Wallace, C. L. Fiore, R. S. Granetz, A. E. Hubbard, J. H. Irby, E. S. Marmar, S. Shiraiwa, S. M. Wolfe, S. J. Wukitch, M. Bitter, K. Hill, and J. R. Wilson, Nucl. Fusion **49**, 025004 (2009).
[40] R. M. McDermott, C. Angioni, R. Dux, A. Gude, T. P. Ütterich, F. Ryter, G. Tardini, and ASDEX Upgrade Team, Plasma Phys. Controlled Fusion **53**, 035007 (2011).
[41] M. Yoshida, Y. Sakamoto, H. Takenaga, S. Ide, N. Oyama, T. Kobayashi, Y. Kamada, and JT-60 Team, Phys. Rev. Lett. **103**, 065003 (2009).
[42] J. S. deGrassie, K. H. Burrell, L. R. Baylor, W. Houlberg, and J. Lohr, Phys. Plasmas **11**, 4323 (2004).
[43] B. S. DeWitt, Phys. Rev. **85**, 653 (1952).
[44] The derivation is similar to that of Eq. (42) in Ref. 18.
[45] See, e.g., Ref. 13. Appealing to a quantum analogy is also common[9] but unsubstantiated, as there is no reason to believe *a priori* that quantization of collective oscillations in classical plasma is physically meaningful. Quantization is always a leap of logic and as such requires verification.