Transition between inverse and direct energy cascades in multiscale optical turbulence

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Multiscale turbulence naturally develops and plays an important role in many fluid, gas, and plasma phenomena. Statistical models of multiscale turbulence usually employ Kolmogorov hypotheses of spectral locality of interactions (meaning that interactions primarily occur between pulsations of comparable scales) and scale-invariance of turbulent pulsations. However, optical turbulence described by the nonlinear Schrodinger equation exhibits breaking of both the Kolmogorov locality and scale-invariance. A weaker form of spectral locality that holds for multi-scale optical turbulence enables a derivation of simplified evolution equations that reduce the problem to a single scale modeling. We present the derivation of these equations for Kerr media with random inhomogeneities. Then, we find the analytical solution that exhibits a transition between inverse and direct energy cascades in optical turbulence.

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I. INTRODUCTION

In focusing Kerr media, coherent laser pulses of powers much exceeding the so-called "critical power" [1–4] cannot propagate, without transverse filamentation, much longer than the distance of self-focusing. The propagation distance can be greatly increased in a weak turbulence regime where the nonlinear filamentation is prevented by a stronger transverse dispersion of incoherent laser pulses [5]. There is a larger propagation length limit imposed by stimulated 4-photon scattering. This process tends to produce an "inverse cascade" of photons, directed toward small transverse wave numbers, in contrast to the classical Richardson-Kolmogorov-Obukhov cascade [6–10] directed to large wave numbers. At small transverse wave numbers, the transverse dispersion is too small to sustain the phase randomness. The coherent Bose-Einstein condensate thus formed exhibits the transverse filamentation.

The propagation limit could be overcome by suppressing the inverse cascade. What enables the inverse cascade in weak optical turbulence is an additional integral of motion. It is the "transverse energy" of photons conserved in homogeneous media, in addition to the total energy of photons. The transverse energy flux is directed to large transverse wave numbers. It is carried by a small fraction of photons, which results in the transverse cooling of the most photons in the pulse. It was suggested recently [11] that the transverse heating of the pulse through photon scattering on random inhomogeneities of the medium can stop the nonlinear transverse cooling before the unstable condensate formation and thus enable extended propagation of powerful laser pulses.

Below, we analytically find the spectra of weak optical turbulence in randomly inhomogeneous Kerr media. These results significantly refine heuristic estimates of Ref. [11] and broaden the class of analytically tractable regimes of optical turbulence previously found for homogeneous media [5]. Apart from importance for the optical turbulence itself and its direct applications, these results may be useful for other kinds of turbulence described by nonlinear Schrodinger equations (see, for example, Refs. [12,13]). In broader contexts, these results may be conceptually important for less tractable cases including Navier-Stokes turbulence. This relates both to general issues associated with possible deviations from Kolmogorov locality and scale-invariance in turbulence, and to issues associated with transitions between different kinds of cascades, like two-dimensional inverse cascades [14,15] and direct or bidirectional cascades in thicker fluid layers [16,17].

The outline of this paper is as follows. In Sec. II, we start from the Liouville kinetic equation for photons with 4-photon collisional integral. Statistical averaging of this equation over random inhomogeneities of the medium captures the effect of inhomogeneities into a term describing diffusion of photons in wave-vector space. In Sec. III, we review basic properties of this kinetic equation. In Sec. IV, we explain reduction of the 4-photon collisional integral to a quasilocal in wave vectors form suggested in Ref. [5]. In Sec. V, we find solutions of the kinetic equation at transverse wave-numbers much smaller the largest one, where the 4-photon scattering is fast enough to establish a quasistationary spectrum. In Sec. VI, we derive an evolution equation for photon spectrum around its upper boundary in transverse wave-numbers. In Sec. VII, we solve this equation with proper boundary conditions at smaller transverse wave numbers. In Sec. VIII, we use the solution to determine in which parameter space domain the diffusion stops the condensation before the filamentation occurs. In Sec. IX, we discuss the results.

II. KINETIC EQUATION FOR PHOTONS IN KERR MEDIA WITH RANDOM INHOMOGENEITIES

Consider an incoherent laser pulse paraxially propagating in z direction with a fixed group velocity v_g. The transverse motion of photons will be described by Hamilton equations for rays,

$$\frac{d\boldsymbol{k}_\perp}{dt} = -\frac{\partial\omega(\boldsymbol{k},\boldsymbol{r},t)}{\partial\boldsymbol{r}_\perp}, \quad \frac{d\boldsymbol{r}_\perp}{dt} = \frac{\partial\omega(\boldsymbol{k},\boldsymbol{r},t)}{\partial\boldsymbol{k}_\perp},$$

with frequency taken in the form

$$\omega(\boldsymbol{k},\boldsymbol{r},t)=\omega_0+\omega_r(\boldsymbol{r})+\omega_l(k_\perp)+\omega_{nl}(I),$$

where $\omega_r(\boldsymbol{r})$, $\omega_l=\beta k_\perp^2$ and $\omega_{nl}=\alpha I$ are small additions to the pulse carrying frequency ω_0, associated with the medium inhomogeneities, transverse dispersion, and Kerr effect proportional to the local field intensity I, respectively.

Statistical averaging over random phases of the rays leads to an evolution equation for the photon spectral density $I_{\boldsymbol{k}_\perp}(\boldsymbol{r},t)$,

$$\left[\frac{\partial}{\partial t}+v_g\frac{\partial}{\partial z}+2\beta\boldsymbol{k}_\perp\cdot\frac{\partial}{\partial\boldsymbol{r}_\perp}-\frac{\partial(\omega_r+\alpha I)}{\partial\boldsymbol{r}_\perp}\cdot\frac{\partial}{\partial\boldsymbol{k}_\perp}\right]I_{\boldsymbol{k}_\perp}=\mathbb{S}_{\boldsymbol{k}_\perp},\tag{1}$$

which can be viewed as the Liouville kinetic equation for photons with the "collisional integral" $\mathbb{S}_{\boldsymbol{k}_\perp}$. For the 4-photon scattering of interest here,

$$\begin{aligned}\mathbb{S}_{\boldsymbol{k}_\perp}=U\int d\boldsymbol{k}_{1\perp}d\boldsymbol{k}_{2\perp}d\boldsymbol{k}_{3\perp}\delta(\boldsymbol{k}_\perp+\boldsymbol{k}_{1\perp}-\boldsymbol{k}_{2\perp}-\boldsymbol{k}_{3\perp})\\ \times\delta\left(k_\perp^2+k_{1\perp}^2-k_{2\perp}^2-k_{3\perp}^2\right)\\ \times I_{\boldsymbol{k}_\perp}I_{\boldsymbol{k}_{1\perp}}I_{\boldsymbol{k}_{2\perp}}I_{\boldsymbol{k}_{3\perp}}\left(I_{\boldsymbol{k}_\perp}^{-1}+I_{\boldsymbol{k}_{1\perp}}^{-1}-I_{\boldsymbol{k}_{2\perp}}^{-1}-I_{\boldsymbol{k}_{3\perp}}^{-1}\right);\end{aligned}\tag{2}$$

see, for example, Refs. [18–21]. For the spectral density normalization $\int d\boldsymbol{k}_\perp I_{\boldsymbol{k}_\perp}=2\pi I$, the coefficient is

$$U=\alpha^2/\beta.$$

Consider now statistical averaging over random medium inhomogeneities. The photon spectral density fluctuations, $\tilde I_{\boldsymbol{k}_\perp}=I_{\boldsymbol{k}_\perp}-\bar I_{\boldsymbol{k}_\perp}$, can be approximately expressed through the averaged value, $\bar I_{\boldsymbol{k}_\perp}$, by keeping just leading terms in the linear part of Eq. (1). This gives the formula

$$\tilde I_{\boldsymbol{k}_\perp}\approx v_g^{-1}\int_0^\infty d\zeta\,\frac{\partial\omega_r(z-\zeta)}{\partial\boldsymbol{r}_\perp}\frac{\partial}{\partial\boldsymbol{k}_\perp}\bar I_{\boldsymbol{k}_\perp}(\boldsymbol{r}_\perp,z-\zeta,t).$$

Substituting $I_{\boldsymbol{k}_\perp}=\tilde I_{\boldsymbol{k}_\perp}+\bar I_{\boldsymbol{k}_\perp}$ into Eq. (1) and averaging gives

$$\begin{aligned}\left(\frac{\partial}{\partial t}+v_g\frac{\partial}{\partial z}+2\beta\boldsymbol{k}_\perp\frac{\partial}{\partial\boldsymbol{r}_\perp}\right)\bar I_{\boldsymbol{k}_\perp}\\ \approx\overline{\frac{\partial\omega_r}{\partial\boldsymbol{r}_\perp}\frac{\partial\tilde I_{\boldsymbol{k}_\perp}}{\partial\boldsymbol{k}_\perp}}+\alpha\frac{\partial\bar I}{\partial\boldsymbol{r}_\perp}\frac{\partial\bar I_{\boldsymbol{k}_\perp}}{\partial\boldsymbol{k}_\perp}+\mathbb{S}_{\boldsymbol{k}_\perp}.\end{aligned}\tag{3}$$

For media statistically isotropic in $\boldsymbol{r}_\perp$ plane,

$$\overline{\frac{\partial\omega_r(z)}{\partial\boldsymbol{r}_\perp}\frac{\partial\omega_r(z-\zeta)}{\partial\boldsymbol{r}_\perp}}=\frac12\begin{pmatrix}1&0\\0&1\end{pmatrix}\overline{\frac{\partial\omega_r(z)}{\partial\boldsymbol{r}_\perp}\frac{\partial\omega_r(z-\zeta)}{\partial\boldsymbol{r}_\perp}},$$

Eq. (3) takes the form

$$\begin{aligned}\left(\frac{\partial}{\partial t}+v_g\frac{\partial}{\partial z}+2\beta\boldsymbol{k}_\perp\frac{\partial}{\partial\boldsymbol{r}_\perp}\right)\bar I_{\boldsymbol{k}_\perp}\\ \approx D\frac{\partial^2\bar I_{\boldsymbol{k}_\perp}}{(\partial\boldsymbol{k}_\perp)^2}+\alpha\frac{\partial\bar I}{\partial\boldsymbol{r}_\perp}\frac{\partial\bar I_{\boldsymbol{k}_\perp}}{\partial\boldsymbol{k}_\perp}+\mathbb{S}_{\boldsymbol{k}_\perp},\\ D=\frac12\int_0^\infty d\zeta\,\overline{\frac{\partial\omega_r(z)}{\partial\boldsymbol{r}_\perp}\frac{\partial\omega_r(z-\zeta)}{\partial\boldsymbol{r}_\perp}}.\end{aligned}\tag{4}$$

For the kinetic equation applicability, the rate of phase mixing should exceed the rate of nonlinear interaction. In statistically uniform media, this condition also ensures stability of spectra uniform in the transverse directions to small transverse modulations.

For a single-scale initial spectrum with the typical spread of transverse dispersion corrections to wave frequencies $\omega_l\sim\beta k_{\perp0}^2$, the kinetic equation applicability condition is

$$\beta k_{\perp0}^2\gg UI^2/k_{\perp0}^2\iff\beta k_{\perp0}^2\gg\alpha I.$$

Generalizations for multiscale spectra will be considered below.

Kinetic Eq. (4) is further simplified in the coordinate frame moving with the group velocity v_g in z direction. For wave spectra uniform in the transverse directions, the simplified equation takes the form

$$\frac{\partial I_{\boldsymbol{k}_\perp}}{\partial t}=D\frac{\partial^2 I_{\boldsymbol{k}_\perp}}{(\partial\boldsymbol{k}_\perp)^2}+\mathbb{S}_{\boldsymbol{k}_\perp}.\tag{5}$$

The averaging bar over I is omitted here and further (also, "=" is used instead of "≈" where it is not confusing).

III. BASIC PROPERTIES

Kinetic Eq. (5) conserves the number of photons,

$$I=(2\pi)^{-1}\int d\boldsymbol{k}_\perp I_{\boldsymbol{k}_\perp}.\tag{6}$$

The conservation can be formally verified by noticing that the expression under integral in Eq. (2) is antisymmetric to the switching variables $k_\perp,k_{\perp1}\rightleftarrows k_{\perp2},k_{\perp3}$.

The "transverse energy" of photons linearly grows with time,

$$\mathcal{E}(t)\equiv(2\pi)^{-1}\int d\boldsymbol{k}_\perp k_\perp^2 I_{\boldsymbol{k}_\perp}=4ID t+\mathcal{E}_0.\tag{7}$$

This can be verified by integrating Eq. (5) multiplied by $k_\perp^2$ and noticing that $\int d\boldsymbol{k}_\perp k_\perp^2\mathbb{S}_{\boldsymbol{k}_\perp}=0$ (due to the symmetries which allow to produce the combination $k_\perp^2+k_{\perp1}^2-k_{\perp2}^2-k_{\perp3}^2=0$ under the integral).

Since $\int d\boldsymbol{k}_\perp k_\perp^2\mathbb{S}_{\boldsymbol{k}_\perp}=0$, the quantity $k_\perp^2\mathbb{S}_{\boldsymbol{k}_\perp}$ can be presented in the form of divergence of a vector field. The physical meaning of this field is the density of transverse energy spectral flux associated with the stimulated 4-photon scattering. For axisymmetric spectra, this vector field is radial, so that Eq. (5) can be rewritten in the form

$$\frac{\partial}{\partial t}k_\perp^2 I_{k_\perp}=Dk_\perp\frac{\partial}{\partial k_\perp}k_\perp\frac{\partial}{\partial k_\perp}I_{k_\perp}-\frac{1}{k_\perp}\frac{\partial}{\partial k_\perp}\mathbb{J}_{k_\perp},\tag{8}$$

$$\begin{aligned}\mathbb{J}_{k_\perp}=-\frac{U}{2\pi}\int\Theta(k_\perp-k_{4\perp})k_{4\perp}^2 dk_{4\perp}d\boldsymbol{k}_{1\perp}d\boldsymbol{k}_{2\perp}d\boldsymbol{k}_{3\perp}\\ \times\delta(\boldsymbol{k}_{4\perp}+\boldsymbol{k}_{1\perp}-\boldsymbol{k}_{2\perp}-\boldsymbol{k}_{3\perp})\\ \times\delta\left(k_{4\perp}^2+k_{1\perp}^2-k_{2\perp}^2-k_{3\perp}^2\right)\\ \times I_{k_{4\perp}}I_{k_{1\perp}}I_{k_{2\perp}}I_{k_{3\perp}}\left(I_{k_{4\perp}}^{-1}+I_{k_{1\perp}}^{-1}-I_{k_{2\perp}}^{-1}-I_{k_{3\perp}}^{-1}\right),\\ \Theta(x)=\begin{cases}1, & x\geqslant0\\ 0, & x<0\end{cases}.\end{aligned}\tag{9}$$

We are interested here in regimes for which the linear diffusive scattering is initially negligible compared to the

stimulated 4-photon scattering. For single-scale initial spectra ($k_\perp \sim k_{\perp 0}$), this means

$$D \ll UI^2.$$

As long as the linear diffusive scattering remains negligible, the stimulated 4-photon scattering accumulates waves at small transverse wave numbers $k_\perp \ll k_{\perp 0}$. For $D = 0$, it would ultimately make the kinetic equation inapplicable at the small $k_\perp$. However, a small finite D could stop the wave accumulation at small transverse wave numbers still within the kinetic equation applicability range. This may keep the turbulence weak through the entire evolution at each $\boldsymbol{k}_\perp$.

The spectra dominated by stimulated 4-photon scattering have the wave-populated domain $k_\perp \lesssim k_{\perp M}$ expanding with the rate of 4-photon scattering at $k_\perp \sim k_{\perp M}$. Due to a much higher rate of 4-photon scattering at $k_\perp \ll k_{\perp M}$, a nearly equilibrium spectrum is established there,

$$I_{k_\perp} = \frac{T_{k_\perp}}{k_\perp^2 + k_{\perp m}^2}. \tag{10}$$

A slow variation of the "transverse temperature" $T_{k_\perp}$ across the spectrum is kept here to enable a nonzero spectral flux $\mathbb{J}_{k_\perp}$ of the transverse energy.

IV. REDUCTION OF THE TRANSVERSE ENERGY SPECTRAL FLUX INTEGRAL

As seen from the symmetries of Eq. (9), a nonzero contribution to this integral comes only from such photon quartets in which $\max\{k_{4\perp}^2, k_{1\perp}^2, k_{2\perp}^2, k_{3\perp}^2\} > k_\perp^2$. By a proper renaming of the indexes, $\mathbb{J}_{k_\perp}$ can be presented in the form

$$\begin{aligned}\mathbb{J}_{k_\perp} = \frac{U}{2\pi}\int d\boldsymbol{k}_{4\perp} d\boldsymbol{k}_{1\perp} d\boldsymbol{k}_{2\perp} d\boldsymbol{k}_{3\perp} \Theta(k_{1\perp} - k_\perp)\Theta(k_{1\perp} - k_{2\perp})\Theta(k_{1\perp} - k_{3\perp}) \\ \times \left[\Theta(k_\perp - k_{2\perp})k_{2\perp}^2 + \Theta(k_\perp - k_{3\perp})k_{3\perp}^2 - \Theta(k_\perp - k_{4\perp})k_{4\perp}^2\right] \\ \times \delta(\boldsymbol{k}_{4\perp} + \boldsymbol{k}_{1\perp} - \boldsymbol{k}_{2\perp} - \boldsymbol{k}_{3\perp})\delta\left(k_{4\perp}^2 + k_{1\perp}^2 - k_{2\perp}^2 - k_{3\perp}^2\right) I_{k_{4\perp}} I_{k_{1\perp}} I_{k_{2\perp}} I_{k_{3\perp}} \left(I_{k_{4\perp}}^{-1} + I_{k_{1\perp}}^{-1} - I_{k_{2\perp}}^{-1} - I_{k_{3\perp}}^{-1}\right).\end{aligned} \tag{11}$$

For $k_\perp \gg k_{\perp m}$, the major contribution to this integral comes from the domain $k_{4\perp} \ll k_\perp$, and is large in the parameter $\ln(k_\perp^2/k_{\perp m}^2) \gg 1$. This allows us to simplify Eq. (11) as follows:

$$\begin{aligned}\mathbb{J}_{k_\perp} \approx U\mathcal{I}_{k_\perp}\int d\boldsymbol{k}_{1\perp} d\boldsymbol{k}_{2\perp} d\boldsymbol{k}_{3\perp} \Theta(k_{1\perp} - k_\perp)\left[\Theta(k_\perp - k_{2\perp})k_{2\perp}^2 + \Theta(k_\perp - k_{3\perp})k_{3\perp}^2\right] \\ \times \delta(\boldsymbol{k}_{1\perp} - \boldsymbol{k}_{2\perp} - \boldsymbol{k}_{3\perp})\delta\left(k_{1\perp}^2 - k_{2\perp}^2 - k_{3\perp}^2\right) I_{k_{1\perp}} I_{k_{2\perp}} I_{k_{3\perp}} \left(I_{k_{1\perp}}^{-1} - I_{k_{2\perp}}^{-1} - I_{k_{3\perp}}^{-1}\right),\end{aligned} \tag{12}$$

$$\mathcal{I}_{k_\perp} = \int_0^{k_\perp} dk_{4\perp} k_{4\perp} I_{k_{4\perp}}. \tag{13}$$

This shows explicitly that the spectral flux of transverse energy $\mathbb{J}_{k_\perp}$ does not satisfy the Kolmogorov locality hypothesis. Rather, each of scales $k_{4\perp} < k_\perp$ equally contributes to $\mathbb{J}_{k_\perp}$. The nonlocality is encapsulated within the quantity $\mathcal{I}_{k_\perp}$. This can be viewed as a weaker form of locality, where the flux $\mathbb{J}_{k_\perp}$ is produced by a spectrally local 3-wave scattering with the probability proportional to $\mathcal{I}_{k_\perp}$ slowly varying across the spectrum.

Equation (12) can be rewritten in the form

$$\begin{aligned}\mathbb{J}_{k_\perp} \approx 2\pi U\mathcal{I}_{k_\perp}\int_0^\infty\int_0^\infty dk_{2\perp} dk_{3\perp} \Theta(k_{2\perp}^2 + k_{3\perp}^2 - k_\perp^2) \\ \times \left[\Theta(k_\perp - k_{2\perp})k_{2\perp}^2 + \Theta(k_\perp - k_{3\perp})k_{3\perp}^2\right] \\ \times I_{\sqrt{k_{2\perp}^2+k_{3\perp}^2}} I_{k_{2\perp}} I_{k_{3\perp}} \left(I^{-1}_{\sqrt{k_{2\perp}^2+k_{3\perp}^2}} - I_{k_{2\perp}}^{-1} - I_{k_{3\perp}}^{-1}\right).\end{aligned} \tag{14}$$

The major contribution to this integral comes from the domain $k_{3\perp} \sim k_{2\perp} \sim k_\perp$.

This formula can be further simplified in the domain where the transverse temperature $T_{k_\perp} \approx k_\perp^2 I_{k_\perp}$ just slowly varies with $k_\perp$,

$$\left|\frac{\partial \ln T_{k_\perp}}{\partial \ln k_\perp}\right| \ll 1. \tag{15}$$

There,

$$\begin{aligned}&I^{-1}_{\sqrt{k_{2\perp}^2+k_{3\perp}^2}} - I_{k_{2\perp}}^{-1} - I_{k_{3\perp}}^{-1} \\ &\approx k_{2\perp}^2\left(T^{-1}_{\sqrt{k_{2\perp}^2+k_{3\perp}^2}} - T_{k_{2\perp}}^{-1}\right) + k_{3\perp}^2\left(T^{-1}_{\sqrt{k_{2\perp}^2+k_{3\perp}^2}} - T_{k_{3\perp}}^{-1}\right) \\ &\approx \frac{\partial T_{k_\perp}^{-1}}{\partial \ln k_\perp}\left(k_{2\perp}^2 \ln\sqrt{1 + k_{3\perp}^2/k_{2\perp}^2} + k_{3\perp}^2 \ln\sqrt{1 + k_{2\perp}^2/k_{3\perp}^2}\right),\end{aligned}$$

so that Eq. (14) reduces to

$$\begin{aligned}\mathbb{J}_{k_\perp} \approx -A\, U\, \mathcal{I}_{k_\perp} T_{k_\perp} \frac{\partial T_{k_\perp}}{\partial \ln k_\perp}, \\ A = \frac{\pi}{2}\int_0^\infty du \frac{[u^2 \ln(1 + u^{-2}) + \ln(1 + u^2)]^2}{(1 + u^2)u^2} \approx 3.0.\end{aligned} \tag{16}$$

Equation (16) was first suggested in Ref. [5].

V. QUASISTATIONARY SPECTRUM AT $k_\perp \ll k_{\perp M}$

As the upper boundary $k_{\perp M}$ of the photon-populated domain grows, the spectrum evolves, but slowly compared to the 4-photon scattering rate at $k_\perp \ll k_{\perp M}$, which is much larger than

at $k_\perp \sim k_{\perp M}$. Therefore, the time-derivative term in Eq. (8) can be neglected at $k_\perp \ll k_{\perp M}$, and the quasistationary spectrum Eq. (10) establishes there. The diffusion term in Eq. (8) is approximately equal to $4DT_{k_\perp}/k_\perp^2$, for $k_{\perp m} \ll k_\perp \ll k_{\perp M}$. Thus, Eq. (8) takes the form

$$\frac{\partial \mathbb{J}_{k_\perp}}{\partial \ln k_\perp} \approx 4DT_{k_\perp}. \tag{17}$$

The integration and using Eq. (13) gives

$$\mathbb{J}_{k_\perp} \approx 4D\,\mathcal{I}_{k_\perp}. \tag{18}$$

The solution of Eqs. (16) and (18) is

$$T_{k_\perp} \approx \sqrt{T_*^2 + \frac{8D}{AU}\ln\frac{k_{\perp M}}{k_\perp}}, \tag{19}$$

where T_*^2 is the integration constant. The applicability condition Eq. (15) reduces then to

$$\frac{4D}{AUT_{k_\perp}^2} \ll 1. \tag{20}$$

Equation (13) takes the form

$$\mathcal{I}_{k_\perp} \approx \int_{k_{\perp m}}^{k_\perp} T_{k_\perp}\frac{dk_\perp}{k_\perp} \approx \frac{AU}{12D}\left(T_{k_{\perp m}}^3 - T_{k_\perp}^3\right). \tag{21}$$

According to the Eq. (19), there are three domains in the parameter space where the spectrum looks different:

(1) For $AUT_*^2 \gg 8D\ln(k_{\perp M}/k_{\perp m})$, the transverse temperature is approximately the same for all populated states, $T_{k_\perp} \approx T_*$. The conservation law $I \approx T_*\ln(k_{\perp M}/k_{\perp m})$ allows to reduce the above inequality to $AUT_*^2 \gg 8DI/T_*$.

(2) For $8DI/T_* \gtrsim AUT_*^2 \gg 8D$, the diffusion reshapes the spectrum at $8D\ln(k_{\perp M}/k_\perp) \gtrsim T_*^2 AU$, while at $8D\ln(k_{\perp M}/k_\perp) \ll T_*^2 AU$ the temperature profile stays flat, $T_{k_\perp} \approx T_*$.

(3) For $AUT_*^2 \lesssim 8D$, the diffusion reshapes the entire spectrum.

VI. REDUCED EVOLUTION EQUATION

The analytical expression of the spectrum at $k_\perp \ll k_{\perp M}$ through parameters at its upper boundary allows to reduce the problem of multiscale spectrum evolution to a much simpler problem of solving Eqs. (8) and (9) just for photons with $k_\perp \sim k_{\perp M}$. The problem can be further simplified by using, instead of Eq. (9), the reduced Eq. (14) for the transverse energy flux $\mathbb{J}_{k_\perp}$. Also, the approximation $\mathcal{I}_{k_\perp} \approx I$ can be used in Eq. (14) at $k_\perp \sim k_{\perp M}$, since nearly all photons are located at $k_\perp \ll k_{\perp M}$. In the Eq. (8) at $k_\perp \sim k_{\perp M}$, the diffusion term (which is about $\sim DT_{k_{\perp M}}/k_{\perp M}^2$) can be neglected compared to the 4-photon scattering term (which is about $\mathbb{J}_{k_{\perp M}}/k_{\perp M}^2 \gtrsim 4DI/k_{\perp M}^2 \gg DT_{k_{\perp M}}/k_{\perp M}^2$). Thus simplified, Eqs. (8) and (9) can be presented in the form

$$\frac{\partial}{\partial t}k_\perp T_{k_\perp} \approx -\frac{\partial}{\partial k_\perp}\mathbb{J}_{k_\perp}, \tag{22}$$

$$\mathbb{J}_{k_\perp} \approx UIT_*^2\mathcal{J}(x), \quad x = \frac{k_\perp}{k_{\perp M}}, \quad y(x) = \frac{T_{k_\perp}}{T_*}, \tag{23}$$

$$\mathcal{J}(x) = 4\pi\int_0^x dx_2 x_2^2\int_{\sqrt{x^2-x_2^2}}^\infty dx_3\left[\frac{y(x_2)y(x_3)}{x_2^2x_3^2} - \frac{y\left(\sqrt{x_2^2+x_3^2}\right)y(x_3)}{\left(x_2^2+x_3^2\right)x_3^2} - \frac{y\left(\sqrt{x_2^2+x_3^2}\right)y(x_2)}{\left(x_2^2+x_3^2\right)x_2^2}\right]. \tag{24}$$

For a moderately small $x = k_\perp/k_{\perp M}$, the solution of these equations should merge with the quasi-stationary spectrum found above. This provides the boundary condition for Eqs. (22)–(24) at $x \ll 1$. According to the Eqs. (18) and (19), it can be presented in the form

$$\mathcal{J}(x)|_{x\ll 1} \approx \frac{4D}{UT_*^2}, \quad y(x)|_{x\ll 1} \approx \sqrt{1 + \frac{8D}{AUT_*^2}\ln\frac{1}{x}}. \tag{25}$$

Since nearly all transverse energy is located at $k_\perp \sim k_{\perp M}$, its variation should be correctly described by the simplified equations. This can be verified by integrating Eq. (22) over $k_\perp \sim k_{\perp M}$, which gives

$$\frac{d}{dt}\int dk_\perp k_\perp T_{k_\perp} \approx 4ID, \tag{26}$$

in agreement with Eq. (7). Equation (7) now takes the form

$$k_{\perp M}^2 T_* F_1 \approx \mathcal{E}_0 + 4IDt, \quad F_1 = \int_0^\infty dx x y(x). \tag{27}$$

Integration of the multiplied by $k_\perp$ Eq. (22) over $k_\perp \sim k_{\perp M}$ gives

$$\frac{d}{dt}k_{\perp M}^3 T_* F_2 \approx UIk_{\perp M}T_*^2 F_3, \tag{28}$$

$$F_2 = \int_0^\infty dx x^2 y(x), \quad F_3 = \int_0^\infty dx\,\mathcal{J}(x). \tag{29}$$

So far, $k_{\perp M}$ was defined just up to a factor of the order of 1. This factor can be selected now such that

$$F_3 = F_2. \tag{30}$$

VII. SPECTRUM AT THE UPPER BOUNDARY

A. The stage $AUT_*^2 \gg 8D \iff \mathcal{E}_0 \gg 16IDt$

During the stage $AUT_*^2 \gg 8D$, the boundary condition Eq. (25) can be approximately presented in the form

$$\mathcal{J}(+0) \approx 0, \qquad y(+0) \approx 1. \tag{31}$$

The condition $AUT_*^2 \gg 8D$ is equivalent to $\mathcal{E}_0 \gg 16IDt$ (see below), so that the transverse energy variation can be neglected and

$$k_{\perp M}^2 T_* F_1 \approx \mathcal{E}_0. \tag{32}$$

The functions $y(x)$ and $\mathcal{J}(x)$ appear to be time-independent at this stage. The variable separation in Eq. (22) and integration of the split equations gives

$$\mathcal{J}(x) \approx x^2 y(x), \tag{33}$$

$$k_{\perp M}^4 \approx k_{\perp M0}^4 + 4UI\mathcal{E}_0 t/F_1. \tag{34}$$

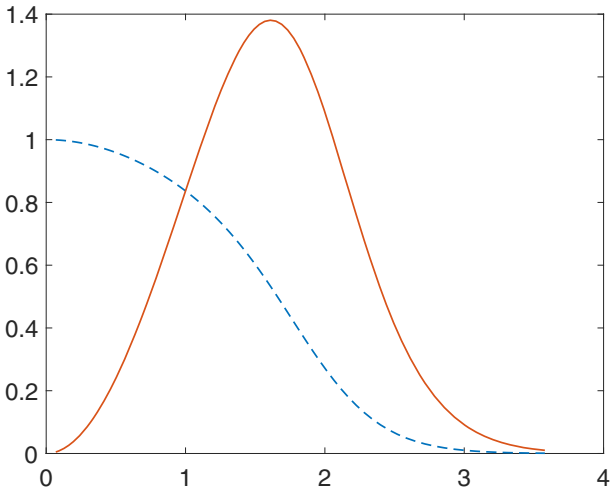

FIG. 1. The normalized transverse temperature of photons $y(x)$ (dashed line) and spectral flux of transverse energy $\mathcal{J}(x)$ (solid line) around the spectrum upper boundary $x = k_\perp/k_{\perp M} \sim 1$ at early evolution stage.

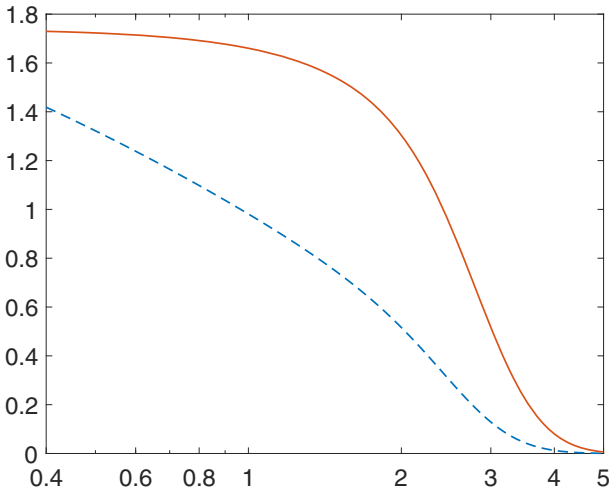

FIG. 2. The normalized transverse temperature of photons $y(x)$ (dashed line) and spectral flux of transverse energy $\mathcal{J}(x)$ (solid line) around the spectrum upper boundary $x = k_\perp/k_{\perp M} \sim 1$ at advanced evolution stage.

Solution of Eqs. (24) and (33) with the boundary condition Eq. (31) is shown in Fig. 1. For this solution,

$$F_1 \equiv F_{1-} \approx 1.5, \qquad F_2 \equiv F_{2-} \approx 2.0. \tag{35}$$

Using Eqs. (32), (34) (at $k_{\perp M}^4 \gg k_{\perp M0}^4$), (35), and $A \approx 3.0$, the condition $AUT_*^2 \gg 8D$ can be presented in the form

$$1 \gg \frac{8D}{AUT_*^2} \approx \frac{8DF_1^2k_{\perp M}^4}{AU\mathcal{E}_0^2} \approx \frac{16IDt}{\mathcal{E}_0}; \tag{36}$$

i.e., $\mathcal{E}_0 \gg 16IDt$, as mentioned above.

B. The stage $4IDt \gg \mathcal{E}_0$

During this stage T_* is constant. The functions $y(x)$ and $\mathcal{J}(x)$ are also time-independent, but not the same as found above for the earlier stage, and the integrals F_1 and F_2 are not the same as in Eq. (35). It follows from Eqs. (27)–(30) that

$$T_*^2 \approx \frac{6D}{UF_1}, \qquad k_{\perp M}^2 \approx It\sqrt{\frac{8UD}{3F_1}}. \tag{37}$$

The boundary condition Eq. (25) now takes the form

$$\mathcal{J}(+0) \approx 2F_1/3, \qquad y(x)|_{x\ll 1} \approx \sqrt{1 + \frac{4F_1}{3A}\ln\frac{1}{x}}. \tag{38}$$

Equation (22) reduces to

$$x^2\frac{d}{dx}y(x) \approx 3\frac{d}{dx}\mathcal{J}(x). \tag{39}$$

By integration from x to ∞, it can be presented in the form

$$x^2y(x) + 2\int_x^\infty dx_1x_1y(x_1) \approx 3\mathcal{J}(x), \tag{40}$$

which automatically gives correct $\mathcal{J}(+0)$. Solution of Eqs. (24) and (40) with $y(x)|_{x\ll 1}$ given by Eq. (38) is shown in Fig. 2. For this solution,

$$F_1 \equiv F_{1+} \approx 2.6, \qquad F_2 \equiv F_{2+} \approx 4.4. \tag{41}$$

VIII. WEAK TURBULENCE APPLICABILITY DOMAIN

To keep the multiscale turbulence weak at all $k_\perp$, the rate of phase mixing should be much larger than the rate of 4-photon scattering at each $k_\perp$. This requirement is the most restrictive at $k_\perp \sim k_{\perp m}$ where the rate of phase mixing is the smallest, while the rate of 4-photon scattering is the largest. The condition for turbulence be weak at $k_\perp \sim k_{\perp m}$ is

$$\beta k_{\perp m}^4 \gg UT_{k_{\perp m}}^2 \iff \beta k_{\perp m}^2 \gg \alpha T_{k_{\perp m}}. \tag{42}$$

According to Eqs. (19) and (21),

$$\ln\frac{k_{\perp M}}{k_{\perp m}} \approx \frac{AU}{8D}\left(T_{k_{\perp m}}^2 - T_*^2\right), \tag{43}$$

$$T_{k_{\perp m}} \approx \left(\frac{12ID}{AU} + T_*^3\right)^{1/3}. \tag{44}$$

According to Eqs. (32) and (34), the quantity $T_*^2 \propto k_{\perp M}^{-4}$ decreases inversely with time during the stage $AUT_*^2 \gg 8D$. As long as $T_*^3 \gg 4ID/U$, it follows from Eqs. (43) and (44) that $\ln(k_{\perp M}/k_{\perp m}) \approx I/T_*$, the diffusion is negligible, and $k_{\perp m}$ decreases exponentially in time like at $D = 0$. The decrease of $k_{\perp m}$ continues, but already slowed down by the diffusion, at $T_*^3 \ll 4ID/U \approx T_{k_{\perp m}}^3$, when Eq. (43) takes the form

$$\ln\frac{k_{\perp M}}{k_{\perp m}} \approx \frac{3}{4}\left(\frac{2UI^2}{D}\right)^{1/3} - \frac{3UT_*^2}{8D}. \tag{45}$$

The weak turbulence applicability condition Eq. (42) then takes the form

$$\frac{\beta k_{\perp M}^2}{\alpha I} \gg \left(\frac{4D}{UI^2}\right)^{1/3}\exp\left[3\left(\frac{UI^2}{4D}\right)^{1/3} - \frac{3UT_*^2}{4D}\right]. \tag{46}$$

The decrease of $k_{\perp m}$ stops at $AUT_*^2 \sim 8D$, i.e., at $16IDt \sim \mathcal{E}_0$. Later, at $4IDt \gg \mathcal{E}_0$, both $k_{\perp M}^2$ and $k_{\perp m}^2$ grow linearly in the time, while T_* and $T_{k_{\perp m}}$ no longer change. For $4IDt \gtrsim \mathcal{E}_0$ (when $\mathcal{E} \sim 4IDt$), the weak turbulence applicability condition

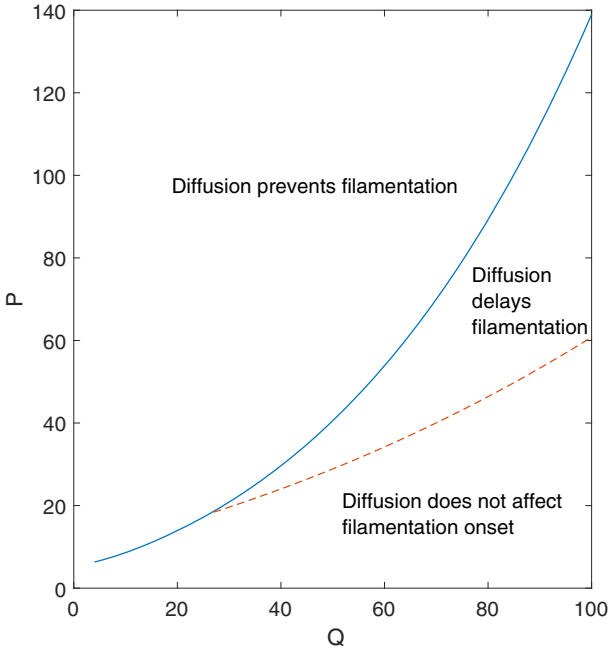

FIG. 3. Divisions of the parameter plane (Q, P).

Eq. (46) reduces, taking into account Eqs. (27), (37), and (41), to

$$\frac{\beta\mathcal{E}}{\alpha I^2} \gg \left(\frac{UI^2}{D}\right)^{-5/6} \exp\left[3\left(\frac{UI^2}{4D}\right)^{1/3}\right]. \tag{47}$$

It is the strictest at $\mathcal{E} \sim \mathcal{E}_0$, and gets softer, as $\mathcal{E}$ grows. For a single-scale initial spectrum $\mathcal{E}_0 \sim Ik_{\perp 0}^2$, so that the quantity

$$P \equiv \frac{\beta\mathcal{E}_0}{\alpha I^2} = \frac{\beta k_{\perp 0}^2}{\alpha I} \gg 1$$

has the physical meaning of the ratio of the initial phase mixing rate to the initial nonlinear frequency shift. The physical meaning of the quantity

$$Q \equiv \frac{UI^2}{D} \gg 1$$

is the ratio of the 4-photon scattering initial rate to the initial rate of ray diffusion on random inhomogeneities. Divisions of the parameter plane (Q, P) are shown schematically in Fig. 3. The diffusion basically prevents the filamentation at

$$P > Q^{-5/6} \exp[3(Q/4)^{1/3}].$$

If this is not satisfied, but

$$Q^{-5/6} \exp[3(Q/4)^{1/3}] > P > 6(2Q)^{-2/3} \exp[(2Q)^{1/3}],$$

then the diffusion starts modifying the spectrum within the weak turbulence regime and thus somewhat delays the condensate formation and filamentation. For an even smaller P, the diffusion basically does not affect the filamentation onset.

IX. DISCUSSION

We described analytically the evolution of transverse spectrum of photons paraxially propagating in a Kerr medium with small random inhomogeneities of refractive index. This evolution can be outlined as follows. As long as the photon diffusive scattering on random inhomogeneities remains negligible compared to the nonlinear 4-photon scattering, the lower spectrum boundary $k_{\perp m}$ decreases exponentially in the propagation time (or distance). The diffusive scattering first manifests at the lower spectrum boundary and starts reshaping the spectrum from there. The reshaping then spreads across the spectrum toward its upper boundary $k_{\perp M}$. During this stage, $k_{\perp M}^4$ keeps growing linearly in time, while $k_{\perp m}$ keeps decreasing though slower than earlier. The Bose-Einstein condensation associated with the decreasing $k_{\perp m}$ stops when the reshaping front reaches the upper spectrum boundary $k_{\perp M}$. The further spectrum evolution occurs in a self-similar regime with both $k_{\perp M}^2$ and $k_{\perp m}^2$ growing linearly in time.

The analytical theory developed here may be useful in suggesting improvements in possible applications, in particular, in backward Raman amplification of powerful laser pulses in plasmas [22]. The transverse filamentation instability of powerful coherent lasers in plasmas is a major factor that may limit the extended backward Raman amplification regimes proposed in Refs. [23,24]. This limitation can be overcome by transverse randomization of amplified pulses. From the theoretical viewpoint, the most straightforward would be the scheme that splits amplification and randomization processes. In such a scheme, the amplification would occur in uniform plasma layers, while the randomization would occur in relatively thin transverse sheets of denser randomly inhomogeneous plasma inserted periodically in the uniform plasma. These sheets would act similarly to standard random phase plates [25–27] (which obviously cannot withstand nearly relativistically intense laser pulses in plasmas). More practical might be, however, to have the amplification and randomization going together in the same plasma with statistically more or less uniform random density inhomogeneities. Such inhomogeneities may significantly modify the amplification process. An accurate theory for this modification still has to be developed. Since inhomogeneities reduce Raman coupling between amplified and pump laser pulses, the amplification to a given intensity would likely take a longer distance. Assuming complete pump depletion, a longer output pulse carrying a larger fluence could be expected.

In a broader context, the substantially extended propagation of powerful laser pulses in plasma without filamentation could make feasible quite new laser amplification regimes, including plasma-based powerful random lasers [28–30].

Beyond plasma-based applications, our theory may be used, for example, to describe how small random inhomogeneities affect the experimentally observed condensation of classical waves in photonic crystals [31]. These experiments were carried out in defocussing crystals, but there is no difference between focusing and defocussing regimes within the applicability range of weak turbulence theory. Even when the condensate forms, it does not make much difference, as long as the number of photons in weakly turbulent component of

spectrum is larger than in condensate. The level of inhomogeneities in this kind of experiments can be controlled [32], which allows to verify how it affects the condensation. For very small inhomogeneity levels, most of photons could condensate before the effect of transverse heating of the photon beam through the scattering on random inhomogeneities is manifest. Yet, even that small transverse heating can ultimately lead to the condensate evaporation; and the theory can be extended to describe this regime as well.

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