Physics of Plasmas ARTICLE scitation.org/journal/php

Understanding turbulence in compressing plasma as a quasi-EOS

Cite as: Phys. Plasmas **26**, 062709 (2019); doi: 10.1063/1.5098790
Submitted: 3 April 2019 · Accepted: 27 May 2019 ·
Published Online: 18 June 2019

View Online Export Citation CrossMark

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Note: This paper is part of the Special Collection: Papers from the 60th Annual Meeting of the APS Division of Plasma Physics.
Note: Paper TI3 1, Bull. Am. Phys. Soc. **63** (2018).
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ABSTRACT

Inspired by experimental Z-pinch results, we investigate plasma turbulence undergoing compression. In addition to Z-pinches, plasma turbulence can be compressed in a range of natural and laboratory settings, including inertial fusion experiments and astrophysical molecular clouds. The plasma viscosity, when modeled as described by Braginskii, depends strongly on both temperature and ionization state, giving it the possibility to have a large range of behavior. Here, we highlight the importance of viscous variation in these settings, as well as various insights that can be gained by considering this variation. Included are a "sudden viscous dissipation" effect that leads to a new concept for inertial fusion or X-ray bursts and a bound on turbulent energy behavior under compression. This bound, which was previously applied in inviscid molecular cloud turbulence, is here shown in an application to turbulence that transitions from inviscid to viscous regimes. The task of understanding turbulence under compression can be cast as the process of seeking a "quasi equation of state" for turbulent energy under compression.

I. INTRODUCTION

Nonradial hydrodynamic motion in the hot spots (gas fill) of inertial-fusion experiments may be seeded by interfacial instabilities (e.g., Rayleigh–Taylor or Richtmeyer-Meshkov instabilities) or by implosion asymmetry generated by a variety of possible sources.[1,2] In Magnetized Liner Inertial Fusion experiments,[3] a laser is used to heat the fusion fuel prior to compression by a Z-pinch. Spatial nonuniformity of the laser heating may lead to nonradial flows in the fuel, which will then be compressed, representing a possible source for nonradial flow in these experiments beyond interface instabilities or implosion asymmetry. Other mechanisms capable of generating nonradial or turbulent flow may be at play in compression experiments; experiments in gas-puff Z-pinches suggest significant, and likely turbulent, nonradial hydrodynamic motion at stagnation,[4–6] the source of which is still unclear.

Here, nonradial hydrodynamic motion refers to motion not associated with the compression itself; this motion may be regarded as "wasted energy" to the extent that it does not contribute to heating in the stagnation process. If a substantial fraction of implosion energy is converted into such nonradial flow, the drive energy required to compress to targeted temperatures and densities may be larger than expected. The presence of a large quantity of hydrodynamic motion can have impact beyond the gross energetics. For example, if the energy in the hydrodynamic motion is comparable to, or larger than, the thermal energy, the flows can be highly compressible, inducing density fluctuations. This appears to be the case in a gas-puff Z-pinch stagnation, where these density fluctuations must be accounted for in order for a correct, self-consistent, analysis of spectroscopic measurements.[6] Even if the amount of energy deposited into nonradial flows is small enough so as to be negligible from the standpoint of overall implosion energetics, such flows can still have important impacts on experiments. Small quantities of nonradial motion may degrade implosion performance,[2,7–9] for example, through inducing mix of colder or nonfuel capsule material into the hot-spot.

The possibility of nonradial flows in compression experiments, combined with their broad range of impact, drives a need to understand the effects of compression on the production and evolution of such flows. For the rest of this paper, we will use turbulent flow and nonradial flow interchangeably, and focus on 3D compression of homogeneous, isotropic turbulence. However, we may also consider the conceptual framework presented here for the case when the nonradial flow is not truly turbulent, and various observed effects (such as the sudden viscous dissipation effect touched on later) can still occur for nonturbulent flows. Further, even the behavior of turbulent nonradial flows during compression may exhibit initial-condition dependence.[10] Here we focus solely on the effects of compression on a volume of flow. In particular, no interface between materials is needed for any growth of the turbulent kinetic energy (TKE) discussed here. Further, we will consider constant velocity compressions, where there is no acceleration of the compression velocity, and thus no acceleration is necessary for the effects discussed here. Then, the compression effects here could be considered to exist on top of whatever interface or other instabilities exist in a compression experiment.

The effects of compression on turbulence have been studied extensively, in neutral fluids and gases (e.g., Refs. 11–23), largely with applications in aerodynamics and combustion (e.g., Refs. 24–26) in mind. The gravitational compression of supersonic neutral-gas turbulence has also been investigated, for application in astrophysics.[27] While these prior works form an important knowledge base, there are many properties and processes that are different between plasmas and neutral gases, motivating the study of compressing turbulence specifically in the plasma case. Studying the compression of turbulence including even a single difference between plasmas and gases, for example, the fact that a plasma and a neutral gas have different viscosity dependencies on temperature, is enough to give new effects in the plasma case, such as a sudden viscous dissipation of turbulent energy.[28] For a plasma, the dependency of viscosity on temperature is far more sensitive. This single difference alone will also mean that new models are needed to predict the behavior of plasma turbulence under compression.[10,29,30] Although there are many differences between neutral gas and plasma that can be considered, initial work on compressing plasma turbulence has focused on the impact of including a plasma viscosity. This is particularly important for inertial fusion experiments, with or without a magnetic field, because viscous effects can start to become relevant for a plasma of hydrogen isotopes in the density and temperature regime sought for fusion gain.[31] Thus, the fuel in inertial fusion experiments can cross between inviscid and viscous regimes during the compression,[1,32,33] making it important to study the effects of changing viscosity on compressing turbulence. While initial studies of new viscous phenomena, like the sudden dissipation effect, focused on zero-Mach turbulence limit, this effect has also been studied in the finite-mach-number case.[34]

Beyond differences in viscosity, or other transport coefficients (e.g., thermal conductivity), the dynamics of turbulent plasma under compression may also differ from neutral gases due to the presence of electromagnetic fields. The effect of compression on plasma turbulence including such fields has been investigated only to a limited degree, in the context of astrophysical plasmas.[35] At a less fundamental, but nevertheless important, level, plasma compression experiments, such as those for inertial fusion or X-ray generation, aim for much larger compression ratios than typical neutral gas applications. For example, internal combustion engines compress by a factor of ~ 10 in volume, while experiments at the National Ignition Facility aim to compress the volume by a factor of $\sim 10^4$. These larger compression factors create more opportunity for the compression to impact the turbulence evolution, including more opportunity to input energy into the turbulence.

We can frame the problem of modeling the evolution of plasma turbulence under compression as the search for a "quasi equation of state" ("quasi-EOS"), by analogy with the thermodynamics of gases or plasmas. This analogy will be presented here, and the limitations of this framing (reflected in the qualifier "quasi") will be discussed.

While, as introduced above, there are a number of ways in which turbulence could be an impediment to achieving design targets in magnetized or unmagnetized inertial fusion experiments, or X-ray generating experiments, it is also worthwhile to consider whether there are ways to *utilize* turbulent flows (or more generally nonradial flows) to achieve design targets. The possibility for utilization arises because the impact on an experiment of having energy in these flows is quite different from having the same energy as thermal energy.

To see this, consider two plasmas with the same energy, partitioned differently: one with all thermal energy (temperature), and the other with most of the energy in some sort of nonradial (possibly turbulent) flow, which is therefore a much colder plasma. While, as discussed above, the plasma with flow energy may be more affected by mix, we should also expect that the energy loss rate from this flowing plasma will be different than the loss rate from the thermal plasma. For instance, the colder, flowing plasma will have lower radiation losses. Thus, in certain circumstances, it may be advantageous, in order to reach a higher energy density, to compress the cold flowing plasma rather than the thermal one. In order to convert this flow back into thermal energy late in the compression, one could imagine utilizing the sudden viscous dissipation mechanism. Some basic discussion of this possibility, and challenges confronting such a scheme, are presented in prior work,[28,29,31] but it remains to be determined whether there exists a practicable design. Even if we are unable to use the sudden dissipation mechanism to convert flow energy to thermal energy, its presence as an energy reservoir with different loss mechanisms than the thermal energy could still make it advantageous for certain applications. In other words, it may not always be appropriate to consider such flows wasted energy; apparently there are large quantities of such flow in gas-puff Z-pinch experiments optimized for K-shell radiation yields[4–6] although alternate explanations have not been exhausted.[36] In any event, there is strong motivation, for the sake of trying to avoid or mitigate such flows where they are likely deleterious, to study and model the compression of plasma turbulence. Doing so also enables us to consider its utilization.

The rest of the paper is organized as follows. In Sec. II, we describe the analogy between seeking to study and model the compression of plasma turbulence and the search for a quasi-EOS. Then, in Sec. III, we show the application of a bounding technique,[37] to bound the turbulent energy of a plasma as it compresses and crosses between inviscid and viscous regimes. This bound serves effectively as a type of model for the quasi-EOS. It was first applied to show that a model for the turbulent velocity in compressing molecular cloud turbulence is likely too dissipative[37] due to saturation of the turbulent length scale.[30] In Sec. IV, we summarize our results and discuss them in a broader context.

II. QUASI-EOS

A. Setup

To illustrate how compressed turbulence can be described by a quasi-EOS, we work from a plasma treatment similar to that considered in prior work.[28–30] For a more complete discussion of the treatment, refer to Davidovits and Fisch.[29] Although this treatment operates in the limit where the turbulent flow speeds are small compared to the sound speed (zero-Mach limit), so that compressibility effects in the turbulence can be disregarded, the same ideas can also be applied to the compressible case. We consider a plasma modeled by the Navier-Stokes (NS) equations, in a cubic domain, of initial side length L_0. The plasma has a viscosity that in principle depends on the temperature and charge state. We break the NS velocity flowfield, $\mathbf{u}$, into two components

$$\mathbf{u} = \mathbf{u}_{\rm background} + \mathbf{v}. \tag{1}$$

The background flow generates the compression

$$\mathbf{u}_{\rm background} = \frac{\dot{L}}{L}\mathbf{x}, \tag{2}$$

where the side length of the cube contracts at a constant velocity

$$L(t) = L_0 - 2U_b t \tag{3}$$

and the overdot indicates a time derivative. The compression generated by the background flow is such that, if we place a cube of initial side length L_0 in the flow, and the cube is advected by the flow, it will remain a cube with a side length that shrinks in time as $L(t)$.

It is convenient then to work in coordinates, $\mathbf{X}$, that move with the flow; these are related to the original NS coordinates, $\mathbf{x}$, by $\mathbf{x} = \bar{L}\mathbf{X}$. Here, we have defined

$$\bar{L} = \frac{L}{L_0} = 1 - 2\frac{U_b}{L_0}t. \tag{4}$$

In the zero-Mach compression limit, any density fluctuations drop out, and the continuity equation will give us the expected result for the density, $\rho = \rho_0(L_0/L)^3$.

Our interest is to solve for the rest of the flow, $\mathbf{v}$. Working from the definitions above, we can write down a NS equation for $\mathbf{V}(\mathbf{X}, t) = \mathbf{v}(\mathbf{x}, t)$. This is

$$\frac{\partial \mathbf{V}}{\partial t} + \bar{L}^{-1}\mathbf{V}\cdot\nabla\mathbf{V} - \frac{2U_b}{L}\mathbf{V} + \bar{L}^2\frac{\nabla P}{\rho_0} = \frac{\bar{L}\mu(T,Z)}{\rho_0}\nabla^2\mathbf{V}, \tag{5}$$

where μ is the (dynamic) viscosity, which in principle can depend on both the temperature T and charge state Z.

In the moving frame, the energy density in the (nonradial) flow is $E = \rho_0\mathbf{V}^2/2$. The total energy is then $E_{\rm total} = \int\int\int_{-L_0/2}^{L_0/2} {\rm d}\mathbf{X}E$. This total energy is the same as in the laboratory frame (in the laboratory frame, the density increases, but the volume to be integrated decreases in a manner that balances it). Using Eq. (5), we can write an equation for $\partial E/\partial t$, which can then be integrated in space to arrive at an evolution equation for ${\rm d}E_{\rm total}/{\rm d}t$. This equation will be

$$\frac{{\rm d}E_{\rm total}}{{\rm d}t} = \frac{4U_b}{L}E_{\rm total} - \bar{L}\bar{\mu}(t)\epsilon_\mu, \tag{6}$$

where ϵ_μ is associated with the viscous dissipation (and will generally be positive, so this term overall is a damping), and the first term on the right of the equals sign comes from the compressive forcing. Here, we have assumed that the temperature and charge state, and therefore the viscosity, depend only on time, and have, therefore, written $\mu(T(t), Z(t)) = \mu_0\bar{\mu}(t)$. The viscous dissipation is

$$\epsilon_\mu = -\int\int\int_{-L_0/2}^{L_0/2} {\rm d}\mathbf{X}\mu_0\mathbf{V}\cdot\nabla^2\mathbf{V}. \tag{7}$$

Rewriting Eq. (6) in terms of the volume, ${\rm d}V = 3L^2\dot{L}{\rm d}t$, we find

$$-\frac{{\rm d}E_{\rm total}}{{\rm dV}} = \frac{2}{3}\frac{E_{\rm total}}{V} - \frac{1}{6U_bL_0L}\bar{\mu}(t)\epsilon_\mu. \tag{8}$$

B. Quasi-EOS and discussion

From Eq. (8), it is natural to define a "turbulent" or "nonradial" flow pressure

$$p_{\rm TKE} = \frac{2}{3}\frac{E_{\rm total}}{V} \tag{9}$$

since this quantity relates the infinitesimal energy injection (into the flow) in a compression to the volume increment, ${\rm d}E_{\rm total,injected} = -p_{\rm TKE}{\rm d}V$. Although the system is not in thermodynamic equilibrium, this then represents a kind of quasi-EOS, relating the flow energy to an effective pressure. In general, we are interested in trying to capture the full evolution of $E_{\rm total}$ (and therefore also $p_{\rm TKE}$) in the compression. That is, we would like to solve Eq. (8), and this modeling task is subsumed in the quasi-EOS description. If we had a solution for the evolution of $E_{\rm total}$, we could plug it into Eq. (9), to give us the quasi-EOS in terms of quantities more primitive than $E_{\rm total}$. This will not be a true EOS because this solution for $E_{\rm total}$, and therefore the effective pressure, will in general depend on the thermodynamic trajectory (compression history) of the system, and thus not only on state variables of the system. However, while the turbulent pressure is not generally a function only of state variables of the system, in special circumstances it can be included in a new variable which is, as now described.

We see that the relationship of $p_{\rm TKE}$ with system flow energy density is the same as that of the pressure of an ideal gas with the thermal energy density; $p_{\rm gas} = \frac{2}{3}U_{\rm gas}/V$, with $U_{\rm gas} = (3/2)Nk_bT$ the ideal (monatomic in this case) gas thermal energy. As a result, if we considered as a system both the total nonradial flow energy and the thermal energy, within an adiabatic compression so that there is no net energy loss (but flow energy can dissipate into thermal energy through the viscosity), we will find that the *total energy* (flow plus thermal, $E_{\rm total} + U_{\rm gas}$) or *total pressure* is still a state-function of the compression; it depends only on $\bar{L}$, and the total energy will grow as $\bar{L}^{-2}$, as shown in Davidovits and Fisch.[30] This state-function property of the total energy in an adiabatic compression need not hold in general for nonisotropic compressions, such as two-dimensional Z-pinch compressions.[30]

The fact that this total energy is a state function of the compression when the compression is adiabatic and isotropic (and boundary effects are ignored) means that, for a given compression, the final amount of energy present, and therefore the difficulty of compressing the plasma, depends only on the initial energy present, not on the partition of this energy between turbulent and thermal. It also means that, in adiabatic compressions that experience the sudden dissipation effect, there will again be a sudden effect only on the partition of the energy, and not on the amount of energy present or the difficulty of compressing the plasma. Of course, this is only in the adiabatic limit, where there is no net energy loss from the compressing plasma. In compression experiments such as those for fusion or X-ray production, there are typically large energy loss mechanisms at play, which will break this state function property and end up giving a (time-dependent) compressibility and overall energy behavior that depends on the initial energy partition. For the thermal energy, these loss mechanisms include radiation and thermal conduction. Boundary effects and inflows or outflows of mass into the turbulent region considered are two possible direct effects on the flow energy dissipation.

There are other flow properties we may be interested in modeling beyond the total turbulent energy, and two states with the same p_{TKE} could have differences in such properties (e.g., characteristic length scale); thus, this quasi-EOS for p_{TKE} is not a full descriptor of the state of the system with respect to the flow, and one may add relations to it.

We can continue the thermodynamic analogy by computing the effective polytropic index for p_{TKE}. The polytropic relation, $pV^n = C$, implies

$$n = \frac{\partial \ln p}{\partial \ln \rho}. \tag{10}$$

Using Eq. (9) in Eq. (10), we can find the polytropic index for p_{TKE}, assuming the turbulent evolution equation, Eq. (8). We find

$$n = \frac{5}{3} - \frac{L_0}{6U_b}\frac{\bar{L}^2\bar{\mu}\epsilon_\mu}{E_{\mathrm{total}}}. \tag{11}$$

Defining a compression time scale as $\tau_c = L/2U_b$, and a turbulent turnover time scale as $\tau_t = 2E_{\mathrm{total}}/\bar{L}\bar{\mu}\epsilon_\mu$, we can write the polytropic index as

$$n = \frac{5}{3} - \frac{2}{3}\frac{\tau_c}{\tau_t}. \tag{12}$$

The dimensionless quantity τ_c/τ_t can also be written as a Reynolds number, as follows. First, define a lengthscale $l^2 = -\int\int\int_{-L_0/2}^{L_0/2} \mathrm{d}\mathbf{X}\mathbf{V}^2/\int\int\int_{-L_0/2}^{L_0/2} \mathrm{d}\mathbf{X}\mathbf{V}\cdot\nabla^2\mathbf{V}$, then define a related lengthscale $\mathcal{L} = Ll^2/L_0^2$. With these definitions, we can define $\mathrm{Re}_c = 2U_b\mathcal{L}\rho/\mu$, and write

$$n = \frac{5}{3} - \frac{2}{3}\frac{1}{\mathrm{Re}_c}. \tag{13}$$

For certain viscosity behavior ($\bar{\mu}$), it can be shown there is a steady state solution to Eq. (6).[29,38] For this (statistical) steady state solution, $\tau_t = \tau_c$ ($\mathrm{Re}_c = 1$), and we will find $n = 1$ in Eq. (12), an "isoturbulent" compression, as expected. For times when the compression is very fast compared to the turbulent turnover time scale ($\tau_c \ll \tau_t$ or, alternatively, $\mathrm{Re}_c \gg 1$), Eq. (12) shows that the turbulence compresses "adiabatically" with $n \lesssim 5/3$. In general, the turbulent time scale, τ_t will evolve during the compression (except for the steady state case). Since τ_c is known, finding the polytropic index evolution requires finding the evolution of τ_t (or, Re_c). This is a recasting of the quasi-EOS modeling task described above, where we must solve Eq. (8). Recent works modeling the evolution of compressing turbulence with a plasma viscosity serve as example solutions for the quasi-EOS.[10,30] In compressions where the turbulent energy first grows and then dissipates, the peak energy corresponds to zero total energy derivative with respect to compression, and therefore to $n = 1$, and $\tau_c = \tau_t$ or $\mathrm{Re}_c = 1$.

In the presentation here, we have restricted our consideration of the turbulent evolution under compression to the effects included in Eq. (6). In general, we could consider additional effects which act either as further forcing or dissipation terms on the turbulence; such effects would then enter into the evolution equation for E_{total}, and therefore also into the equation for the polytropic index. We could also relax assumptions, such as the zero-Mach limit assumption or 3D isotropic compression assumption, and still define a turbulent pressure [which need not be identical to Eq. (9)] and a quasi-EOS.

III. BOUND

It is possible to bound the TKE (turbulent kinetic energy) behavior of a compressing flow governed by Eq. (5). The bounding technique which we present here was first developed in an application to the 3D, isothermal, compression of supersonic turbulence, such as occurs in molecular clouds. In this context, it was applied as a validation tool for modeling and simulation.[37] Because the molecular cloud is treated as compressing isothermally, the viscosity is constant during the compression, and the plasma will stay inviscid. The bound can also be applied to compressing turbulence in the zero-Mach limit of Eq. (5), including in the case when the viscosity changes during the compression. We briefly show this application here, but readers are referred to Ref. 37 for more detailed discussion of the bounding technique, and to Ref. 30 for some follow-up discussion of the likely underlying mechanism for the apparent violation of the bound by a model demonstrated in Ref. 37. In addition to showing the application of the bound to the varying-viscosity, zero-Mach-limit case, we motivate a new kind of bound on the TKE behavior of compressing turbulence. Whereas the previous bound is based on consideration of the viscous dissipation term, the new bound is based on consideration of the nonlinear term. We will therefore refer to these two bounds as a "dissipation" bound and a "nonlinear" bound.

A. Dissipation bound

With $\nu_0 = \mu_0/\rho_0$, and again introducing $\bar{\mu} = \mu(t)/\mu_0$, Eq. (5) is

$$\frac{\partial \mathbf{V}}{\partial t} + \bar{L}^{-1}\mathbf{V}\cdot\nabla\mathbf{V} - \frac{2U_b}{L}\mathbf{V} + \bar{L}^2\frac{\nabla P}{\rho_0} = \nu_0\bar{L}\bar{\mu}\nabla^2\mathbf{V}. \tag{14}$$

We rescale the velocity, pressure, and time as[28,29,39]

$$\mathbf{V} = \bar{L}^\delta\hat{\mathbf{V}}, \tag{15}$$

$$P = \bar{L}^\eta\hat{P}, \tag{16}$$

$$\mathrm{d}\hat{t} = \bar{L}^\tau\mathrm{d}t. \tag{17}$$

If we then pick $\delta = -1$, $\tau = -2$, and $\eta = -5$, the rescaled momentum equation becomes

$$\frac{\partial \hat{\mathbf{V}}}{\partial \hat{t}} + \hat{\mathbf{V}}\cdot\nabla\hat{\mathbf{V}} + \frac{\nabla\hat{P}}{\rho_0} = \nu_0\bar{L}^3\bar{\mu}\nabla^2\hat{\mathbf{V}}. \tag{18}$$

Associated with Eq. (14) is an energy equation, Eq. (6), for which we seek the solution. Call this solution E_{sol}. We can equivalently solve the energy equation associated with Eq. (18); the solution of this equation, $\hat{E}_{\mathrm{sol}}$, will be simply related, through the transformed velocity, to the sought for solution, $E_{\mathrm{sol}} = \bar{L}^{-2}\hat{E}_{\mathrm{sol}}$.

Now, consider the alternate momentum equation

$$\frac{\partial \hat{\mathbf{V}}}{\partial \hat{t}} + \hat{\mathbf{V}}\cdot\nabla\hat{\mathbf{V}} + \frac{\nabla\hat{P}}{\rho_0} = \nu_0\nabla^2\hat{\mathbf{V}} \tag{19}$$

which is Eq. (18), but with the time dependent coefficient of the viscosity (dissipation) term removed, $\bar{L}^3\bar{\mu} = 1$. Equation (19) is the usual Navier-Stokes momentum equation. Turbulence governed by this equation will decay, and this decay is well studied. Define $\hat{E}_{\text{decay}}$ to be the solution to the energy equation associated with Eq. (19).

Suppose now that, for all times during the flow evolution, the time-dependent portion of the viscosity coefficient in Eq. (18), $\bar{L}^3\bar{\mu}$, is less than or equal to 1. The (unproven) basis for the bound is that the TKE dissipation in this case should be no greater than the case where $\bar{L}^3\bar{\mu}=1$ for all time, Eq. (19), and therefore we should have $\hat{E}_{\text{sol}} \geq \hat{E}_{\text{decay}}$. By the transformation, we can then say

$$E_{\text{sol}} \geq \bar{L}^{-2}\hat{E}_{\text{decay}}; \quad (\bar{L}^3\bar{\mu}(t) \leq 1) \tag{20}$$

and the decay solution gives us a lower bound of the energy behavior of the compressing case. If, instead, we had $\bar{L}^3\bar{\mu} \geq 1$ for all times, then we expect the decay solution to instead give an upper bound on the energy behavior.

B. Nonlinear bound

In Sec. III A, we rescaled Eq. (14) so that only the viscous dissipation term has a time-dependent coefficient. We can instead rescale so that only the nonlinear term ($\sim\mathbf{V}\cdot\nabla\mathbf{V}$) has a time-dependent coefficient. For simplicity, as in prior work, assume $\bar{\mu}=\bar{L}^{-2\beta}$, where β is an exponent for the viscosity growth (or decrease) under compression, which is determined by the net effects of heating (or cooling) and ionization (or recombination) during compression.[30] Now, rescaling as before, but with $\delta=-1$, $\eta=-2(1+\beta)$, and $\tau=1-2\beta$, we find the transformed momentum equation

$$\frac{\partial\hat{\mathbf{V}}}{\partial\hat{t}} + \bar{L}^{-3+2\beta}\hat{\mathbf{V}}\cdot\nabla\hat{\mathbf{V}} + \frac{\nabla\hat{P}}{\rho_0} = \nu_0\nabla^2\hat{\mathbf{V}}. \tag{21}$$

As before, the solution to the energy equation associated with Eq. (21), which we again call $\hat{E}_{\text{sol}}$, will be related to the solution we seek, E_{sol}, as $E_{\text{sol}}=\bar{L}^{-2}\hat{E}_{\text{sol}}$. Also, as before, we consider the solution $\hat{E}_{\text{decay}}$, to the energy equation associated with Eq. (19).

To the extent that the effect of the nonlinear term in decaying 3D Navier-Stokes turbulence is to cascade energy to the dissipation scales, then we may expect that, for a given flowfield, if the coefficient of the nonlinear term is made smaller, the (instantaneous) transfer of energy to the dissipation scales will be reduced. We may hypothesize that, if $\bar{L}^{-3+2\beta} \leq 1$ for all time, we will have $\hat{E}_{\text{sol}} \geq \hat{E}_{\text{decay}}$, so that the decay solution represents a lower bound of the energy behavior in this case

$$E_{\text{sol}} \geq \bar{L}^{-2}\hat{E}_{\text{decay}}; \quad (\bar{L}^{-3+2\beta}(t) \leq 1). \tag{22}$$

On the other hand, if $\bar{L}^{-3+2\beta} \geq 1$ for all time, the decay solution should instead give an upper bound on the energy behavior.

C. Bounds discussion

If we substitute $\bar{\mu}=\bar{L}^{-2\beta}$ into the condition for the dissipation lower bound in Eq. (20), and compare to the condition for the nonlinear lower bound in Eq. (22), we see that, with the exception of when $\beta=3/2$, if one condition is met, the other will not be met. Thus, when the dissipation bound is a lower bound, the nonlinear bound will be an upper bound, and when the dissipation bound is an upper bound, the nonlinear bound will be a lower bound. Thus, if the bounds hold, we can bound on both sides the TKE behavior of a turbulent flow undergoing compression using the solution for that field's TKE decay. The case $\beta=3/2$ is special, and in this case both bounds will be *tight*, since then the equation of interest can be transformed into exactly the decaying turbulence case.

In Fig. 1, we use direct numerical simulation results to test both the dissipation and nonlinear bounds for the cases $\beta=5/2$ and $\beta=3/2$. These simulations, carried out in the pseudospectral code Dedalus,[40,41] are the same as those in previous work, but with a

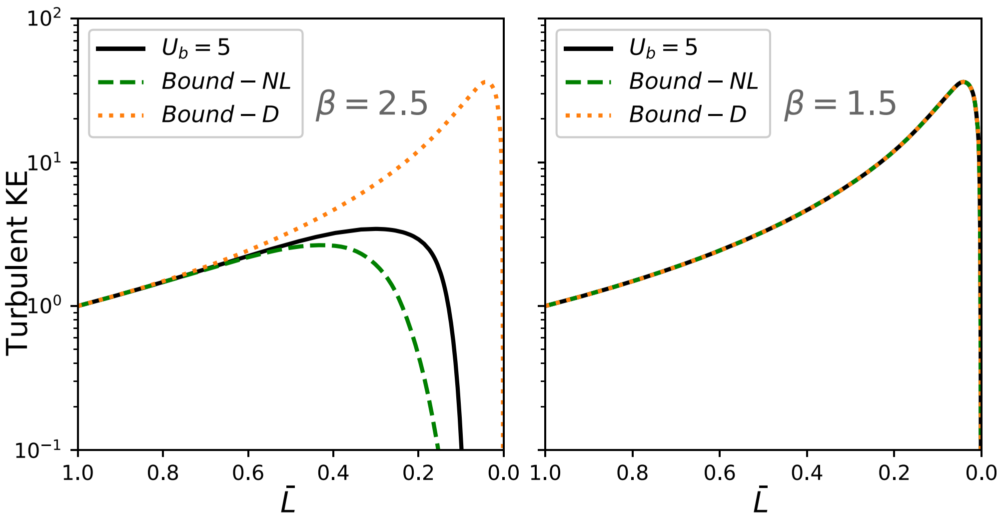

FIG. 1. A comparison of the dissipation based bound (Bound - D) and nonlinear based bound (Bound - NL) with simulation results (solid black line) for the turbulent kinetic energy (TKE) during an isotropic, initially-rapid, compression of a turbulent flowfield in three dimensions. The initial TKE is normalized to 1, and the compression progresses from left to right in the plots, with $\bar{L}$ the compression ratio (side length of the flowfield). The left and right plots display simulation results (and associated bounds) for an identical compression, save for the growth rate of the viscosity with compression, $\mu \sim \bar{L}^{-2\beta}$. In the case where $\beta=3/2$ (the right plot), it can be seen that the bounds are tight (all lines coincide). See Sec. III for the more information on the bounds and this figure.

compression rate normalized to the initial TKE dissipation rate of $U_b = 5$; see Sec. IV in Ref. 30 and lines 1 and 3 of Table I in the same. The case $\beta = 5/2$ corresponds to a fully ionized plasma with a Braginskii viscosity, and obeying an ideal gas equation of state, which is being compressed adiabatically, so that the temperature growth is $T \sim \bar{L}^{-2}$. This temperature growth rate ignores the effect of the dissipated TKE, which is a good approximation in the zero-Mach flow limit. When $\beta = 5/2$, the condition in the nonlinear bound, Eq. (22), reads $\bar{L}^2 \leq 1$, which will be satisfied, since $\bar{L} = 1$ for $t = 0$, and then monotonically decreases in time. Then, the solution is bounded from below using the nonlinear bound in this case. On the other hand, the condition for the dissipation lower bound, Eq. (20), is not satisfied. Instead, the condition for the dissipation bound to be an upper bound, $\bar{L}^3\bar{\mu} \geq 1$, is satisfied for this case ($\bar{L}^{-2} \geq 1$). Comparing with simulations in the left plot in Fig. 1, we see that, at least in this case, both bounds work as expected. The case $\beta = 3/2$, which is trivial for the reason given above, is shown in the figure because it serves to emphasize that the bounds are tight in this situation and may be tight in others, which we discuss below.

In producing Fig. 1, we have used the numerical solution for $\hat{E}_{\rm decay}$ to plot the bounds. While turbulent decay is well studied, for a general initial condition, capturing the TKE behavior during the decay, including possibly from high to low Reynolds number, with a high accuracy is still a complicated modeling problem (e.g., see Ref. 42 for the case where the turbulent length scale is saturated). Nonetheless, even with uncertainty in the decay behavior, this bounding strategy can still inform modeling.[37]

Note that both bounds are motivated and then checked for a few cases, rather than *proven*. The two bounds (nonlinear and dissipative) rest on different assumptions, and it is possible there will be situations in which one or the other, or both, do not hold. The action of the Navier-Stokes nonlinearity is quite complex, and so we should not be surprised if there are turbulent initial conditions, or viscosity time dependencies, for which the nonlinear bound does not end up holding. Note that, in the tested cases, the viscosity monotonically increases in time. Having seen that the bounds become tight as $\beta \to 3/2$, it is interesting to ask whether one or the other bound will be tight in any other scenarios. In the case of the dissipation bound, we may expect that the bound becomes tight in situations in which changes to the viscous dissipation coefficient, or its precise value, do not influence the turbulence dynamics. As seen on the left plot of Fig. 1, this is clearly not the case when the plasma transitions from being inviscid to viscous.

IV. DISCUSSION AND SUMMARY

This work considering the quasi-EOS for turbulence in a plasma compression can be placed in a broader context of considering the behavior of nonthermal energy under compression. As discussed here, Eq. (9) is written for the isotropic compression case, but we can also consider turbulence compressed in a nonisotropic fashion and write an associated quasi-EOS. Beyond turbulent flow states, the dynamics of organized flow profiles, such as solid body rotation, can be considered. Such organized flow profiles display interesting and potentially useful phenomena under compression, and can be at times considered in a true equation-of-state limit.[43–45] We can also consider the utility of organized flow profiles for fusion or X-ray generation schemes utilizing the sudden viscous dissipation phenomenon (which does not require the nonradial flow to be turbulent). However, in the high-Reynolds-number limit, organized flow profiles are not always stable to compression (see, e.g., Refs. 46–48), so that we may end up with at least partially disorganized or turbulent flow profiles even starting from an organized flow.

Similar to storage in various nonradial flow states, nonthermal energy can also be stored in plasma waves during compression. The energy in these waves can exhibit a variety of effects, including ones that are qualitatively similar to certain flow effects, for instance, the wave energy can first amplify and then dissipate suddenly late in the compression.[49–54]

To summarize, plasmas undergo compression in a variety of laboratory experiments, including those targeted at fusion or X-ray production, as well as in natural systems. In these plasma compressions, there are many possible sources for nonradial flow, that is, flow not associated with the compression itself. This nonradial flow can then impact experiment performance or produce important effects in natural systems. As such, understanding and modeling these flows is important. Here, we show how the process of understanding turbulent nonradial flow can be thought of as the search for a quasi equation of state for turbulent flow under compression. We also show two types of bound, which can be useful as a check on models or simulations, or as a type of model (quasi-EOS) in their own right.

ACKNOWLEDGMENTS

This work was supported by the U.S. Department of Energy Fusion Energy Sciences Postdoctoral Research Program administered by the Oak Ridge Institute for Science and Education (ORISE) for the DOE. ORISE is managed by Oak Ridge Associated Universities (ORAU) under DOE Contract No. DE-SC0014664. This work was based in part on an invited presentation for the American Physical Society's Marshall N. Rosenbluth Outstanding Doctoral Thesis Award. It was also supported by NSF Contract Nos. PHY-1506122 and NNSA 67350-9960 (Prime No. DOE DE-NA0001836).

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