

Nonlinear ohmic dissipation in axisymmetric DC and RF driven rotating plasmas

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J. M. Rax,¹ E. J. Kolmes,² I. E. Ochs,² N. J. Fisch,² and R. Gueroult³

AFFILIATIONS

¹ Departments of Physics, University of Paris XI & Ecole Polytechnique, LOA, 91128 Palaiseau, France

² Princeton Plasmas Physics Laboratory, Princeton University, Princeton, New Jersey 08540, USA

³ LAPLACE, Université de Toulouse, CNRS, 31062 Toulouse, France

ABSTRACT

An axisymmetric fully ionized plasma rotates around its axis when a charge separation between magnetic surfaces is produced from DC fields or RF waves. On each magnetic surface, both electrons and ions obey the isorotation law and perform an azimuthal E cross B rotation at the same angular velocity. When Coulomb collisions are taken into account, such a flow displays no Ohmic current short circuiting of the charge separation and thus no linear dissipation. A nonlinear Ohmic response appears when inertial effects are considered, providing a dissipative relaxation of the charge separation between the magnetic surfaces. This nonlinear conductivity results from an interplay between Coriolis, centrifugal, and electron-ion collisional friction forces. This phenomenon is identified, described, and analyzed. In addition, both the quality factor of angular momentum storage and the efficiency of wave driven angular momentum generation are calculated and shown to be independent of the details of the charge separation processes.

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I. INTRODUCTION

Axisymmetric, fully ionized, magnetized plasmas can be set in rotation around their axis through a small breakdown of quasineutrality sustained by DC or RF power. The uncompensated polarizing free charges rapidly rearrange along the magnetic field lines to screen the electric field component along the magnetic field. A steady state electric field \mathbf{E} , perpendicular to the magnetic field \mathbf{B} , drives an E cross B azimuthal flow around the axis.^{1,2} The steady-state sustainment of the charge imbalance between magnetic surfaces can be achieved either (i) with an applied DC radial voltage drop between magnetic surfaces or (ii) through a resonant wave induced radial drift across magnetic surfaces. The first scheme requires a set of polarized end plates intercepting the magnetic field lines at the edge of the discharge, and the second scheme needs the propagation and absorption of a plasma wave with the right dispersion relation carrying a significant amount of angular momentum around the axis. Both schemes are associated with injected power consumption. This power is ultimately dissipated through the nonlinear Ohmic radial current identified and described in this study.

Two classical rotating configurations have been widely investigated in plasma physics: the *Brillouin rigid body rotation*

associated with homogeneous magnetic fields³ and the *Ferraro isorotation* associated with axisymmetric inhomogeneous magnetic fields.⁴ The study of axisymmetric rotating pulsar magnetospheres offers a third,^{5,6} more complex, model of rotating configurations where the uncompensated charge density ρ is called the *Goldreich-Julian charge density* (G-J).^{7,8} Because of the absence of relativistic and radiative effects, the axisymmetric laboratory plasmas analyzed in this paper are much simpler than pulsar magnetospheres. Nevertheless, we will adopt this nomenclature and call *G-J charges* the uncompensated charges driving the rotation to mark the differences with the background quasineutral charges. Besides pulsar magnetospheres, rotating axisymmetric plasmas and the general problem of angular momentum conversion and dissipation with radial electric field, axial magnetic field, wave helicity, and plasma vorticity have received considerable attention within the framework of (i) plasma centrifuge for isotope separation,⁹⁻¹⁴ (ii) nonneutral plasma physics,^{15,16} (iii) thermonuclear magnetic confinement studies with homopolar devices, rotating mirrors, and tokamaks,¹⁷⁻²⁰ and (iv) particle acceleration and magnetic field generation with plasma bubbles and channels.²¹⁻²³

If we consider the classical MHD Ohm's law on an axisymmetric magnetic surface rotating at velocity $\mathbf{v} = \mathbf{E} \times \mathbf{B}/B^2$, such that $\mathbf{E} \cdot \mathbf{B} = 0$, then the current

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (1)$$

where σ is the conductivity,²⁴ cancels. This cancellation of the linear Ohmic response is exact only if the \mathbf{E} cross \mathbf{B} drift is uniform.²⁵ If the \mathbf{E} cross \mathbf{B} drift is accelerated, Eq. (1) must be complemented with inertial forces, and as we will demonstrate, a nonlinear Ohmic response comes into play to ensure G-J charge relaxation and dissipation.

If we study a conducting fluid of free particles with charge q and mass m per particle, rather than Eq. (1), we must consider the relation²⁶

$$\mathbf{j} = \sigma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{m}{q} \mathbf{v} \cdot \nabla \mathbf{v} \right). \quad (2)$$

For electrons in metals, the effects associated with the last term of the right hand side of Eq. (2) are called *excitation of current by acceleration* and Eq. (2) provides the right tool to describe rotating metallic conductors.²⁶

For ions and electrons in fully ionized plasmas, these effects are called *inertial effects* and Eq. (2) must be revisited with a two fluid model.^{25,27} Inertial effects, associated with centrifugal and Coriolis forces, are usually small, but, as we will demonstrate, they must be taken into account to describe the collisional Ohmic short circuiting of the G-J charges both for Brillouin and Ferraro flows.

Besides inertial effects, *finite Larmor radius effects*, driving dissipative diamagnetic flows, and *ion-ion collisions*, driving viscous momentum transfer,^{25,27} are also responsible for a small dissipative current short circuiting the G-J charges. For plasmas considered in this study, the order of magnitude of these finite Larmor radius and viscous effects is evaluated and shown to be typically smaller than that of inertial effects for Brillouin and quasi-Brillouin flows.

To analyse the impact of the inertial effects, rather than the one fluid MHD relation Eq. (2), we will use a two fluid model all along this study. This paper is organized as follows: in Sec. II, we consider a rigid body, fully ionized, collisional Brillouin flow and set up the two coupled, fourth order, algebraic equations fulfilled by the ion and electron vorticities. These equations are solved for the slow branch of Brillouin rotation modes through an expansion with respect to collisionality, and a dissipative nonlinear current is found to provide G-J charge relaxation. The conductivity is nonlinear and displays a quadratic scaling with respect to the electric field. The origin of this conduction is rather intricate as it involves an interplay between Coriolis, centrifugal, and Coulomb friction forces. A step by step physical analysis of this response is proposed in Sec. IV. In Sec. III, to prepare the study of Sec. IV, we review the basic electrodynamics of rotating axisymmetric plasmas. We derive the Ferraro isorotation law and some new expressions for the G-J charges.

In Sec. IV, we calculate the conductivity induced by electron-ion collisions combined with inertial drifts for a fully

ionized discharge fulfilling isorotation in an axisymmetric magnetic configuration. Then, in Sec. V, this new result is used (i) to evaluate the *quality factor* of energy storage in a rotating plasma and, in Sec. VI, (ii) to describe and analyse the efficiency of wave driven rotation. We calculate the efficiency of wave orbital angular momentum conversion into plasma orbital angular momentum and show that it is independent of the details of the wave dispersion and the wave-particle resonance. These new results on the quality factor and the efficiency clearly display the interest of schemes with wave sustainment of plasma rotation which have been put forward both for isotope separation and magnetic confinement.^{13,17,20}

The impact of diamagnetic flows is then analyzed in Sec. VII. This finite Larmor radius effect induces a linear electric conduction, and this current is shown to be smaller than the inertial nonlinear current. Section VII is also devoted to the comparison between viscous damping, resulting from the coupling between slow and fast magnetic surfaces, and Ohmic dissipation. The original results of this study are summarized in Sec. VIII concluding this study.

II. NONLINEAR OHMIC CONDUCTION IN A BRILLOUIN FLOW

In this section, we set up the two coupled nonlinear algebraic equations, Eqs. (17a) and (17b), fulfilled by electrons and ions vorticities describing a collisional, fully ionized, Brillouin flow.^{3,15} Then, we expand the solutions of these equations with respect to the collision frequency and express the nonlinear conductivity resulting from this expansion. The extension and confirmation of the final result of this section to a general axisymmetric flows are considered and analyzed in Secs. III and IV.

Consider a rotating, cylindrical, fully ionized, uniform, magnetized plasma illustrated in Fig. 1 where a set of concentric polarized electrodes provides a radial voltage drop ($\phi_0 > \phi_1 > \phi_2 \dots$) between the various cylindrical magnetic surfaces. We use cylindrical polar coordinates (r, θ, z) associated with the cylindrical polar basis $[\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z]$. The magnetic field \mathbf{B} is uniform and directed along the z axis, and the electric field \mathbf{E} is radial and increases linearly with respect to r

$$\mathbf{B} = B\mathbf{e}_z, \quad \mathbf{E} = E(r)\mathbf{e}_r. \quad (3)$$

As $\nabla \cdot \mathbf{B} = 0$, we can introduce the flux function $2\pi\Psi(r)$ to describe the magnetic field. As $\nabla \times \mathbf{E} = 0$, we can introduce the electrostatic potential $\phi(r)$ to describe the electric field. With these flux and potential functions, the expressions of the fields become

$$\mathbf{B} = \nabla\Psi \times \frac{\mathbf{e}_\theta}{r}, \quad \mathbf{E} = -\nabla\phi. \quad (4)$$

Introducing the low frequency permittivity, $\epsilon_\perp = 1 + \omega_{pi}^2/\omega_i^2 \approx \omega_{pi}^2/\omega_i^2$ ($\omega_{pi} = nq^2/\epsilon_0 m_i$ and $\omega_i = qB/m_i$ are the ion plasma and ion cyclotron frequencies), these flux and potential functions can be expressed as

$$\Psi = B \frac{r^2}{2}, \quad \phi = -\frac{\rho}{\epsilon_0 \epsilon_\perp} \frac{r^2}{4}, \quad (5)$$

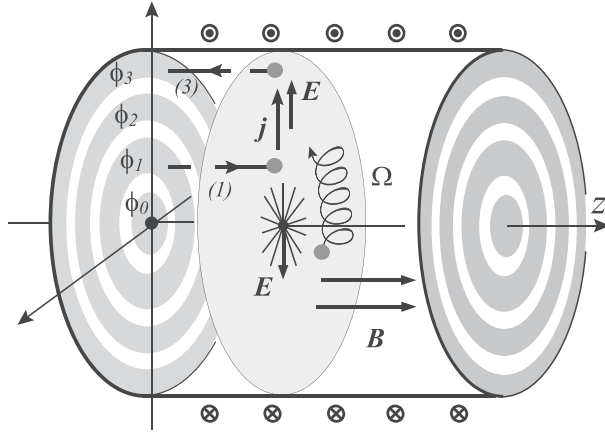


FIG. 1. Fully ionized Brillouin flow driven by a set of concentric electrodes sustaining a radial electric field \mathbf{E} ; the current path from ϕ_1 to ϕ_3 is made of two highly conducting axial field lines (1) and (3) short circuited by the radial nonlinear Ohmic current \mathbf{j} .

where ρ is the uniform guiding center charge density such that $\nabla \cdot \epsilon_{\perp} \epsilon_0 \mathbf{E} = \rho$. We assume that heating and fuelling systems, such as waves and pellet injections, complemented by a particle and power exhaust system, ensure a steady state complete ionization and a steady state flat density profile. The small breakdown of quasineutrality ρ is short circuited by a collisional current \mathbf{j} and is sustained either by wave absorption or DC radial polarization. For the latter case, a set of concentric electrodes, illustrated in Fig. 1, provides a voltage drop ($\phi_0 > \phi_1 > \phi_2 > \phi_3$) between the various cylindrical magnetic surfaces as the conductivity along the field lines is very large.

The collisional depletion of ρ through \mathbf{j} is continuously compensated by the power supply driving these electrodes, or through wave induced charge separation, in order to ensure steady state rotation. In this section, we will demonstrate that \mathbf{j} is directed along \mathbf{E} but is not a linear function of \mathbf{E} .

In a fully ionized plasma, with electron velocity \mathbf{v}_e and ion velocity \mathbf{v}_i , electron-ion collisions are the source of friction forces, $\mathbf{F}_{i \rightarrow e} = -\mathbf{F}_{e \rightarrow i}$, between electrons and ions. At the single particle level, these dissipative forces can be expressed as

$$\mathbf{F}_{i \rightarrow e} = -m_e \nu_e (\mathbf{v}_e - \mathbf{v}_i), \quad \mathbf{F}_{e \rightarrow i} = -m_i \nu_i (\mathbf{v}_i - \mathbf{v}_e), \quad (6)$$

where we have introduced ν_e and ν_i , the momentum exchange frequencies between the two populations, according to the classical definitions

$$m_e \nu_e = m_i \nu_i = \sqrt{\frac{m_i m_e}{m_e + m_i}} \frac{n q^4 \log \Lambda}{6 \pi \epsilon_0^2 (2 \pi)^{\frac{1}{2}} (k_B T)^{\frac{3}{2}}} = m \nu, \quad (7)$$

where q is the electron charge, m_e and m_i are the electron and ion masses, and the other notations are standard.²⁷ We have

considered a fully ionized hydrogen plasma with a quasineutral density n such that $\rho \ll nq$. The temperature $T = T_e = T_i$ is uniform, and the effective mass is defined as $m = \sqrt{m_i m_e}$.

The relation $m_e \nu_e = m_i \nu_i$ describes the fact that the momentum lost by one population is gained by the other, strictly speaking that the two friction forces are not *dissipative* as there is no *entropy production* because the macroscopic momentum is not dispersed into a large number of microscopic degrees of freedom but transferred, at the macroscopic level, from the slow population to the fast one if $\mathbf{v}_e \neq \mathbf{v}_i$. The steady-state momentum balance for the electron and ion populations is given by two Euler fluid equations coupled through these Coulomb friction terms $\mathbf{F}_{i \rightarrow e} = -\mathbf{F}_{e \rightarrow i}$.

In this section, we will follow the method used previously to study the weakly ionized collisional Brillouin flow.²⁸ The inertial term of Euler's equations, $d\mathbf{v}/dt = (\mathbf{v} \cdot \nabla)\mathbf{v}$, is expressed through the classical identity: $(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla v^2/2 + (\nabla \times \mathbf{v}) \times \mathbf{v}$. In writing Euler's equations, we explicitly display the axial vorticities $\Omega_{e/i}$ associated with a velocity field $\mathbf{v}_{e/i}$ and defined as $\Omega_{e/i} = (\nabla \times \mathbf{v}_{e/i}) \cdot \mathbf{b}/2$, where $\mathbf{b} = \mathbf{B}/B$. The steady state momentum balances between centrifugal ($\nabla v^2/2$) and Coriolis ($2\Omega \mathbf{b} \times \mathbf{v}$) inertial forces, on the left hand side of Eqs. (8a) and (8b), and electric Coulomb forces, magnetic Laplace forces, and electron-ion friction forces, on the right hand side of Eqs. (8a) and (8b), which is given by

$$\frac{1}{2} \nabla v_e^2 + 2\Omega_e \mathbf{b} \times \mathbf{v}_e = -\frac{q}{m_e} \mathbf{E} - \frac{q}{m_e} \mathbf{v}_e \times B \mathbf{b} - \nu_e (\mathbf{v}_e - \mathbf{v}_i), \quad (8a)$$

$$\frac{1}{2} \nabla v_i^2 + 2\Omega_i \mathbf{b} \times \mathbf{v}_i = \frac{q}{m_i} \mathbf{E} + \frac{q}{m_i} \mathbf{v}_i \times B \mathbf{b} - \nu_i (\mathbf{v}_i - \mathbf{v}_e). \quad (8b)$$

We do not consider viscous dissipation because $\Delta \mathbf{v}_e = \Delta \mathbf{v}_i = \mathbf{0}$ is exactly fulfilled by the rigid body rotation solutions. For a generic rotation considered here, if the temperature, density, and vorticity distributions are rather flat with respect to the radial variable, viscosity effects and diamagnetic responses are pushed toward the boundary of the plasma and are negligible in the core of the column, and at this boundary, the matching of the rotation to the wall is another problem which requires the identification and analysis of electrical, thermal, and mechanical boundary layers; this problem is out of the scope of this study. The typical radial profiles of the various dynamical quantities (flux, potential, vorticity, etc.) describing this flow are illustrated in Fig. 2. Euler's equations Eqs. (8a) and (8b) can be rewritten as

$$\mathbf{v}_e + \alpha_e \mathbf{v}_e \times \mathbf{b} = \mathbf{v}_i - \frac{1}{m \nu} \nabla (m_e v_e^2/2 - q\phi), \quad (9a)$$

$$\mathbf{v}_i + \alpha_i \mathbf{v}_i \times \mathbf{b} = \mathbf{v}_e - \frac{1}{m \nu} \nabla (m_i v_i^2/2 + q\phi), \quad (9b)$$

where we have introduced the electron cyclotron frequency $\omega_e = qB/m_e$ and the ion cyclotron frequency ω_i and defined the generalized Hall parameters, α_e and α_i ,²⁸ according to the definitions

$$\nu_e \alpha_e = \omega_e - 2\Omega_e, \quad \nu_i \alpha_i = -\omega_i - 2\Omega_i. \quad (10)$$

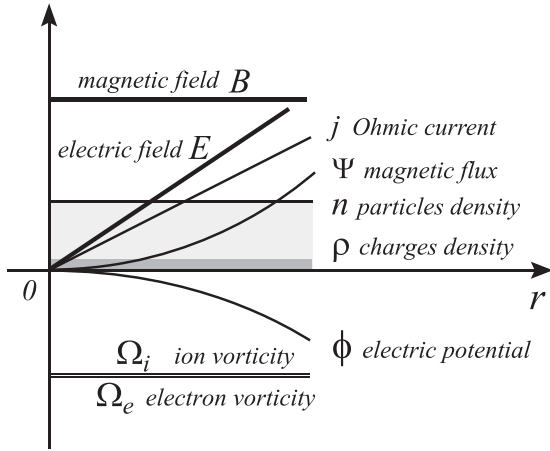


FIG. 2. Electric potential ϕ , magnetic flux Ψ , vorticity Ω , charge density ρ , and current j profiles for a Brillouin flow.

In order to find the rigid body rotation solutions of Eqs. (9a) and (9b), we take (i) the rotational and then (ii) the divergence of these two equations. We consider that all the variations of the $\mathbf{v}_{e/i}$ and ϕ fields are radial so that $\nabla \times (\mathbf{v} \times \mathbf{b}) = -(\nabla \cdot \mathbf{v})\mathbf{b}$ and $\nabla \cdot (\mathbf{v} \times \mathbf{b}) = (\nabla \times \mathbf{v}) \cdot \mathbf{b}$. With the help of these relations, we obtain

$$2\Omega_e - \alpha_e \nabla \cdot \mathbf{v}_e = 2\Omega_i, \quad 2\Omega_i - \alpha_i \nabla \cdot \mathbf{v}_i = 2\Omega_e, \quad (11)$$

by taking the rotational of Eqs. (9a) and (9b). Then, taking the divergence of Eqs. (9a) and (9b) leads to the equations fulfilled by the vorticities $\Omega_{e/i}$

$$(\Omega_e - \Omega_i) \left(\frac{1}{\alpha_e} + \frac{1}{\alpha_i} \right) + \alpha_e \Omega_e = -\frac{1}{2m\nu} \Delta(m_e v_e^2/2 - q\phi), \quad (12a)$$

$$(\Omega_i - \Omega_e) \left(\frac{1}{\alpha_e} + \frac{1}{\alpha_i} \right) + \alpha_i \Omega_i = -\frac{1}{2m\nu} \Delta(m_i v_i^2/2 + q\phi). \quad (12b)$$

Under the rigid body rotation hypothesis,²⁸ the azimuthal velocities of ions and electrons are given by $v_{\theta e} = \Omega_e r$ and $v_{\theta i} = \Omega_i r$, and the relations Eq. (11) give the following radial components after integration: $v_{re} = (\Omega_e - \Omega_i)r/\alpha_e$ and $v_{ri} = (\Omega_i - \Omega_e)r/\alpha_i$. Note that $\Delta \mathbf{v}_e = \Delta \mathbf{v}_i = \mathbf{0}$ is exactly fulfilled. This allows us to express the radial current $j = nq(v_{ri} - v_{re})$ as

$$\mathbf{j} = nq(\Omega_i - \Omega_e) \left(\frac{1}{\alpha_i} + \frac{1}{\alpha_e} \right) \mathbf{r} e_r. \quad (13)$$

On the basis of these components, the electron and ion terms associated with the centrifugal force, $\Delta v^2 = \Delta(v_r^2 + v_\theta^2)$, can be expressed as

$$\frac{\Delta v_e^2}{4} = \Omega_e^2 + \frac{(\Omega_e - \Omega_i)^2}{\alpha_e^2}, \quad \frac{\Delta v_i^2}{4} = \Omega_i^2 + \frac{(\Omega_i - \Omega_e)^2}{\alpha_i^2}. \quad (14)$$

The final relations fulfilled by the vorticities $\Omega_{e/i}$ as a function of the electric potential drive ϕ are thus given by

$$\begin{aligned} (\Omega_e - \Omega_i) \left(\frac{1}{\alpha_e} + \frac{1}{\alpha_i} \right) + \alpha_e \Omega_e + \frac{m_e}{m\nu} \left[\Omega_e^2 + \frac{(\Omega_e - \Omega_i)^2}{\alpha_e^2} \right] \\ = \frac{q}{2m\nu} \Delta\phi \end{aligned} \quad (15a)$$

and

$$\begin{aligned} (\Omega_i - \Omega_e) \left(\frac{1}{\alpha_e} + \frac{1}{\alpha_i} \right) + \alpha_i \Omega_i + \frac{m_i}{m\nu} \left[\Omega_i^2 + \frac{(\Omega_i - \Omega_e)^2}{\alpha_i^2} \right] \\ = -\frac{q}{2m\nu} \Delta\phi. \end{aligned} \quad (15b)$$

$\Omega_{i/e}$ are the two unknowns of these two equations, and the right hand side of these relations, $\Delta\phi$, must be independent of the radial position in order to find rigid body rotation solutions to this set of algebraic equations Eqs. (15a) and (15b). This can be achieved if we consider a small uniform space charge $\rho = q(n_i - n_e)$ ($n_i - n_e \ll n$). This uniform space charge is the source of a linear electric field E and a parabolic potential ϕ with respect to r illustrated in Fig. 2.

We define the E cross B angular velocity $\Omega = E/Br$ and the frequency ω_E according to the definition

$$\frac{\omega_E^2}{\omega_i \omega_e} = \frac{mE}{qB^2 r} = \frac{m\Omega}{qB} \ll 1, \quad (16)$$

such that Poisson's equation becomes $\Delta\phi = -2\omega_E^2 m/q$. With this definition of ω_E , we have to solve a set of two coupled fourth order algebraic equations, Eqs. (17a) and (17b), with two unknowns, the vorticities $\Omega_{i/e}$

$$\begin{aligned} (\Omega_e - \Omega_i) \left(\alpha_e^{-1} + \alpha_i^{-1} \right) + \alpha_e \Omega_e + \frac{m_e}{m\nu} \left[\Omega_e^2 + \alpha_e^{-2} (\Omega_e - \Omega_i)^2 \right] \\ + \frac{\omega_E^2}{\nu} = 0, \end{aligned} \quad (17a)$$

$$\begin{aligned} (\Omega_i - \Omega_e) \left(\alpha_e^{-1} + \alpha_i^{-1} \right) + \alpha_i \Omega_i + \frac{m_i}{m\nu} \left[\Omega_i^2 + \alpha_i^{-2} (\Omega_i - \Omega_e)^2 \right] \\ - \frac{\omega_E^2}{\nu} = 0. \end{aligned} \quad (17b)$$

The solutions of this system of two algebraic equations describe the various branches of collisional rigid body rotation in a fully ionized collisional plasma. The solutions of Eqs. (17a) and (17b), completed with relation Eq. (13), provide the exact expression of the Ohmic current short circuiting the G-J charge ρ .

Rather than an exact solution of the fourth order coupled equations [Eqs. (17a) and (17b)], we seek an approximate expression of the current based on a small collisionality expansion of the solution which matches the Brillouin slow mode when $\nu \rightarrow 0$. The ordering between the various time scales involved in these equations is $\nu_i \ll \nu \ll \nu_e \ll \Omega_i \sim \Omega_e \sim \Omega \ll \omega_i \ll \omega_e$; thus, we consider two small parameters: $m\nu/qB = \nu_i/\omega_i = \nu_e/\omega_e \ll 1$ and $m\Omega/qB = \Omega/\sqrt{\omega_e \omega_i} \ll 1$. We will solve Eqs. (17a) and (17b) through an expansion with respect to these two small parameters.

The zero order solution is simply the E cross B flow without inertial and collisional effects. The first order slow Brillouin solutions are the sum of this azimuthal E cross B drift plus the first inertial drift corrections

$$\frac{\Omega_e}{\omega_e} = -\frac{\omega_E^2}{\omega_e \sqrt{\omega_i \omega_e}} - \frac{\omega_E^4}{\omega_e^3 \omega_i} + O\left(\frac{m\Omega}{qB}\right)^3 + O\left(\frac{m\nu}{qB}\right), \quad (18a)$$

$$\frac{\Omega_i}{\omega_i} = -\frac{\omega_E^2}{\omega_i \sqrt{\omega_i \omega_e}} + \frac{\omega_E^4}{\omega_i^3 \omega_e} + O\left(\frac{m\Omega}{qB}\right)^3 + O\left(\frac{m\nu}{qB}\right). \quad (18b)$$

We can plug the expression of $\Omega_i - \Omega_e$ in Eqs. (17a) and (17b) to get the next order and so on, but this level of expansion is sufficient to analyze the nonlinear Ohmic current Eq. (13) as

$$\frac{1}{\alpha_e} + \frac{1}{\alpha_i} = 2 \frac{m\nu}{qB} \left(\frac{\Omega_e}{\omega_e} + \frac{\Omega_i}{\omega_i} \right) \left[1 + O\left(\frac{m\Omega}{qB}\right)^2 \right]. \quad (19)$$

Using the relations Eqs. (18a), (18b), and (19) and the expression Eq. (13), the first order electric current \mathbf{j} , with respect to an $m\Omega/qB \ll 1$ and $m\nu/qB \ll 1$ expansion, is given by

$$\frac{\mathbf{j}}{2nq} = \frac{\nu}{\sqrt{\omega_i \omega_e}} \frac{\omega_E^4}{\omega_i^3 \omega_e} \Omega_i \mathbf{r} \mathbf{e}_r = \frac{\nu_i \Omega^2 \mathbf{E}}{\omega_i^3 B}. \quad (20)$$

As $\Omega \sim E$, this Ohmic current \mathbf{j} is a cubic function of the electric field E , and its origin, analyzed in Sec. IV, is to be traced back to an interplay between Coriolis, centrifugal, and Coulomb friction forces. The current Eq. (20) scales as B^{-6} with respect to the magnetic field strength, and this scaling explains the high efficiency of rotation generation in confined plasmas identified and analyzed within the context of wave driven magneto-electric confinement studies.²⁰

To summarize this section, the weak dissipative current \mathbf{j} , expressed by Eq. (20), continuously short circuits the G-J charges ρ between field lines. To evaluate this space charge ρ and validate the ordering $\rho \ll nq$, we apply Gauss theorem to this cylindrical configuration, $\rho/\varepsilon_0 \varepsilon_\perp = 2E/r$, to get the expression

$$\frac{n_i - n_e}{n_i + n_e} = \frac{\rho}{2nq} = \frac{\varepsilon_0 \varepsilon_\perp \Omega B}{nq} = \frac{\Omega}{\omega_i} \ll 1, \quad (21)$$

which is coherent with the fact that we consider the slow branch ($\Omega \ll \omega_i$) of the two fast and slow Brillouin rotation modes.^{3,15}

Note the Maxwell time τ_M associated with the resistive decay of the uncompensated free charges ρ is no longer given by the classical expression $\tau_M \sim |\rho/\nabla \cdot \mathbf{j}| \sim \nu_e/\omega_p^2$ but is given by the ratio: $\tau_M \sim |\rho/\nabla \cdot \mathbf{j}| \sim \omega_i^2/\Omega^2 \nu_i$. Thus, (i) along the field lines, the relaxation time of free charges scales as ν_e , and (ii) across the field lines as $(\omega_i/\Omega)^2/\nu_i$, this is also the classical behavior, ν_e versus $1/\nu_i$, of both mobility and diffusion (i) along and (ii) across the magnetic field. A complete analysis of the dynamics of the G-J charge relaxation will be presented in Secs. V and VI where the *quality factor* of angular momentum storage and the *efficiency* of angular momentum generation will give two practical characterizations of this relaxation besides the Maxwell time.

Although the occurrence of dissipative inertial effects of the type Eq. (2) in accelerated flows is described in the literature,²⁵⁻²⁷ the analysis and expansion of the inertial term $d\mathbf{v}/dt = (\mathbf{v} \cdot \nabla)\mathbf{v}$ with a rotating two fluid model leading to the general nonlinear conductivity Eq. (20) did not appear in the literature.

The relaxation of a fully ionized collisional Brillouin flow did not attract much attention, and the exact solution of the weakly ionized collisional Brillouin flow appears only recently in the literature.²⁸ Equation (20) is therefore one of the new results of this study. Finally, while Eq. (20) has been derived assuming rigid body rotation and cylindrical geometry, we will expose in Sec. IV that this result can be recovered and generalized in the more general case of an axisymmetric rotating flow fulfilling Ferraro isorotation. As it will be shown, the only difference between Eq. (20) and the generic case is a geometrical factor associated with the radial profile of the magnetic flux function.

III. ISOROTATION OF AN AXISYMMETRIC PLASMA

In order to set the frame for a generalization of the expression of the nonlinear Ohmic current Eq. (20), we consider a fully ionized, steady state, axisymmetric, plasma depicted in Fig. 3. We use cylindrical polar coordinates (r, θ, z) associated with the cylindrical polar basis $[\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z]$. The structure of this general axisymmetric configuration around the z axis is described by the electric field \mathbf{E} and magnetic field \mathbf{B}

$$\mathbf{B}(r, z) = B_r \mathbf{e}_r + B_z \mathbf{e}_z, \quad \mathbf{E}(r, z) = E_r \mathbf{e}_r + E_z \mathbf{e}_z. \quad (22)$$

We introduce the magnetic flux function $2\pi\Psi(r, z)$ and the electrostatic potential $\phi(r, z)$ to describe these magnetic and electric fields. With these flux and potential functions, the expressions of these fields become

$$\mathbf{B}(r, z) = \nabla\Psi \times \frac{\mathbf{e}_\theta}{r}, \quad \mathbf{E}(r, z) = -\nabla\phi. \quad (23)$$

Because of the very large conductivity along the field lines, $\mathbf{E} \cdot \mathbf{B} = 0$, and the electric field is everywhere perpendicular to the magnetic field. Magnetic surfaces $\Psi(r, z) = C^{te}$ are therefore equipotential surfaces $\phi(r, z) = C^{te}$. As a result, ϕ is determined by the magnetic flux Ψ : $\phi(r, z) = \phi(\Psi)$. When we will consider diamagnetic flow, we will also assume that the pressure gradient is perpendicular to the magnetic surfaces, i.e., $\Psi(r, z)$ are isobaric and isopotential surfaces.

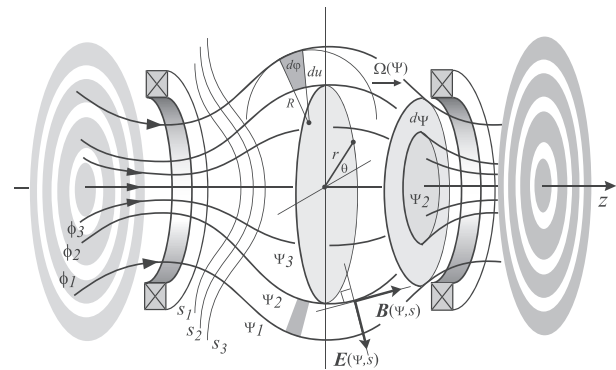


FIG. 3. Rotating axisymmetric magnetized plasma: R is the radius of curvature, ϕ is the electric potential, Ψ is the magnetic flux, and $\Omega(\Psi)$ is the angular velocity; (Ψ, θ, s) or (ϕ, θ, s) provides a set of generalized coordinates.

Both electrons and ions rotate around the z axis with an E cross B velocity \mathbf{v} defining the angular velocity $\Omega(r, z)$

$$\mathbf{v}(r, z) = \frac{\mathbf{E} \times \mathbf{B}}{B^2} = \Omega(r, z) r \mathbf{e}_\theta. \quad (24)$$

We introduce s the curvilinear abscissa along each field line such that (Ψ, θ, s) provide a set of coordinates illustrated in Fig. 3. Let us call du the small perpendicular distance between two neighbouring magnetic surfaces. According to Fig. 3, the conservation of magnetic flux can be written as $B2\pi r du = 2\pi d\Psi$, and the relation between the electric field and potential is $E du = -d\phi$. The elimination of du between these two relations leads to the Ferraro isorotation law

$$\Omega(\Psi) = \frac{d\phi}{d\Psi}. \quad (25)$$

Each magnetic surface can be viewed as an equipotential surface rotating uniformly and the full configuration as a set of nested magnetic surfaces rotating around the z axis.⁴

We assume that the temperature difference $\Delta T = T_1 - T_2$ between two magnetic surfaces Ψ_1 and Ψ_2 is smaller than the voltage drop $\Delta\phi = \phi_1 - \phi_2$ between these two surfaces according to the ordering as follows: $k_B \Delta T \ll q \Delta\phi$. Thus, the pressure force can be neglected in front of the electric force as $|\nabla n k_B T| \ll n q |\nabla\phi|$. This strong ordering is fulfilled by the fast rotating discharges of interest for thermonuclear fusion and allows us to study separately inertial effects and finite Larmor radius effects. For the weak ordering, $|\nabla n k_B T| \sim n q |\nabla\phi|$, the pressure profiles $P(\Psi)$ and $\phi(\Psi)$ provide a set of given data to solve a Grad-Shafranov equation for $\Psi(r, z)$ similar to the profiles $P(\Psi)$ and $I(\Psi)$ for the classical Grad-Shafranov equation. This analysis of the mechanical/electrical MHD equilibrium is far beyond the scope of this study which is only devoted to the analysis of Ohmic dissipation for a given configuration $\Psi(r, z)$ and $\phi(\Psi)$.

Before analyzing the collisional nonlinear short circuiting of the G-J charges ρ , we will demonstrate that they are completely determined by the magnetic flux $\Psi(r, z)$ and the electric potential $\phi(\Psi)$. In the historical papers on the pulsar magnetosphere, the G-J charges were usually derived starting from Ampère and Poisson equations associated with Ψ and ϕ .^{7,8} Here, we will introduce the radius of curvature of the field lines, $R^{-1} = |d\mathbf{b}/ds|$, to get new simple expressions. Let us call $R(\Psi, s)$ the radius of curvature of a field line and φ the angle along the osculating circle. Consider, in Fig. 3, a small grey box ($du, R d\varphi, 2\pi r$) between two magnetic surfaces: du is along \mathbf{E} , $R d\varphi$ along \mathbf{B} , and $2\pi r$ along \mathbf{e}_θ . We use the Gauss theorem, $(E + dE)(R + dR)d\varphi - ER d\varphi = \rho R d\varphi du / \epsilon_0 \epsilon_\perp$, and the relation between the electric field and potential, $E du = -d\phi$, to express the guiding center charge density ρ as

$$\frac{\rho}{\epsilon_0} = -\frac{c^2}{V_A^2} \frac{E}{\Omega R} \frac{\partial RE}{\partial \Psi}, \quad (26)$$

where V_A is the Alfvén's velocity and c is the velocity of light. Consider, in Fig. 3, a small closed loop ($\pm du, \pm R d\varphi$) along two field lines and between two magnetic surfaces, du along \mathbf{E} , and

$R d\varphi$ along \mathbf{B} . Ampère's theorem can be written as $BR d\varphi - (B + dB)(R + dR)d\varphi = -\mu_0 R d\varphi du \rho E/B$, where we have neglected the diamagnetic currents in front of the E cross B convection current. The guiding center charge density ρ is thus given as a function of the magnetic field as

$$\frac{\rho}{\epsilon_0} = -c^2 \frac{B}{\Omega R} \frac{\partial RB}{\partial \Psi}. \quad (27)$$

Note that $R = \infty$ for the Brillouin flow.

To summarize this brief presentation of a generic axisymmetric rotating plasma, the magnetic field is structured by magnetic surfaces $\Psi(r, z)$ such that $\mathbf{B}(\Psi, s) = \nabla\Psi \times \mathbf{e}_\theta/r$, the electric field $\mathbf{E}(\Psi, s) = -\Omega(\Psi)\nabla\Psi$ is everywhere perpendicular to the magnetic field, and the plasma flow fulfils the Ferraro isorotation law $\mathbf{v}(\Psi, s) = \Omega(\Psi)r\mathbf{e}_\theta$. For this classical flow, as the velocities of electrons and ions are equal, there is no Coulomb friction between electrons and ions and thus no Ohmic collisional dissipation of this equilibrium described by the two given functions $[\Psi(r, z), \phi(\Psi)]$ or equivalently by $[\Psi(r, z), \Omega(\Psi)]$.

IV. NONLINEAR OHMIC CONDUCTION IN A FERRARO FLOW

In this section, we consider the two fluid model, but we adopt a Lagrangian point of view, different from the Eulerian point of view used in Sec. II. Moreover, taking advantage of the Ferraro isorotation law Eq. (25), we consider the dynamics on a magnetic surface Ψ in the frame corotating at the angular velocity $\Omega(\Psi)$, different from the laboratory frame point of view used in Sec. II.

In this (co)rotating frame, the electron and ion velocities are \mathbf{v}_e and \mathbf{v}_i , the electric field cancels everywhere on the Ψ magnetic surface, but we have to take into account the Coriolis and centrifugal forces to write down that electron and ion forces balance

$$m_e \Omega^2 \mathbf{r}_e + 2m_e \mathbf{v}_e \times \Omega \mathbf{e}_z - q \mathbf{v}_e \times \mathbf{B} \mathbf{b} = m_e \nu_e (\mathbf{v}_e - \mathbf{v}_i), \quad (28a)$$

$$m_i \Omega^2 \mathbf{r}_e + 2m_i \mathbf{v}_i \times \Omega \mathbf{e}_z + q \mathbf{v}_i \times \mathbf{B} \mathbf{b} = m_i \nu_i (\mathbf{v}_i - \mathbf{v}_e), \quad (28b)$$

where the local unit vector $\mathbf{b} = \mathbf{B}/B$ is given by

$$B r \mathbf{b}(r, z) = \frac{\partial \Psi}{\partial r} \mathbf{e}_z - \frac{\partial \Psi}{\partial z} \mathbf{e}_r. \quad (29)$$

The first terms on the left hand side of Eqs. (28a) and (28b) are the centrifugal and Coriolis forces and the terms on the right hand side describe collisional friction. In these equations, we do not consider the pressure gradient across the magnetic surfaces as we assumed in Sec. III that it is smaller than the electric force. Along the magnetic field line, the dynamics of electrons and ions is controlled by the parallel component of the centrifugal force and the diamagnetic and parallel friction forces. This parallel dynamics does not interfere with the perpendicular one and is regulated by an ambipolar field and a pressure gradient. These components are hence not included here in the analysis of the perpendicular dynamics. Introducing the cyclotron frequencies,

we have to solve the following set of equations for the velocities \mathbf{v}_e and \mathbf{v}_i on the flux surface Ψ in the frame rotating at angular velocity $\Omega(\Psi)$

$$2\mathbf{v}_e \times \frac{\Omega}{\nu_e} \mathbf{e}_z - \mathbf{v}_e \times \frac{\omega_e}{\nu_e} \mathbf{b} - \mathbf{v}_e + \mathbf{v}_i = -\frac{\Omega^2}{\nu_e} \mathbf{r} \mathbf{e}_r, \quad (30a)$$

$$2\mathbf{v}_i \times \frac{\Omega}{\nu_i} \mathbf{e}_z + \mathbf{v}_i \times \frac{\omega_i}{\nu_i} \mathbf{b} - \mathbf{v}_i + \mathbf{v}_e = -\frac{\Omega^2}{\nu_i} \mathbf{r} \mathbf{e}_r, \quad (30b)$$

under the strong ordering: $\nu_i \ll \nu \ll \nu_e \ll \Omega \ll \omega_i \ll \omega_e$, $\nu_e \sim \nu_i \ll \Omega r$. Note that in the lab frame, $\nu_i \ll \nu \ll \nu_e \ll \Omega \ll \omega_i \ll \omega_e$, but $\nu_e \sim \nu_i \sim \Omega r$. Following the ordering, the ion dynamics Eq. (30a) is dominated by the balance between the magnetic Laplace force and the inertial centrifugal force: $\Omega^2 \mathbf{r} \mathbf{e}_r = -\omega_i \mathbf{v}_i \times \mathbf{b}$. The last equation is solved as a guiding center centrifugal force drift

$$\mathbf{v}_i = -\frac{\Omega^2}{\omega_i} \mathbf{r} \mathbf{b} \times \mathbf{e}_r. \quad (31)$$

Equation (31) describes an ion collisionless azimuthal cross-field flow driven by the centrifugal force. The same azimuthal collisionless flow for the electrons is smaller by an m_e/m_i mass ratio factor. Then, we have to consider the Coulomb collisions and the friction force of the ions with azimuthal velocities \mathbf{v}_i with the electrons with azimuthal velocities $\mathbf{v}_e = m_e \mathbf{v}_i / m_i$. This collisional coupling is the source of an azimuthal force \mathbf{F}_i on the ions and $\mathbf{F}_e (= -\mathbf{F}_i)$ on the electrons

$$\mathbf{F}_i = -m_i \nu_i \mathbf{v}_i = m_i \nu_i \frac{\Omega^2}{\omega_i} \mathbf{r} \mathbf{b} \times \mathbf{e}_r. \quad (32)$$

These azimuthal friction forces are the source of a cross-field ion flow \mathbf{V}_i and a cross-field electron flow $\mathbf{V}_e (= \mathbf{V}_i)$; thus, no net electric current is observed at this level of analysis

$$\mathbf{V}_i = \frac{\mathbf{F}_i \times \mathbf{B}}{qB^2} = \frac{\nu_i \Omega^2}{\omega_i^2} \mathbf{r} (\mathbf{b} \times \mathbf{e}_r) \times \mathbf{b}. \quad (33)$$

Now, we take into account the Coriolis force, which has been neglected until now according to the ordering. The two small collisional flows \mathbf{V}_e and \mathbf{V}_i are responsible for small azimuthal Coriolis forces \mathbf{f}_i and $\mathbf{f}_e (\ll \mathbf{f}_i)$

$$\mathbf{f}_i = 2m_i \mathbf{V}_i \times \Omega \mathbf{e}_z = 2m_i \Omega \nu_i \frac{\Omega^2}{\omega_i^2} \mathbf{r} [(\mathbf{b} \times \mathbf{e}_r) \times \mathbf{b}] \times \mathbf{e}_z. \quad (34)$$

This Coriolis/collisional force drives a cross-field ion flow \mathbf{w}_i and a cross-field electron flow $\mathbf{w}_e (\ll \mathbf{w}_i)$

$$\mathbf{w}_i = \frac{\mathbf{f}_i \times \mathbf{B}}{qB^2} = 2\nu_i \frac{\Omega^3}{\omega_i^3} \mathbf{r} [(\mathbf{b} \times \mathbf{e}_r) \times \mathbf{b}] \times \mathbf{e}_z \times \mathbf{b}. \quad (35)$$

The direction of this flow is along the electric field, and it is proportional to the momentum exchange frequency

$$\begin{aligned} [(\mathbf{b} \times \mathbf{e}_r) \times \mathbf{b}] \times \mathbf{e}_z \times \mathbf{b} &= \left(\frac{1}{rB} \frac{\partial \Psi}{\partial r} \mathbf{E} \times \mathbf{e}_z \right) \times \mathbf{b} \\ &= \frac{1}{r^2 B^2} \left(\frac{\partial \Psi}{\partial r} \right)^2 \frac{\mathbf{E}}{E}. \end{aligned} \quad (36)$$

The associated electric current $\mathbf{j} = nq\mathbf{w}_i$ is the nonlinear Ohmic current providing G-J charge relaxation. Its final expression is given by

$$\frac{\mathbf{j}}{2nq} = \frac{\nu_i}{E^2 B} \frac{\Omega^4}{\omega_i^3} \left(\frac{\partial \Psi}{\partial r} \right)^2 \frac{\mathbf{E}}{E} = \frac{\nu_i \Omega^2 E_r^2}{\omega_i^3 E^2 B} \mathbf{E}, \quad (37)$$

where we have used the Ferraro isorotation law Eq. (25) to express the right hand side of this relation.

Moving back to the laboratory frame does not change this expression of the radial Ohmic current Eq. (37) which is one of the main new results of this study. For a Brillouin flow $E_r = E$, we recover exactly the result Eq. (20) obtained at the end of Sec. II through a careful expansion of the solutions of the exact fourth order coupled equations fulfilled by the vorticities in the laboratory frame. As $\Omega \sim E$, the conductivity scales as E^2 and is clearly nonlinear.

To compare this new result $\mathbf{j} = \sigma_{NL} \mathbf{E}$ described by Eq. (37) with the usual picture on Ohmic dissipation, we introduce the classical Braginsky expression for the electric conductivity perpendicular to the magnetic field,²⁹ $\sigma_B = nq^2 \nu_e / m_e \omega_e^2$, to get the scaling and ordering

$$\frac{\sigma_{NL}}{\sigma_B} = 2 \left(\frac{\Omega}{\omega_i B} \right)^2 \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right)^2 \sim \left(\frac{\Omega}{\omega_i} \right)^2 \ll 1. \quad (38)$$

To summarize Secs. III and IV, we have demonstrated that a small breakdown of quasineutrality described by the G-J charges Eq. (26) leads to a (iso)rotation of the axisymmetric plasma and that these charges are short circuited by a nonlinear Ohmic current Eq. (37) resulting from an interplay between (i) Coriolis, (ii) centrifugal, and (iii) Coulomb friction forces. To complete this description of charges, we have also identified through Eq. (33) an outward ambipolar particle flow $n\mathbf{V}_i + n\mathbf{V}_e$, perpendicular to the magnetic surfaces such that the associated mass flow \mathbf{J} is given by

$$\frac{\mathbf{J}}{nm_i} = \nu_i \frac{\Omega^2}{\omega_i^2} \mathbf{r} (\mathbf{b} \times \mathbf{e}_r) \times \mathbf{b} = s_{\pm} \frac{\nu_i \Omega B_z \mathbf{E}}{\omega_i^2 B}, \quad (39)$$

where s_{\pm} is the sign of $-(\Omega \cdot \mathbf{e}_z)(\mathbf{E} \cdot \mathbf{e}_r)(\mathbf{B} \cdot \mathbf{e}_z)$. Note that for the Brillouin flow, the expression $(\mathbf{b} \times \mathbf{e}_r) \times \mathbf{b} = \mathbf{e}_r$ clearly shows that this ambipolar mass flux is radially outward. Moving back to the laboratory frame does not change the expression of this collisional radial ambipolar flow Eq. (39).

V. QUALITY FACTOR OF ANGULAR MOMENTUM STORAGE

As depicted in Fig. 3, axisymmetric plasma rotation can be sustained through the polarization of each magnetic surface Ψ with a system of concentric conductive electrodes intercepting the magnetic field lines at the left and right edges of the plasma. A voltage generator is used to sustain a voltage drop $\Delta\phi$ in between the magnetic surfaces Ψ and $\Psi + \Delta\Psi$, resulting in a plasma rotation $\Omega = \Delta\phi / \Delta\Psi$ of these surfaces according to the Ferraro isorotation law Eq. (25).

This voltage drop $\Delta\phi$ is short circuited by the inertial Ohmic current Eq. (37), and a steady state active power

consumption takes place to sustain the rotation. The reactive power is associated with energy storage during the initial transient buildup of the polarization and rotation. The rotating plasma discharge can be viewed either as an electrostatic energy storage or as a kinetic energy storage, with total energy U_Ω given by

$$U_\Omega = \iiint \frac{\rho\phi}{2} d\tau = \iiint \varepsilon_\perp \varepsilon_0 \frac{E^2}{2} d\tau = \iiint nm_i \frac{\Omega^2 r^2}{2} d\tau, \quad (40)$$

where $d\tau$ is the volume element and the integrations run over the volume of the plasma. Note that $\Omega r = E/B$ and $\varepsilon_\perp = \omega_{pi}^2/\omega_i^2$ lead directly to the last identity of Eq. (40).

This global energy content U_Ω is dissipated through electron-ion collisions. If the power and control systems are switched off, rotation slows down and the density decays because (i) on each magnetic surface, the angular momentum around z is continuously destroyed by the resistive Ohmic torque associated with the force $\mathbf{j} \times \mathbf{B}$, Eq. (37), and (ii) energy leaks out of the magnetic surface through the outward convection of energy associated with the ambipolar mass flow \mathbf{J} , Eq. (39).

At a given point r , the decay of the density of kinetic energy is the sum of two terms: (i) the resistive work of the density of force $\mathbf{j} \times \mathbf{B}$ and (ii) the convective ambipolar mass flux described by \mathbf{J}

$$\frac{\partial}{\partial t} \left(n \frac{m_i}{2} \Omega^2 r^2 \right) = nm_i \Omega \frac{\partial \Omega}{\partial t} r^2 + \frac{\partial n}{\partial t} m_i \Omega^2 r^2. \quad (41)$$

The decay of the angular velocity $\partial \Omega/\partial t$ is due to the dissipative work of the torque $\mathbf{r} \times (\mathbf{j} \times \mathbf{B})$, and the decay of the density $m_i \partial n/\partial t = -\nabla \cdot \mathbf{J}$ is due to the ambipolar mass flux \mathbf{J}

$$nm_i \Omega \frac{\partial \Omega}{\partial t} r^2 = -\mathbf{r} \times (\mathbf{j} \times \mathbf{B}) \cdot \Omega, \quad (42a)$$

$$\frac{m_i}{2} \frac{\partial n}{\partial t} \Omega^2 r^2 = -\frac{\Omega^2 r^2}{2} \nabla \cdot \mathbf{J}. \quad (42b)$$

The rate of collisional energy dissipation dU_Ω/dt is thus given as a function of \mathbf{j} and \mathbf{J} , Eqs. (37) and (39), by

$$\frac{dU_\Omega}{dt} = - \iiint \left[\mathbf{r} \times (\mathbf{j} \times \mathbf{B}) \cdot \Omega + \frac{\Omega^2 r^2}{2} \nabla \cdot \mathbf{J} \right] d\tau, \quad (43)$$

where integration runs over the volume of the plasma. In this section, to establish the expression of the quality factor, we will use the Brillouin flow approximation $\partial \Psi/\partial r = rB_z \approx rB$ or $\partial \Psi/\partial r = E_r/\Omega \approx E/\Omega$ so that the expressions of the collisional electric current flow Eq. (37) and the collisional ambipolar particle flow Eq. (39) become

$$\mathbf{j} = 2nq \frac{\nu_i \Omega^2}{\omega_i^3} \frac{\mathbf{E}}{B}, \quad (44a)$$

$$\mathbf{J} = nm_i \frac{\nu_i \Omega}{\omega_i^2} \frac{\mathbf{E}}{B} \mathbf{e}_r. \quad (44b)$$

Note that $\nabla \cdot \mathbf{J} = 2nm_i \nu_i \Omega^2/\omega_i^2$ and $\nabla \cdot \mathbf{j} = 4nq \nu_i \Omega^3/\omega_i^3$ are independent of the radius r . The generalization of the results

obtained in this section to a generic Ferraro flow requires the inclusion of $\partial \Psi/\partial r$ and $\partial \Psi/\partial z$ factors associated with the full expressions given by Eqs. (37) and (39). With these simple expressions, the power of the resistive torque and the energy convection by the ambipolar mass flow are given by

$$\mathbf{r} \times (\mathbf{j} \times \mathbf{B}) \cdot \Omega = 2qn \nu_i \frac{\Omega^4 B}{\omega_i^3} r^2, \quad (45a)$$

$$\frac{\Omega^2 r^2}{2} \nabla \cdot \mathbf{J} = qn \nu_i \frac{\Omega^4 B}{\omega_i^3} r^2. \quad (45b)$$

Thus, one third of the energy decay is due to the collisional particles losses and two thirds due to the nonlinear Ohmic conductivity. The ratio of the energy decay during one turn $-(dU_\Omega/dt)(2\pi/\Omega)$ to the stored energy U_Ω defines the quality factor Q of this energy storage system

$$\frac{1}{Q} = -\frac{2\pi dU_\Omega/dt}{\Omega U_\Omega} = 12\pi \frac{\iiint q \nu_i \Omega^4 B \omega_i^{-3} r^2 d\tau}{\Omega \iiint m_i \Omega^2 r^2 d\tau} = 12\pi \frac{\nu_i \Omega}{\omega_i^2}. \quad (46)$$

The time scale for collisional relaxation is not $1/\nu_i$ but far longer by a factor $(\omega_i/\Omega)^2$ already identified for the evaluation of the Maxwell time at the end of Sec. II.

Rather than using the resistive torque $\mathbf{r} \times (\mathbf{j} \times \mathbf{B})$ Eq. (45a) to evaluate the power of the Ohmic current Eq. (37) short circuiting the G-J charges, we can consider the rate of resistive charge depletion $\partial \rho/\partial t$ given by $\partial \rho/\partial t = -\nabla \cdot \mathbf{j}$. This Ohmic charge decay is then used to express the power

$$- \iiint \phi \frac{\partial \rho}{\partial t} d\tau = \iiint 2nq \nu_i \frac{\Omega^4}{\omega_i^3} B r^2 d\tau, \quad (47)$$

where the electrostatic potential is $\phi = -Er/2 = -B\Omega r^2/2$. We recover Eq. (45a) which confirms that, as expected, the electrical and mechanical power balances give the very same result.

VI. EFFICIENCY OF RF ANGULAR MOMENTUM GENERATION

Section V was devoted to the study of the free decay of a rotating plasma column after all the sustaining and control systems have been switched off.

In this section, we consider instead a situation where the plasma column is sustained in steady state rotation (i) by an RF wave system, providing angular momentum injection, and (ii) by a pellet, or a gas puffing, system providing particle fuelling to maintain a steady state density profile. We thus assume that there is a source of particles on each magnetic surface $S(\Psi)$ such that $\partial n/\partial t = 0$. When an ion at rest appears on the surface Ψ from neutral ionization, it starts to rotate at the velocity $r\Omega(\Psi) = E/B$ and compensates the collisional ambipolar depletion associated with \mathbf{J} given in Eq. (39).

Instead of polarizing end plate electrodes with a voltage generator, the angular momentum of the axisymmetric discharge depicted in Fig. 3 can be sustained using RF waves carrying angular momentum. To ensure continuous angular momentum input, resonant and dispersive conditions for these

waves to propagate and be absorbed within the plasma can be identified.

To simplify the analysis, we limit ourselves here to the Brillouin flow approximation $\partial\Psi/\partial r = rB_z \approx rB$ or $\partial\Psi/\partial r = E_r/\Omega \approx E/\Omega$. Note although that the generalization of the results to a generic Ferraro flow simply requires including the $\partial\Psi/\partial r$ and $\partial\Psi/\partial z$ factors. The resistive torque $\mathbf{r} \times (\mathbf{j} \times \mathbf{B})$ slows down rotation, while the absorption of angular momentum from the wave spins up rotation. If we call dL/dt the rate of wave angular momentum absorption per particle, the angular momentum balance between absorption and dissipation is given by

$$\frac{dL}{dt} = \mathbf{r} \times \frac{\mathbf{j} \times \mathbf{B}}{n} \cdot \mathbf{e}_z = 2q\nu_i \frac{\Omega^3 B}{\omega_i^3} r^2. \quad (48)$$

When a wave is absorbed by a plasma, (i) energy, (ii) linear momentum, and (iii) angular momentum are transferred from the wave to the particles. Consider (i) a cylindrical wave, $f(r)\sin(l\theta - \omega t)$, with azimuthal mode number l and frequency $\omega/2\pi$, and (ii) an ion at radial position r with energy \mathcal{E} and guiding center orbital angular momentum $L = m_i\Omega r^2$ with respect to the z axis. Each time a quantum of energy $\delta\mathcal{E} = \hbar\omega$ is absorbed by this ion, a quantum of angular momentum $\delta L = \hbar l$ is also absorbed. This angular momentum gain is then dissipated as a result of the collisional torque Γ Eq. (45a). We call $w_{l\omega}$ the steady state power, per particle, needed to sustain the angular momentum L

$$w_{l\omega} = \frac{d\mathcal{E}}{dt} = \frac{\omega}{l} \frac{dL}{dt} = \frac{\omega}{l} \Gamma = 2q\nu_i \frac{\omega\Omega^3 B}{l\omega_i^3} r^2, \quad (49)$$

where we have balanced the wave driven angular dL/dt momentum gain by the nonlinear Ohmic loss Eq. (45a). Starting from this single particle power absorption $w_{l\omega}$, the efficiency of angular momentum generation can be easily expressed.

Before expressing this efficiency, we will recover this result Eq. (49) with the full picture of the Hamiltonian dynamics along the lines used in the recent studies on wave driven rotational transform and transport driven current generation in centrally fuelled discharges.^{20,30}

Consider an electromagnetic wave displaying orbital angular momentum around z and linear momentum along z , that is to say, with typical space-time periodicity of the type $f(r)\sin(l\theta + k_{\parallel}z - \omega t)$, where l is the azimuthal mode number and k_{\parallel} is the parallel wave vector. Near a given point r , this cylindrical wave can be viewed locally as a plane wave $\sin(k_{\perp}x + k_{\parallel}z - \omega t)$ where (x, y, z) is a local set of Cartesian coordinates centered on r such that y is directed along \mathbf{e}_r , x is directed along \mathbf{e}_θ , and $k_{\perp} = l/r$ is the local perpendicular wave vector. When this electromagnetic wave propagates and is absorbed in a rotating plasma, a certain amount of (i) energy, (ii) linear momentum along the z axis, and (iii) angular momentum around the z axis are exchanged between the wave and the particles.

The energy \mathcal{E} , the parallel momentum mv_{\parallel} along z , and the cyclotron velocity v_c around \mathbf{B} of resonant particles are changed through the wave interaction, and also, the guiding center positions y_g across \mathbf{B} as part of the momentum are no longer free

but bound to the static magnetic field through the invariance of the canonical momentum along y .^{20,31,32} This variation of the guiding center positions y_g drives a radial charge separation between magnetic surfaces and provides orbital angular momentum deposition (rotation around z) inside the magnetized plasma column in addition to energy (heating) and linear momentum (current generation along z) depositions.^{33,34}

This picture of the resonant wave-particle interaction has already been presented in previous studies, both for plane waves and for cylindrical waves.^{20,30-32} It has been demonstrated, on the basis of Hamilton's equations, that the increments of the various dynamical variables \mathcal{E} , v_{\parallel} , v_c , and y_g are not independent but fulfill

$$\frac{dy_g}{d\mathcal{E}} = \frac{k_{\perp}}{m_i\omega_i\omega}, \quad \frac{dv_c}{d\mathcal{E}} = \frac{\omega_i}{m_iv_c\omega}, \quad \frac{dv_{\parallel}}{d\mathcal{E}} = \frac{k_{\parallel}}{m_i\omega}. \quad (50)$$

During a small time dt , the steady state power transferred from the wave to the particle is simply $d\mathcal{E} = w_{l\omega}dt$, and for continuous absorption, this power induces a radial particle velocity

$$\frac{dr}{dt} = \frac{dy_g}{d\mathcal{E}} \frac{\partial\mathcal{E}}{\partial t} = \frac{l}{qB\omega r} w_{l\omega}, \quad (51)$$

where we have used the relationship between global cylindrical and local plane waves $k_{\perp} = l/r$. This wave induced radial current is continuously short circuited by the collisional velocity w_i given by Eq. (35). At a radius r , the power needed to sustain the RF driven G-J charge separation against collisional relaxation is locally given by the balance $w_i = e_r dr/dt$ between Eqs. (35) and (51)

$$2\nu_i \frac{\Omega^2 E}{\omega_i^3 B} = \frac{lw_{l\omega}}{qB\omega r}. \quad (52)$$

Thus, as $\Omega = E/Br$, we recover is the result Eq. (49) obtained on the basis of a simple quantum transition argument.

These results are more conveniently presented if we consider the total power $W_{l\omega}$ needed to sustain the full plasma column rotation. This global power is proportional to the energy U_{Ω} stored in the rotation, and this relation can be expressed by the efficiency

$$\frac{W_{l\omega}}{U_{\Omega}} = \frac{2 \int n w_{l\omega} 2\pi r dr}{\int n m_i \Omega^2 r^2 2\pi r dr} = \frac{4\omega \Omega \nu_i}{l \omega_i^2}. \quad (53)$$

Equation (53) is one of the main new results of this paper. It expresses the efficiency of angular momentum generation when the wave (ω, l) sustains a rigid body rotation Ω in a magnetized fully ionized plasma (ν_i, ω_i) . It is to be noted that the electron-ion collision frequency must be replaced by an effective decorrelation frequency if dissipation and transport are dominated by the turbulent activity of the plasma. In this turbulent regime, more power is needed to sustain rotation compared to the collisional regime.

According to Eq. (49), to sustain a rigid body rotation in a uniform plasma, the power deposition profile $w_{l\omega}(r)$ must be quadratic with respect to the radius r . To sustain a rigid body

rotation in an axisymmetric configuration such as the one depicted in Fig. 3, the power deposition profile $w_{\omega}(\Psi)$ must follow the scaling of the ratio $\nu_i r^2 B_z^2 / B^2$ as a function of Ψ .

Note that this analysis of the angular momentum generation process is simpler than the wave driven current generation because the relaxation does not involve a kinetic description of the resonant particles.^{33,34} Here, the short circuiting of the wave driven charge separation is ensured by a bulk inertial ionic current Eq. (37) even if the wave power is absorbed by a minority population.

Equation (53) is a new universal result independent of the particular wave branch and wave-particle resonance. The relation Eq. (53) indicates that low frequency waves with a high azimuthal number are more efficient to drive rotation through orbital angular momentum deposition.

VII. FINITE LARMOR RADIUS AND ION-ION DISSIPATIONS

Up to this point, we assumed that the temperature difference $k_B \Delta T$ between two magnetic surfaces Ψ_1 and Ψ_2 is smaller than the voltage drop $q \Delta \phi$ between these two surfaces, and thus, the pressure force can be neglected in front of the electric force as $|\nabla n k_B T| \ll n q |\nabla \phi|$. This ordering is fulfilled by the fast rotating discharges of interest for thermonuclear fusion and allows us to study separately *inertial effects* and *finite Larmor radius diamagnetic effects* which are smaller. Besides diamagnetic effects, ion-ion Coulomb collisions drive a friction force such that an additional dissipation takes place if $d\Omega/d\Psi \neq 0$ as the fast magnetic surfaces will transfer angular momentum to the slow ones through ion-ion friction. This effect can be minimized down to a very small value since a well-shaped power deposition profile $w_{\omega}(\Psi)$ can be identified from the relation Eq. (49) in order to ensure that $d\Omega/d\Psi = d^2\phi/d\Psi^2 \sim 0$.

In this section, we set up the frame to identify the basic scaling and ordering of thermal and viscous effects providing collisional relaxations of G-J charges. A deeper analysis of the impact of these finite Larmor radius effects and ion-ion collision effects is left to a future work.

Consider a fully ionized axisymmetric plasma such that the ion pressure is equal to the electron one everywhere and the total pressure is $P(\Psi)$. The ion and electron diamagnetic velocities, \mathbf{v}_i^* and \mathbf{v}_e^* , are azimuthal, and they flow in opposite directions $\mathbf{v}_i^* (= -\mathbf{v}_e^*)$ can be expressed as

$$\mathbf{v}_i^* = \mathbf{b} \times \frac{\nabla \Psi}{2nqB} \frac{dP}{d\Psi}. \quad (54)$$

The collisional friction between the electron and ion flows is the source of an azimuthal force \mathbf{F}_i^* on the ions and $\mathbf{F}_e^* (= -\mathbf{F}_i^*)$ on the electrons

$$\mathbf{F}_i^* = -2m_i \nu_i \mathbf{v}_i^* = -\frac{\nu_i}{\omega_i} \frac{dP}{d\Psi} \mathbf{b} \times \frac{\nabla \Psi}{n}. \quad (55)$$

The factor 2 comes from the relation $\mathbf{v}_i^* - \mathbf{v}_e^* = 2\mathbf{v}_i^*$. These azimuthal friction forces are the source of a cross-field ion flow \mathbf{V}_i^* and a cross-field electron flow $\mathbf{V}_e^* (= \mathbf{V}_i^*)$

$$\mathbf{V}_i^* = \frac{\mathbf{F}_i^* \times \mathbf{B}}{qB^2} = \frac{\nu_i}{nqB\omega_i} \frac{dP}{d\Psi} \mathbf{b} \times (\mathbf{b} \times \nabla \Psi). \quad (56)$$

Now, we consider the Coriolis force in the (co)rotating frame. The two small collisional flows \mathbf{V}_e^* and \mathbf{V}_i^* Eq. (56) are responsible for small azimuthal Coriolis forces \mathbf{f}_i^* and $\mathbf{f}_e^* (\ll \mathbf{f}_i^*)$

$$\mathbf{f}_i^* = 2m_i \Omega \mathbf{V}_i^* \times \mathbf{e}_z = 2 \frac{\nu_i \Omega}{n\omega_i^2} \frac{dP}{d\Psi} [\mathbf{b} \times (\mathbf{b} \times \nabla \Psi)] \times \mathbf{e}_z. \quad (57)$$

This Coriolis collisional force drives a cross-field ion flow and a smaller cross-field electron flow. The current associated with these flows is

$$\mathbf{j}^* = n \frac{\mathbf{f}_i^* \times \mathbf{b}}{B} = 2 \frac{\nu_i \Omega}{B\omega_i^2} \frac{dP}{d\Psi} ([\mathbf{b} \times (\mathbf{b} \times \nabla \Psi)] \times \mathbf{e}_z) \times \mathbf{b}. \quad (58)$$

The direction of this flow is along the electric field as

$$([\mathbf{b} \times (\mathbf{b} \times \nabla \Psi)] \times \mathbf{e}_z) \times \mathbf{b} = -\frac{\partial \Psi}{\partial r} \frac{\mathbf{E}}{E}, \quad (59)$$

and its amplitude is proportional to the momentum exchange frequency. The final diamagnetic current providing G-J charge relaxation is thus given by

$$\mathbf{j}^* = 2\nu_i \frac{\Omega}{\omega_i^2 B} \left| \frac{\partial P}{\partial r} \right| \frac{\mathbf{E}}{E}. \quad (60)$$

The associated conduction is linear, and the conductivity, independent of the electric field, displays a fourth power scaling with respect to the magnetic field. The finite Larmor radius linear conductivity associated with Eq. (60) is smaller than the non-linear inertial one described by Eq. (37) since

$$\frac{j}{j^*} = \frac{nq\Omega^3}{E\omega_i} \left(\frac{\partial \Psi}{\partial r} \right)^2 / \frac{\partial P}{\partial r} \sim \left(\frac{\Omega r}{\omega_i \rho_i} \right)^2. \quad (61)$$

For a small Larmor radius and significant angular velocity, which is the regime needed for isotope separation, Ohmic dissipation is thus larger than collisional/diamagnetic dissipation. It is to be noted that the parameter $\Omega r / \omega_i \rho_i$ is the Mach number M which is an important control parameter when the thermal energy content of the plasma becomes of the order of the mechanical energy contents Eq. (40). Besides this small diamagnetic effect, ion-ion collisions offer also a possibility to relax the G-J charges. If we assume that the ion viscosity is given by the Braginsky relation,^{27,29} then the viscous ion-ion friction force per particle is

$$\mathbf{F}_{i \rightarrow i} = \frac{3k_B T}{10\sqrt{2}} \frac{\nu}{\omega_i^2} \Delta \mathbf{v}_i. \quad (62)$$

This force cancels for the Brillouin flow, but for a Ferraro flow on each magnetic surface, the isorotation law $\Omega(r, z) = \Omega(\Psi)$ is fulfilled, and if $d\Omega/d\Psi \neq 0$, the fast magnetic surfaces accelerate the slow ones and the slow magnetic surfaces slow down the fast ones through ion-ion momentum transfer $\mathbf{F}_{i \rightarrow i}$. The azimuthal component of the force $\mathbf{F}_{i \rightarrow i}$ describing this momentum exchange is given by the classical expression for the azimuthal viscous stress component

$$\mathbf{e}_\theta \cdot \mathbf{F}_{i \rightarrow i} = \frac{3k_B T}{10\sqrt{2}} \frac{\nu}{\omega_i^2} \left(\frac{1}{r^2} \frac{\partial}{\partial r} r^3 \frac{\partial \Omega}{\partial r} + r \frac{\partial^2 \Omega}{\partial z^2} \right), \quad (63)$$

where we have assumed flat density and temperature profiles. The azimuthal component of the viscous force drives a cross-field flow $\mathbf{F}_{i-i} \times \mathbf{B}/qB^2$ which is colinear with the electric field \mathbf{E} , and the viscous current $\mathbf{j}_v = nq\mathbf{F}_{i-i} \times \mathbf{B}/qB^2$ is given by

$$\mathbf{j}_v = \frac{3nk_B T}{10\sqrt{2}} \frac{\nu B}{\omega_i^2 E^2} \Omega^2 \left(\frac{\partial}{\partial r} r^3 \frac{\partial \Omega}{\partial r} + r^3 \frac{\partial^2 \Omega}{\partial z^2} \right) \frac{\mathbf{E}}{E}. \quad (64)$$

This conduction process is linear with respect to the electric field but is *non-local* as it involves $\partial E/\partial r$, $\partial^2 E/\partial r^2$, and $\partial^2 E/\partial z^2$ through the derivatives of $\Omega = E/Br$.

In order to set up an ordering between Ohmic dissipation and viscous damping, we simply have to compare the azimuthal Coriolis force \mathbf{f}_i given by Eq. (34) with the azimuthal viscous stress component $\mathbf{e}_\theta \cdot \mathbf{F}_{i-i}$ given by Eq. (63). Viscous damping is unimportant if $\mathbf{e}_\theta \cdot \mathbf{F}_{i-i} < \mathbf{e}_\theta \cdot \mathbf{f}_i$, which rewrites

$$\left| \frac{1}{\Omega^3 r^3} \frac{\partial}{\partial r} \left(r^3 \frac{\partial \Omega}{\partial r} \right) \right| < 10 \frac{\sqrt{m_e m_i}}{k_B T}. \quad (65)$$

Note that this ordering is easily fulfilled: (i) for rigid body rotations $\Omega(r, z) = \Omega_0$, (ii) for Keplerian rotation $\Omega(r, z) = L/r^2$, and (iii) if the power deposition profile $w_{l\omega}(\Psi)$ is tailored such that $d\Omega/d\Psi \sim 0$. To explore this nearly non-viscous behavior $d\Omega/d\Psi \sim 0$, we consider power law profiles $\Omega = \Omega_0(r/r_0)^2$. With this scaling, the previous condition becomes

$$10M > |\alpha(\alpha + 2)| \sqrt{m_i/m_e}, \quad (66)$$

where M is the Mach number defined as the ratio of the angular velocity to the thermal velocity. For a rigid body rotation, α is a small number and Ohmic dissipation dominates over viscous dissipation if $\alpha < 5M\sqrt{m_e/m_i}$. For a strongly sheared rotation $d\Omega/d\Psi \neq 0$, we have to identify the critical radius where the transition from Ohmic to viscous dissipation occurs since $\partial\Omega/\partial r = 0$ on axis at $r = 0$. This deeper analysis of viscous damping is left for a future study.

To summarize this section, we have addressed the issues of the finite Larmor radius and ion-ion collisions as candidate mechanisms to relax G-J charges in rotating plasmas, and both effects have been quantified and shown to be smaller than the nonlinear Ohmic dissipation Eq. (37) for typical plasmas of interest such as Brillouin rotation and generic isorotation with $d\Omega/d\Psi \sim 0$.

VIII. SUMMARY AND CONCLUSION

The main three new results of this study are given by Eqs. (37), (46), and (53) obtained in Secs. IV, V, and VI.

The relation Eq. (37) provides the general expression of the Ohmic nonlinear current short circuiting G-J charges, Eq. (26), in an axisymmetric rotating plasma described by the magnetic flux $\Psi(r, z)$ and electric potential $\phi(\Psi)$. Compared to the classical Braginsky expression for the conductivity perpendicular to the magnetic field, the axisymmetric rotating plasma displays a B^{-6} scaling with respect to the magnetic field strength instead of the classical B^{-2} scaling. This scaling and the small value of the current for typical plasma parameters explain the high RF

efficiency of rotation generation in fully ionized magnetically confined plasmas.

This efficiency and the quality factor for energy storage Q are given by Eqs. (46) and (53). To keep these expressions simple, we have neglected the curvature R^{-1} of the field lines through the approximation, $\partial\Psi/\partial r = rB_z \approx rB$ or $\partial\Psi/\partial r = E_r/\Omega \approx E/\Omega$. Taking into account the curvature leads to the expression of the power associated with the Ohmic torque

$$\iiint [\mathbf{r} \times (\mathbf{j} \times \mathbf{B})] \cdot \Omega d\tau = 2q \iiint \nu_i \frac{\Omega^4 E_r^2}{\omega_i^3 E^2} Br^2 nd\tau, \quad (67)$$

in the place of Eq. (45a). The general relations, with a full account of finite curvature R , can be then easily derived on the basis of Eq. (37). Whether or not curvature is accounted for, the quality factor and efficiency will still be written as the ratio of two integrals with geometrical factors $\partial\Psi/\partial r$ and will display the same final scaling as Eqs. (46) and (53) with respect to the plasma parameters.

The original result given by Eq. (37) completes the results of a previous study where the relaxation of a Brillouin flow in a weakly ionized rotating plasma was analyzed on the basis of the exact solution of the uncoupled fourth order algebraic equations describing electron and ion vorticities.²⁸ This problem was much simpler than the present one since the geometry was simpler, and the electron and ion dynamics were uncoupled because neutral collisions, rather than Coulomb collisions, ensured relaxation. The relation Eq. (37) is directly relevant to the analysis of the power balance in innovative magneto-electric toroidal traps in the limit of a very large aspect ratio.^{20,35,36}

Equation (53) gives the efficiency of wave orbital angular momentum conversion into plasma orbital angular momentum. It shows that the conversion efficiency does not depend on the details of the wave dispersion and the wave-particle resonance. This analysis completes the results obtained on wave driven rotational transform where the problem of poloidal rotation sustainment was addressed for the purpose of toroidal magnetic confinement.²⁰ There the efficiency was directly compared to classical current generation efficiency^{33,34} and was constrained from the very beginning of the study by the goal of a large rotational transform to counteract the vertical magnetic drift of the toroidal trap. The high efficiency of angular momentum generation in a closed toroidal configuration predicted in these earlier studies is thus confirmed and extended here to open axisymmetric configurations of the mirror type. It is to be noted that because of the finite aspect ratio, viscous damping is more important for toroidal traps than for the mirror type discharges considered here.

Although this study was restricted to hydrogen plasmas, the generalization to $Z \neq 1$ single species plasmas is straightforward. On the other hand, the generalization to a multiple ion species plasma is less straightforward. Yet, this extension appears necessary given the practical interest of these discharges.³⁷

This work also provides an electrical engineering point of view on axisymmetric rotating plasmas. The magnetic surfaces $\Psi_1, \Psi_2, \Psi_3, \dots$ in Fig. 3 can be viewed as a set of nested conducting cylindrical shells. The equivalent network associated with two neighbouring shells is described by a capacitor with capacitance $C = \epsilon_{\perp} \epsilon_0 S/d$ and nonlinear resistance $\mathcal{R} = d/S\sigma_{NL}$

$$C = \frac{\omega_{pi}^2 \epsilon_0 S}{\omega_i^2 d}, \quad \mathcal{R} = \frac{\omega_i^4 d}{\omega_{pi}^2 \Omega^2 2\epsilon_0 \nu_i S}, \quad (68)$$

where d is the radial gap between the two neighboring magnetic surfaces and S is their surfaces, and we have used Eq. (20) to evaluate the conductivity σ_{NL} . We recover the scaling of the relaxation time for this equivalent elementary RC cell, $RC = \omega_i^2/2\nu_i\Omega^2$, already identified in Secs. II and V. Thus, we can set up an operational transmission line model of axisymmetric discharge as a radially distributed circuit with a well-defined impedance per unit length along the radial direction: the inverse capacitance per unit length is $\omega_i^2/\epsilon_0 S\omega_{pi}^2$ and the resistance per unit length is $\omega_i^4/2\epsilon_0\nu_i S\omega_{pi}^2\Omega^2$. The RF driven charge separation or the DC voltage drop sustainment, described in Secs. V and VI, can be viewed as a continuous recharging of this distributed capacity to compensate the short circuiting associated with the distributed conductance.

Besides this electrical engineering model, the analysis and description of the interplay between Coriolis, centrifugal, and Coulomb friction forces, offered in Sec. IV, provide a clear description of the G-J charge relaxation process in fully ionized plasmas.

Finally, a promising feature identified in this study is the possibility to tailor the power deposition profile $w_{\omega}(\Psi)$ in order to reach approximately a Brillouin regime $d^2\phi/d\Psi^2 = 0$ in an axisymmetric configuration $\Psi(r, z)$, that is to say, to design a rotating plasma free of rotational shear. This is particularly interesting given that velocity shear is typically a source of both instabilities and damping.

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REFERENCES

- ¹B. Lehnert, *Nucl. Fusion* **11**, 485 (1971).
- ²J. M. Wilcox, *Rev. Mod. Phys.* **31**, 1045 (1959).
- ³L. Brillouin, *Phys. Rev.* **67**, 260 (1945).
- ⁴V. C. A. Ferraro, *Mon. Not. R. Astron. Soc.* **97**, 458 (1937).
- ⁵V. S. Beskin, A. V. Gurevich, and Y. N. Istomin, *Sov. Phys. Usp.* **29**, 946 (1986).
- ⁶V. S. Beskin, *Phys. Usp.* **40**, 659 (1997).
- ⁷P. Goldreich and W. H. Julian, *Astrophys. J.* **157**, 869 (1969).
- ⁸W. H. Julian, *Astrophys. J.* **183**, 967 (1973).
- ⁹B. Bonnevier, *Ark. Fys.* **33**, 255 (1966).
- ¹⁰T. Ohkawa and R. L. Miller, *Phys. Plasmas* **9**, 5116 (2002).
- ¹¹S. Shinohara and S. Horii, *Jpn. J. Appl. Phys.* **46**, 4276 (2007).
- ¹²R. Gueroult, J.-M. Rax, and N. J. Fisch, *Phys. Plasmas* **21**, 020701 (2014).
- ¹³A. J. Fetterman and N. J. Fisch, *Phys. Plasmas* **18**, 094503 (2011).
- ¹⁴R. R. Prasad and M. Krishnan, *Phys. Fluids* **30**, 3496 (1987).
- ¹⁵R. C. Davidson, *Physics of Nonneutral Plasmas* (Imperial College Press, 2001).
- ¹⁶R. C. Davidson and N. A. Krall, *Phys. Rev. Lett.* **22**, 833 (1969).
- ¹⁷A. J. Fetterman and N. J. Fisch, *Phys. Rev. Lett.* **101**, 205003 (2008).
- ¹⁸A. A. Bekhtenev, V. I. Volosov, V. E. Pal'chikov, M. S. Pekker, and Y. N. Yudin, *Nucl. Fusion* **20**, 579 (1980).
- ¹⁹A. J. Fetterman and N. J. Fisch, *Phys. Plasmas* **18**, 055704 (2011).
- ²⁰J. M. Rax, R. Gueroult, and N. J. Fisch, *Phys. Plasmas* **24**, 032504 (2017).
- ²¹G. Shvets, N. J. Fisch, and J.-M. Rax, *Phys. Rev. E* **65**, 046403 (2002).
- ²²I. Y. Kostyukov, G. Shvets, N. J. Fisch, and J. M. Rax, *Phys. Plasmas* **9**, 636 (2002).
- ²³C. Thauray, E. Guillaume, S. Corde, R. Lehe, M. L. Bouteiller, K. T. Phuoc, X. Davoine, J. M. Rax, A. Rousse, and V. Malka, *Phys. Rev. Lett.* **111**, 135002 (2013).
- ²⁴L. J. Spitzer, *Astrophys. J.* **116**, 299 (1952).
- ²⁵V. Rozhansky, "Mechanism of transverse conductivity and generation of self-consistent electric fields in strongly ionized magnetized plasma" in *Reviews of Plasma Physics* (Springer-Verlag, Berlin-Heidelberg, 2008), pp. 1-52.
- ²⁶L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics of Continuous Media* (Pergamon Press, Oxford, 1984).
- ²⁷P. Helander and D. J. Sigmar, *Collisional Transport in Magnetized Plasma* (Cambridge University Press, Cambridge, 2002).
- ²⁸J. M. Rax, A. Fruchtman, R. Gueroult, and N. J. Fisch, *Phys. Plasmas* **22**, 092101 (2015).
- ²⁹S. I. Braginskii, *Rev. Plasma Phys.* **1**, 205 (1965).
- ³⁰J. M. Rax, J. Robiche, R. Gueroult, and C. Ehrlacher, *Phys. Plasmas* **25**, 072503 (2018).
- ³¹N. J. Fisch and J.-M. Rax, *Phys. Rev. Lett.* **69**, 612 (1992).
- ³²N. J. Fisch and J.-M. Rax, *Nucl. Fusion* **32**, 549 (1992).
- ³³N. J. Fisch, *Phys. Rev. Lett.* **41**, 873 (1978).
- ³⁴N. J. Fisch, *Rev. Mod. Phys.* **59**, 175 (1987).
- ³⁵T. H. Stix, *Phys. Rev. Lett.* **24**, 135 (1970).
- ³⁶T. H. Stix, *Phys. Fluids* **14**, 692 (1971).
- ³⁷E. J. Kolmes, I. E. Ochs, and N. J. Fisch, *Phys. Plasmas* **25**, 032508 (2018).