Gromov ground state in phase space engineering for fusion energy

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Phase space engineering by rf waves plays important roles in both thermal D-T fusion and nonthermal advanced fuel fusion, but not all phase space manipulation is allowed; certain fundamental limits exist. In addition to Liouville's theorem, which requires the manipulation to be volume preserving, Gromov's nonsqueezing theorem imposes another constraint. The Gardner ground state is defined as the ground state accessible by smooth volume-preserving maps. However, the extra Gromov constraint should produce a higher-energy ground state. An example of a Gardner ground state forbidden by Gromov's nonsqueezing theorem is given. The challenge question is "What is the Gromov ground state, i.e., the lowest energy state accessible by smooth symplectic maps?" This is a difficult problem. As a simplification, we conjecture that the linear Gromov ground state problem is solvable.

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I. PHASE SPACE ENGINEERING FOR FUSION ENERGY AND GARDNER GROUND STATE

Phase space engineering by rf waves plays important roles in both thermal D-T fusion and nonthermal advanced fuel fusion, from plasma heating [1,2] and current drive [3–5] to instability suppression [6–9] and α-particle energy channeling [10–15]. Particle accelerator technologies frequently employ phase space engineering to modify the characteristics of charged particles [16–19]. Notably, methods exchanging the transverse and longitudinal emittance of charged particle beams have been designed to enhance the beam quality [20–24].

Advanced fuel fusion using p-B11 or D-He3 must be conducted in a nonthermalized environment and thus requires significant power circulation within the system to keep particles in nonequilibrium energy states [25]. While this might seem inefficient, it was recently illustrated [26] that the energy does not have to be lost if the power flow is carefully managed. This concept is similar to energy recovery systems used in energy-recovering particle accelerators [27]. Just as successful deuterium-tritium fusion requires near-perfect tritium recirculation with less than 0.1% loss [26,28,29], making advanced fuel fusion work depends on maintaining nonthermal particle distributions through efficient power recirculation in the system [26].

When using rf electromagnetic fields to manipulate charged particles, certain fundamental limits exist. One of the constraints is imposed by Liouville's theorem, which states that the volume particles occupy in phase space must remain constant—you can reshape this volume, but not compress it.

To highlight the importance of this and other constraints, we focus on the energy extraction schemes for aneutronic fusion. While aneutronic fusion has the advantage of releasing energy as charged particles (which can theoretically be converted directly into electricity), there is a catch. The initial fusing ions have much less energy than the fusion products, meaning the released energy spreads out into a larger phase space volume. This volume must be preserved during any electromagnetic energy extraction process.

This raises a key question: Given a distribution of fusion products (like alpha particles in proton-boron fusion), what is the maximum energy we can extract electromagnetically? Put another way, what is the lowest energy state we can reach through electromagnetic interactions? Because of phase space volume conservation, this "ground state" energy cannot be zero. As noted above, this defines a limit on the energy that can be extracted from a plasma using rf waves. It also has applications for understanding instabilities and turbulence, where it quantifies the energy that is available to drive a mode [30–37].

Gardner [38] posed the following problem: for a given phase space distribution of charged particles and an energy function, what is the ground state, i.e., minimum energy state, accessible under volume-preserving maps? He constructed the ground state by minimizing the system energy under the constraint of constant phase space volume. This method became known as the Gardner restacking algorithm [39,40]. For any given two compact, connected sets that are diffeomorphic and of the same volume in phase space, it can be proven (see the Appendix) using a technique known as Moser's trick [41] that there must exist a volume-preserving diffeomorphism between the two sets. Thus the ground state constructed by the Gardner restacking algorithm is accessible by smooth volume-preserving maps.

However, volume preservation is not the only constraint we need to consider. A more stringent constraint is "nonsqueezability," which comes from the underlying symplectic nature

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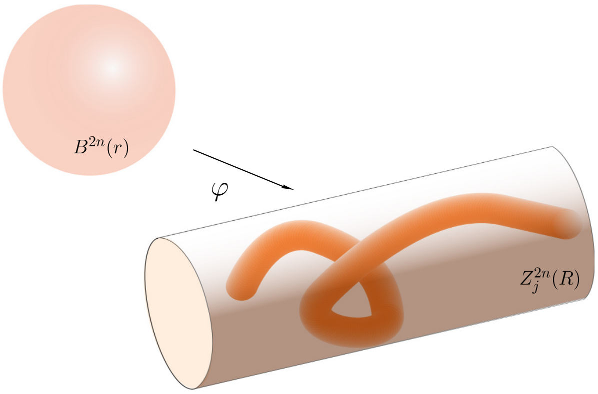

FIG. 1. There exists no smooth symplectic map φ in $\mathbb{R}^{2n}$, sending the ball $B^{2n}(r)$ to a cylinder $Z_j^{2n}(R)$ when $r > R$. But when $r = R$, the ball $B^{2n}(R)$ fits comfortably within the bounds of the cylinder $Z_j^{2n}(R)$ without any squeezing.

of charged particle dynamics in electromagnetic fields. We identify here that this means that the Gardner ground state might not actually be achievable using real electromagnetic fields, whether externally applied or self-generated by the system.

II. GROMOV'S NONSQUEEZING THEOREM

For a canonical Hamiltonian system of n degrees of freedoms in $\mathbb{R}^{2n}$, the dynamics is governed by the familiar Hamilton's equation:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i},$$
$$H = H(q^i, p_i, t),$$
$$i = 1, 2, ..., n. \tag{1}$$

The solution map of Eq. (1), $\varphi_t : \boldsymbol{z}(0) = [q(0), \boldsymbol{p}(0)] \mapsto \boldsymbol{z}(t) = [\boldsymbol{q}(t), \boldsymbol{p}(t)]$ is symplectic; i.e.,

$$(D\varphi_t)^T J (D\varphi_t) = J, \tag{2}$$

$$D\varphi_t \equiv \frac{\partial \varphi_t(z)}{\partial z}, \tag{3}$$

$$J = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix}. \tag{4}$$

Here, $D\varphi_t$ is the Jacobian matrix of the solution map φ_t, and J defines an almost complex structure on ; i.e., $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $J^2 = -1$. Symplecticity is the defining characteristic of Hamiltonian systems, and being symplectic is a much stronger geometric constraint than being volume preserving.

One way to characterize the symplectic constraint is given by Gromov's nonsqueezing theorem [42–50], which states that there exists no smooth symplectic map φ in $\mathbb{R}^{2n}$ sending the ball $B^{2n}(r)$ to a cylinder $Z_j^{2n}(R)$ when $r > R$; see Fig. 1. Here, the ball and the cylinder are defined as

$$B^{2n}(r) \equiv \left\{ (q^1, q^2, ..., q^n, p_1, p_2, ..., p_n) \middle| \sum_{i=1}^{n} (p_i^2 + q^{i2}) < r^2 \right\}, \tag{5}$$

$$Z_j^{2n}(R) \equiv \left\{ (q^1, q^2, ..., q^n, p_1, p_2, ..., p_n) | p_j^2 + q^{j2} < R^2 \right\}. \tag{6}$$

The theorem significantly reduces the space of allowed manipulations in phase space. Because $B^{2n}(R) \subset Z_j^{2n}(R)$, the ball $B^{2n}(R)$ fits comfortably within the bounds of the cylinder $Z_j^{2n}(R)$ without any squeezing. However, if we attempt to expand the ball's radius by even an infinitesimal increment, no symplectic map can force this slightly larger ball into the same cylindrical space. This geometric constraint presents a fundamental challenge—imagine a carpenter trying to fit an ever-so-slightly enlarged piece into a predefined space, only to find it mathematically impossible.

With this constraint in mind, we pose the following problem of the Gromov ground state [26]: for a given phase space distribution of charged particles and an energy function, what is the ground state under all possible smooth symplectic maps? This Gromov ground state is higher than the Gardner ground state. It determines the theoretical upper limit of electromagnetically extractible energy [30,31,51,52] in aneutronic fusion devices [25,53–60].

Gromov's nonsqueezing theorem suggests transitioning from volume-preserving numerical algorithms [61–68] to symplectic algorithms [69–96] to more accurately simulate the phase space engineering processes of charged particles.

III. GARDNER GROUND STATE FORBIDDEN BY GROMOV'S NONSQUEEZING THEOREM

In this section, we demonstrate an example of a Gardner ground state that is not a Gromov ground state.

Consider a system with two degrees of freedom in $\mathbb{R}^4$ with canonical coordinates (x, y, v_x, v_y). Assume there is an external potential in the x direction,

$$\phi(x) = \tfrac{1}{2}x^2. \tag{7}$$

A particle's energy is

$$\varepsilon = \tfrac{1}{2}\left(x^2 + v_x^2 + v_y^2\right). \tag{8}$$

This system is a simplified model for charged particle dynamics in accelerators or quadrupole ion traps [19].

Suppose the initial distribution function f_0 is uniform inside $B^4(r)$; i.e.,

$$f_0(x, y, v_x, v_y) = \Theta\left[r^2 - \left(x^2 + y^2 + v_x^2 + v_y^2\right)\right] \equiv \begin{cases} 1, & \left(x^2 + y^2 + v_x^2 + v_y^2\right) < r^2. \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

Here, $\Theta(x)$ is the Heaviside step function. Such a distribution function is known as a water-bag distribution. The phase space volume occupied by f_0 is

$$V(f_0) = \int_{B^4(r)} dx dy dv_x dv_y = \frac{\pi^2}{2} r^4. \tag{10}$$

The energy of the system is

$$\begin{aligned} W(f_0) &= \int \frac{1}{2}\left(x^2+v_x^2+v_y^2\right) f_0 dxdydv_xdv_y. \\ &= \int_{B^4(r)} \frac{1}{2}\left(x^2+v_x^2+v_y^2\right) dxdydv_xdv_y \\ &= 2\int_0^{\pi/2} W_3(r\cos\theta) r\cos\theta d\theta \\ &= 2\int_0^{\pi/2} \frac{2\pi}{5} r^6 \cos^6\theta d\theta = \frac{\pi^2}{8} r^6, \end{aligned} \quad (11)$$

where

$$W_3(r) \equiv \int_0^r \frac{a^2}{2} 4\pi a^2 da = \frac{2\pi}{5} r^5 \quad (12)$$

is the energy line density inside the ball $B^3(r) \equiv \{(x, v_x, v_y)|x^2+v_x^2+v_y^2 < r^2\}$.

The energy of the system can be reduced by pushing particles to regions in phase space with lower energy density while preserving the phase space volume, or the volume of the water bag. In particular, we can squeeze, via a volume-preserving map, the initial ball $B^4(r)$ into the following "cylinder" in $\mathbb{R}^4$,

$$\begin{aligned} &B^3(R)\times L \\ &\quad = \left\{(x, y, v_x, v_y)|x^2+v_x^2+v_y^2 < R^2 \text{ and } 0 < y < L\right\}. \end{aligned} \quad (13)$$

Our intention is to mold the cylinder into a long noodle along the y axis by increasing L and decreasing R while maintaining the same phase space volume.

The distribution function is now

$$f(x, y, v_x, v_y) = \begin{cases} 1, & (x, y, v_x, v_y) \in B^3(R)\times L \\ 0, & \text{otherwise} \end{cases}. \quad (14)$$

The phase space volume of the system is

$$V(f) = \int_{B^3(R)\times L} dxdydv_xdv_y = \frac{4\pi}{3} R^3 L. \quad (15)$$

The energy of the system is

$$\begin{aligned} W(f) &= \int \frac{1}{2}\left(x^2+v_x^2+v_y^2\right) f dxdydv_xdv_y. \\ &= \int_{B^3(R)\times L} \frac{1}{2}\left(x^2+v_x^2+v_y^2\right) dxdydv_xdv_y \\ &= W_3(R)L = \frac{2\pi}{5} R^5 L = \frac{3}{10} R^2 V. \end{aligned}$$

Volume-preserving requires that V is a constant; i.e.,

$$V = \frac{\pi^2}{2} r^4 = \frac{4\pi}{3} R^3 L = \text{const}. \quad (16)$$

Under this constraint of constant volume, we can mold the cylinder into a long noodle by letting $R \to 0$ and $L \to \frac{3V}{4\pi R^3}$, which leads to

$$W(f) \to 0. \quad (17)$$

Thus, the ground state energy is 0, which is reachable when $R \to 0$, if volume preserving is the only constraint. Let us call this ground state the Gardner ground state.

However, this Gardner ground state is not reachable by symplectic maps, because this volume-preserving map sends $B^4(r)$ to

$$B^3(R)\times L \subset Z_1^4(R) \equiv \left\{(x, y, v_x, v_y)|x^2+v_x^2 < R^2\right\}. \quad (18)$$

According to Gromov's nonsqueezing theorem, this map cannot be symplectic if $R < r$.

IV. WHAT IS THE GROMOV GROUND STATE?

The unanswered question is "What is the Gromov ground state—the lowest energy state accessible by smooth symplectic maps? Can the energy of the Gromov ground state be close to the energy of the Gardner ground state?"

Assume there is a symplectic map sending $B^4(r)$ to $B^3(R)\times L$ (this assumption is very likely to be wrong). Then the minimum R allowed by Gromov's theorem for symplectic maps is r. If we take this $B^3(r)\times L$ with $L = 3V/4\pi r^3 = 3\pi r/8$ to be an "approximate Gromov ground state," then its energy would be

$$W(f) = \frac{3}{10} r^2 V = \frac{3\pi^2}{20} r^6 = \frac{6}{5} W(f_0).$$

Obviously, this approximate Gromov ground state is not a good approximation at all, because its energy is not reduced from the initial energy of f_0. This demonstrates that Gromov's nonsqueezing theorem puts a strong constraint on the ground state accessible by symplectic maps relative to the volume-preserving maps. The noodle allowed by Gromov's theorem looks more like a ball because $L = 3\pi r/8 \sim R = r$. That is why the energy of the state is not reduced relative to the initial ball. In other words, the volume-preserving constraint allows the ball to be molded into a long noodle. But Gromov's theorem says the footprint in the (x, v_x) plane and the (y, v_y) plane cannot be reduced, and if one must mold the ball into a noodle, then the noodle has to be thick and short.

On the other hand, Gromov's theorem does not prohibit molding the ball into a mushroom with a long stem and a thin cap. This is because this mushroom's footprint in the (x, v_x) plane and the (y, v_y) plane is not reduced relative to the initial ball. This mushroom's energy could be very close to 0, since the cap can be very thin. However, the issue is that Gromov's theorem only says squeezing is not allowed; it does not say what is accessible via symplectic maps. This mushroom may still be inaccessible via symplectic maps, even though it satisfies Gromov's nonsqueezing constraint.

Another argument against this mushroom being a Gromov ground state is the nonsqueezing theorem applied to the stem by itself. The preimage of the stem should be almost the entire ball, which cannot be squeezed into the stem. In this example, the preimage of the stem is a neighborhood of the ball.

It was proven [97] that a smooth symplectic map exists to send a set in phase space to an arbitrarily close neighborhood of another set of the same phase space volume if the derivatives of the maps are allowed to be arbitrarily large. Therefore, the Gromov ground state approaches the Gardner ground state if the symplectic maps are allowed to have arbitrarily large derivatives. Of course, having arbitrarily large derivatives implies that the map is becoming nonsmooth.

Interestingly, this suggests connections with the theory of diffusively accessible free energy, in which Gardner's restacking operation (which exchanges the populations of two elements of phase space) is replaced with a mixing operation (wherein their populations are averaged) [36,40,51,52,98–100]. Suppose there existed a symplectic map that produced very fine-scale structure in some local region in phase space, such that the map appeared to produce diffusion when viewed on a coarser scale in phase space (this is essentially the intuition behind processes like quasilinear diffusion). Suppose, furthermore, that such a map could be applied to different, perhaps overlapping regions of phase space, again with the effect of generating fine-scale structures that appear to produce diffusion on larger scales. Then, in the limit where these fine-scale structures could be made arbitrarily fine, it would be possible to construct a symplectic map to a state arbitrarily close to any state that is accessible through mixing operations. It has been shown [40] that sequences of mixing operations can access states that are arbitrarily close to the Gardner ground state, so this would imply that the free energy accessible through symplectomorphisms is arbitrarily close to the free energy accessible through volume-preserving maps. The interesting point here is that the requirement of arbitrarily fine-scale structure suggests that such a symplectomorphism would have very large derivatives, consistent with the result in Ref. [97].

In any event, the result of Ref. [97] suggests that the free energy accessible through an arbitrary symplectomorphism is arbitrarily close to that accessible through volume-preserving maps, but that the symplectomorphisms needed to accomplish this may not be smooth and therefore may not be appropriate in all scenarios. With this consideration, we should specify the classes of allowed symplectic maps when posing the Gromov ground state problem. Since real fusion devices are of finite size, we could also further require the domain and range of the symplectomorphisms to be bounded.

As an example of practical importance, we can study the linear Gromov ground state problem. Most, if not all, beam optical components for controlling charged particles can be modeled by linear symplectic maps [16–19], even though linear symplectic maps do not describe the wave-induced dynamics for current drive [3] and α channeling [10–15] in tokamaks. It is known [46,50] that linear symplectic maps define a linear symplectic capacity, and it agrees with the symplectic capacity [44,101,102] defined by general symplectic maps for phase space ellipsoid in $\mathbb{R}^{2n}$. Here, the symplectic capacity $C(U)$ of a set $U \subset \mathbb{R}^{2n}$ is defined as the cross section of the largest ball that can be embedded into U by a symplectomorphism, and the linear symplectic capacity $C^{\mathrm{lin}}(U)$ is defined similarly except that symplectomorphisms are constrained to be linear maps in $\mathbb{R}^{2n}$; i.e.,

$$C(U) \equiv \sup\{\pi r^2 | \exists\, \varphi \in \mathrm{Sym} \;\; \text{s.t.} \;\; \varphi[B^{2n}(r)] \subset U\}, \quad (19)$$

$$C^{\mathrm{lin}}(U) \equiv \sup\{\pi r^2 | \exists\, \varphi \in \mathrm{ISp}(2n) \;\; \text{s.t.} \;\; \varphi[B^{2n}(r)] \subset U\}. \quad (20)$$

Intuitively, that $C(U) = C^{\mathrm{lin}}(U)$ when U is an ellipsoid can be interpreted as linear symplectic maps being as flexible as nonlinear symplectic maps. If so, a neighborhood of the Gromov ground state might be approachable by linear symplectic maps. Let us call the minimum energy state accessible via linear symplectic maps the linear Gromov ground state.

In particular, for the counterexample given above, we pose the following problem: for the external potential $\phi(x)$ and f_0 defined in Sec. III, what is the linear Gromov ground state? That is, what is the 4×4 symplectic matrix S that minimizes the energy of $S[B^4(1)]$? We conjecture that this problem is solvable [103].

V. SUMMARY AND DISCUSSION

What we have identified here is an important constraint on the ground state energy of collisionless rearrangement by waves of charged particles in a plasma. Indeed, this constraint applies to any rearrangement by means of Hamiltonian dynamics. In particular, we suggest that applying the nonsqueezing theorem of Gromov could lead to a higher ground state energy, at least in practical cases. The point here that has gone unrecognized in the existing literature on plasma available energy is that Hamiltonian dynamics is always phase-space-volume preserving, but that not all phase-space-volume-preserving transformations are accessible via Hamiltonian dynamics (or more formally, symplectic maps).

Because of the interest in fusion applications of recovering particle energy (particularly fusion byproduct energy) in waves, it is of great interest to know the maximum recoverable amount or the ground state for a given energy distribution. The first question is what the allowable ground states are according to the rules of particle motion. For particles obeying Hamiltonian dynamics, the rules are clear for getting from configuration A to configuration B in the 6D phase space: One, there must be phase space conservation, so configuration B has to have the same phase space densities as configuration A. This leads to Gardner restacking [38–40], but configuration A has also to travel an allowable path to configuration B. That leads to the Gromov nonsqueezing constraint [39,40], which is much harder to quantify. However, in view of the last section, if transformations with arbitrarily large gradients are allowed, then the Gromov ground state approaches the Gardner ground state.

However, not all allowable transformations are practical. In controlling charged particles by waves in the most important fusion applications, it is invariably the case that the waves are arranged to diffuse particles in velocity space (like in current drive) or in the combined velocity-configuration space (like in α channeling). Because wave-particle interactions are a blunt instrument, it is not practically possible to exert extremely fine control over the particle rearrangement. This practical limit is important; in the absence of this limit we might be allowed transformations that are not smooth and therefore allow energy recovery approaching the Gardner free energy, even taking into account the Gromov constraint as indicated in the last section. But with only blunt transformations possible, both the Gardner ground state energy and the Gromov ground state energy must necessarily rise. However, the extent of this rise may not be the same.

This same notion of "bluntness"—in which the realistically accessible states are constrained by a lack of perfect control over which phase space volumes go where—is closely related to the idea of the free energy under diffusive operations [98]. Interestingly, even for that problem, arbitrarily fine control over the regions of phase space undergoing diffusion makes it possible to replicate the Gardner ground state arbitrarily closely [52].

If arbitrary symplectomorphisms can replicate the Gardner ground state arbitrarily well, that still leaves open the problem of finding the accessible energy under the Gromov constraint when infinitely fine-grained control over the map is not possible. In the absence of arbitrarily fine control by the waves, Gardner restacking is also modified; one could call the resulting free energy the *coarse-grained Gardner ground state energy* (one could imagine, for example, the Gardner problem in which phase space is discretized with some finite cell size). Of course, that will depend on how coarsely the plasma is put into bins of constant density in the 6D configuration space. Now adding the Gromov constraint on allowable particle motion, one can define for such a degree of control by waves a *coarse-grained Gromov ground state energy*. The coarse-grained Gromov ground state energy will necessarily be higher than the coarse-grained Gardner ground state energy. However, this is the energy that is hard to compute, except in the case when infinitely fine-grained discretization is allowed, in which case both energies approach the same ground state Gardner energy.

But the practical question—at least for diffusion by waves for extracting energy—is, in fact, "what is the *coarse-grained Gromov ground state energy*?" This energy not only depends on the coarseness, but also on how coarseness is defined. One approach to this problem in which we conjecture a solution may be approachable is by examining the ground state under linear symplectic mappings, which should be solvable.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: PROOF OF ACCESSIBILITY OF GARDNER GROUND STATE VIA VOLUME-PRESERVING MAP

In this Appendix, we show that the Gardner ground state, constructed by minimizing the system energy under the constraint of constant phase space volume, is accessible by smooth volume-preserving maps. It suffices to prove that given any two physically well-defined regions of the same volume in phase space, we can always find a volume-preserving diffeomorphism connecting the two regions. We formulate this result as the following proposition.

Proposition A1. Let A and B be two compact, connected sets of $\mathbb{R}^m$. If A and B are diffeomorphic and have the same volume as measured by a volume form Ω in $\mathbb{R}^m$, then there exists a volume-preserving diffeomorphism $\psi : A \to B$.

Proof. Let $\phi : A \to B$ be the diffeomorphism between A and B. Since ϕ is not necessarily volume preserving, $\phi^*\Omega|_A \neq \Omega|_A$ in general. However, a volume-preserving diffeomorphism $\psi : A \to B$, defined by the property $\psi^*\Omega|_A = \Omega|_A$, can be constructed as follows.

Let $\Omega_1|_A = \phi^*\Omega|_A$. Because A and B have the same volume as measured by a volume form Ω,

$$\int_A \Omega = \int_B \Omega,$$

which implies

$$\int_A \Omega = \int_{\phi(A)} \Omega = \int_A \phi^*\Omega = \int_A \Omega_1.$$

According to Theorem A1, on the compact, connected manifold A, there exists a diffeomorphism $\tau : A \to A$ such that $\Omega = \tau^*\Omega_1$. Let

$$\psi = \phi \circ \tau : A \to B.$$

We have

$$\psi^*\Omega|_A = \tau^* \circ \phi^*\Omega = \tau^*\Omega_1 = \Omega.$$

Thus $\psi : A \to B$ is a volume-preserving diffeomorphism between A and B ■.

Here we require the two sets to be diffeomorphic in addition to having the same phase space volume. This is to rule out situations where the two sets are topologically different. For phase space engineering in the present context, the phase space $\{(q^1, q^2, ..., q^n, p_1, p_2, ..., p_n)\}$ is identified with $\mathbb{R}^m$ ($m = 2n$), and the volume form is the canonical volume form $\Omega = dp_1 \wedge ... \wedge dp_n \wedge dq^1 \wedge ... \wedge dq^n$. The proof of Proposition A1 uses the following theorem.

Theorem A1 (Moser). Let Ω and Ω_1 be two volume forms on a compact, connected manifold M. There exists a diffeomorphism $\tau : M \to M$ such that $\Omega = \tau^*\Omega_1$ iff $\int_M \Omega = \int_M \Omega_1$.

Moser [41] proved Theorem A1 using a technique that is now called Moser's trick.

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