Efficiency of current drive by fast waves

Charles F. F. Karney and Nathaniel J. Fisch
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08544

(Received 9 July 1984; accepted 28 September 1984)

The Rosenbluth form for the collision operator for a weakly relativistic plasma is derived. The formalism adopted by Antonsen and Chu can then be used to calculate the efficiency of current drive by fast waves in a relativistic plasma. Accurate numerical results and analytic asymptotic limits for the efficiencies are given.

I. INTRODUCTION

Currents may be efficiently generated in a plasma by the injection of rf waves whose phase velocities are several times the electron thermal speed.[1] The efficiency, defined as the ratio of current generated to power dissipated, is achieved in this instance because the rf-generated plateau decays at a rate given by the collision frequency for the fast electrons, which is relatively low. In the quest for higher efficiencies, current drive by waves which interact with relativistic electrons has also been considered.[2] Relativistic effects modify the scaling of the efficiency, placing an upper bound on the efficiency achievable by current drive by fast waves. In this paper we do several things: We give a more complete analysis of this problem based on a formalism adopted by Antonsen and Chu.[3] Specifically, we find that the effect of finite electron temperature leads to an enhancement of the efficiency. In order to calculate this effect, we first give expressions for the most important terms in the electron–electron collision integral in the relativistic limit. These expressions are put in Rosenbluth form so as to be amenable to easy implementation on a computer. We imagine that the relativistic Rosenbluth potentials that we identify may be useful in other problems arising in very hot plasmas.

In order to put the present work in perspective, let us briefly review the chief tools used in the study of current drive. The early work used fairly crude analytical models.[1,4] These models were sufficient to obtain the scaling laws for the efficiency of current drive but were unable to provide the coefficients with any accuracy. Therefore the analytical treatment was supplemented by numerical solutions to the two-dimensional (in momentum space) Fokker–Planck equation,[5–8] from which accurate estimates of the efficiency could be found. The first accurate analytical treatment of current drive was based on a Langevin formulation of the electron motion.[9,2] This involved taking the electron temperature to be small, allowing energy scattering to be ignored. The moment hierarchy for the Langevin equations can then be closed, which allows an analytical solution to be obtained. This was followed by a more complete numerical study of the Fokker–Planck equation for current drive in which the problem was reduced to the numerical solution of a one-dimensional (1-D) integro-differential equation with a source caused by the rf.[10] In this work toroidal effects were also included. The results agreed with the Langevin analysis[9] in the limit of large phase velocities (as they should) and gave more accurate numerical data for phase velocities comparable to or smaller than the thermal velocity. More recently, Antonsen and Chu[3] and, independently, Taguchi[11] using methods first used in the study of beam-driven currents[12,13] recognized that it is not necessary to solve the rf-driven Fokker–Planck equation in order to find the rf-induced current. Instead, they showed that the Green's function for the current is the Spitzer–Härm function[14] describing the perturbed electron distribution in the presence of an electric field. This reduces the problem to the determination of a single 1-D function, from which the current generated by any form of rf drive can be calculated by a simple integration.

Up until now, the only reliable analytical results for current drive in a relativistic plasma are those obtained using the Langevin methods by Ref. 2. As we will show, these are only exact for $T_e \ll m_e c^2$ and $p^2 \gg m_e T_e$ (where p is the momentum of the resonant electrons). A more complete analytical or numerical treatment along the lines of that achieved in the nonrelativistic case was hampered by the lack of a convenient form for the relativistic collision operator. This is remedied to some extent by the results of this paper, where we calculate the collision integrals for the first Legendre harmonic of the perturbed electron distribution neglecting electromagnetic effects on the binary interaction (in this approximation the collision integral reduces to the Landau form). Having done this, we are able to generalize the treatment of Antonsen and Chu[3] to the relativistic case. A number of useful results flow from this: We can numerically calculate to high precision the current-drive efficiencies in the relativistic regime. We can perform an asymptotic analysis of the Spitzer–Härm problem to obtain analytic approximations to the efficiencies. In addition, we give higher-order asymptotic corrections to the current-drive efficiencies in the nonrelativistic limit. Throughout this paper, toroidal effects are entirely ignored. Although these effects are important in the study of current drive by low-phase-velocity waves, they play little role in current drive by fast waves. Incorporation of these effects, however, proceeds in exact analogy with the treatment for the nonrelativistic case.[3]

Relativistic effects on rf current drive have also been considered by Hizanidis and Bers.[15] However, their analysis is flawed and their results for the current-drive efficiency are incorrect.

The plan of this paper is as follows: In Sec. II we show how the relativistic collision operator may be reduced to the Landau form. In this form, the collision operator is costly to evaluate numerically; so, in Sec. III we convert the collision integrals to a Rosenbluth form, which may be evaluated very efficiently. The formulation of Antonsen and Chu is generalized to the relativistic case in Sec. IV. The numerical results for the efficiencies are given in Sec. V and the asymptotic results in Sec. VI. Finally, in Sec. VII we examine the asymptotic form of the efficiencies using the full relativistic collision operator.

II. RELATIVISTIC COLLISION OPERATOR

The collision operator for a relativistic plasma is given by Beliaev and Budker.[16] They give the collision operator as

$$\left.\frac{\partial f_a(\mathbf{p})}{\partial t}\right|^{\text{coll}} = \sum_b C(f_a, f_b), \tag{1a}$$

$$C(f_a,f_b) = (q_a^2 q_b^2/8\pi\epsilon_0^2)\log\Lambda^{a/b} \times\frac{\partial}{\partial\mathbf{p}}\cdot\int \mathsf{U}\cdot\left(f_b(\mathbf{p}')\frac{\partial f_a(\mathbf{p})}{\partial\mathbf{p}} - f_a(\mathbf{p})\frac{\partial f_b(\mathbf{p}')}{\partial\mathbf{p}'}\right)d^3\mathbf{p}', \tag{1b}$$

where the kernel U is given by

$$\mathsf{U} = \frac{\gamma_a\,\gamma_b'(1-\boldsymbol{\beta}_a\cdot\boldsymbol{\beta}_b')^2}{c[\gamma_a^2\gamma_b'^2(1-\boldsymbol{\beta}_a\cdot\boldsymbol{\beta}_b')^2-1]^{3/2}} \times\{[\gamma_a^2\gamma_b'^2(1-\boldsymbol{\beta}_a\cdot\boldsymbol{\beta}_b')^2-1]\mathsf{I}-\gamma_a^2\,\boldsymbol{\beta}_a\,\boldsymbol{\beta}_a - \gamma_b'^2\,\boldsymbol{\beta}_b'\,\boldsymbol{\beta}_b' + \gamma_a^2\gamma_b'^2(1-\boldsymbol{\beta}_a\cdot\boldsymbol{\beta}_b')(\boldsymbol{\beta}_a\,\boldsymbol{\beta}_b' + \boldsymbol{\beta}_b'\,\boldsymbol{\beta}_a)]\}. \tag{2}$$

Here a and b are species labels, q_s is the charge of species s, $\log\Lambda^{a/b}$ is the Coulomb logarithm, ϵ_0 is the dielectric constant, $\mathbf{p}$ is the momentum, $\mathbf{v}_s = c\boldsymbol{\beta}_s = \mathbf{p}/m_s\gamma_s$ is the velocity of species s, and $\gamma_s = (1+p^2/m_s^2c^2)^{1/2}$. The distributions are normalized so that

$$\int f_s(\mathbf{p})\,d^3\mathbf{p} = n_s,$$

the number density. We are primarily interested in situations where fast electrons are colliding off a weakly relativistic background. In that case $\beta_b' \ll 1$, and we can approximate U by its nonrelativistic form

$$\mathsf{U} = (u^2\mathsf{I} - \mathbf{uu})/u^3, \quad \mathbf{u} = \mathbf{v}_a - \mathbf{v}_b'. \tag{1c}$$

Since the original form for U was symmetric in the primed and unprimed variables, we could equally well have obtained Eq. (1c) under the assumption that $\beta_a \ll 1$. The relative difference between Eqs. (1c) and (2) is $O(\beta_b')$. However, the error in the collision operator $C(f_a, f_b)$ is smaller than this. This point is examined in more detail in Sec. VII. Equations (1) are precisely the collision operator given by Landau.[17] Indeed, an examination, of his derivation shows that the mechanics of the collisions are treated relativistically; the interaction, however, is calculated nonrelativistically assuming a Coulomb potential. Use of Landau collision operator implies a neglect of the relativistic (i.e., electromagnetic) effects on the binary interaction. What we have shown here is that such an approximation is valid provided at least one of the colliding particles is nonrelativistic.

It is readily established that Eqs. (1) conserve number, momentum, and energy ($\mathscr{E}_s = m_s c^2\gamma_s$), that an H theorem applies, and that the equilibrium solution is a relativistic Maxwellian $f_s(\mathbf{p}) \propto \exp(-\mathscr{E}_s'/T)$, where $\mathscr{E}_s' = (\mathscr{E}_s - \mathbf{v}_d\cdot\mathbf{p})/(1-v_d^2/c^2)^{1/2}$ is the energy in a frame moving at $\mathbf{v}_d$, and T and $\mathbf{v}_d$ are independent of the species s.

Throughout the rest of this paper we will restrict our attention to an electron–ion plasma. We assume the ions are stationary and infinitely massive ($m_i \to \infty$). This allows us to express the electron–ion collision operator in (p,μ) space (where $\mu = p_\parallel/p$ and $\parallel$ and $\perp$ are with respect to the magnetic field) as

$$C(f,f_i) = \Gamma\frac{Z}{2vp^2}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}f(\mathbf{p}), \tag{3}$$

where

$$\Gamma = n_e q_e^4 \log\Lambda^{e/e}/4\pi\epsilon_0^2,$$

$$Z = -q_i \log\Lambda^{e/i}/q_e \log\Lambda^{e/e} \approx -q_i/q_e,$$

and we have assumed neutrality $q_e n_e + q_i n_i = 0$. In Eq. (3) and henceforth we shall omit the species labels from all electron quantities.

III. GENERALIZATION OF THE ROSENBLUTH POTENTIALS

For computational purposes, the Landau operator is not the most convenient form for the collision operator. If the plasma is azimuthally symmetric, a 2-D integration must be performed at each point in momentum space. If the number of grid points is $N\times N$, this requires $O(N^4)$ calculations. This requirement is dramatically reduced in the nonrelativistic case by expressing the collision operator in terms of Rosenbluth potentials.[18] Unfortunately, although the Landau operator can be used without change to describe the collisions in a relativistic Coulomb plasma, the Rosenbluth form no longer applies. [The derivation of the Rosenbluth form from the Landau form requires, for instance, that $(\partial/\partial\mathbf{p})\cdot\mathsf{U} = -(\partial/\partial\mathbf{p}')\cdot\mathsf{U}$, a relation that only holds nonrelativistically.]

However, because the kernel of the collision integral Eq. (1c) has the same form as in the nonrelativistic case, it is possible to borrow some of the techniques of Ref. 18. We convert the $\mathbf{p}'$ integration in Eq. (1b) to $\mathbf{v}'$ space, substitute a particular Legendre component for $f(\mathbf{p}')$, and manipulate the resulting integrals into the form

$$\int |\mathbf{v}-\mathbf{v}'|P_k(\mu')h(v')\,d^3\mathbf{v}'$$

or

$$\int |\mathbf{v}-\mathbf{v}'|^{-1}P_k(\mu')h(v')\,d^3\mathbf{v}',$$

which may be evaluated in the same way as Rosenbluth potentials[18] (P_k is a Legendre polynomial).

Here we give the resulting expressions for collisions off a stationary Maxwellian background, i.e., $C(f,f_m)$, and for collisions of a Maxwellian off the first Legendre component of a background, i.e., $C(f_m, \mu f_1)$. In both cases only electron–electron collisions are considered. These terms are all that are required for the solution of the Spitzer–Härm problem (giving the Green's function for the rf-current drive), and they suffice for an accurate numerical solution of the 2-D Fokker–Planck equation as described in Sec. V.

Beginning with the case of collisions off a Maxwellian, let us start by assuming merely that the background is isotropic $f(\mathbf{p}) = f_0(p)$. The 3-D integrals in Eq. (1b), then reduce to 1-D integrals giving

$$C(f,f_0) = \frac{1}{p^2}\frac{\partial}{\partial p}p^2\left(A(p)\frac{\partial}{\partial p} + F(p)\right)f(\mathbf{p}) + \frac{B(p)}{p^2}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu}f(\mathbf{p}), \tag{4a}$$

where

$$A(p) = \frac{4\pi\Gamma}{3n}\left(\int_0^p p'^2 f_0(p')\frac{v'^2}{v^3}\,dp' + \int_p^\infty p'^2 f_0(p')\frac{1}{v'}\,dp'\right), \tag{4b}$$

$$F(p) = \frac{4\pi\Gamma}{3n}\left(\int_0^p p' f_0(p')\frac{3v' - v'^3/c^2}{v^2}\,dp' + \int_p^\infty p' f_0(p')\frac{2v}{c^2}\,dp'\right), \tag{4c}$$

$$B(p) = \frac{4\pi\Gamma}{3n}\left(\int_0^p p'^2 f_0(p')\frac{3v^2 - v'^2}{2v^3}\,dp' + \int_p^\infty p'^2 f_0(p')\frac{1}{v'}\,dp'\right). \tag{4d}$$

Specializing to the case $f_0 = f_m$ and using the relation $\partial f_m/\partial p = -(v/T)f_m$, we find that

$$F(p) = (v/T)A(p),$$

and the steady-state solution to $C(f,f_m) = 0$ is that f is a relativistic Maxwellian[19] with temperature T,

$$f_m(p) = \frac{n}{4\pi m^2 cTK_2(\Theta^{-1})}\exp\left(\frac{-\mathscr{E}}{T}\right), \tag{5}$$

where

$$\mathscr{E} = mc^2\gamma,$$
$$\Theta = T/mc^2$$

($\Theta = 1$ corresponds to an electron temperature of 511 keV), and K_n is the nth-order modified Bessel function of the second kind.

For later use we define here a thermal momentum

$$p_t = \sqrt{mT},$$

a mean-squared velocity

$$v_t^2 = \frac{1}{3n}\int v^2 f_m(p)\,d^3\mathbf{p} = \frac{T}{m}V_t^2,$$
$$V_t^2 = 1 - \tfrac{5}{2}\Theta + \tfrac{55}{8}\Theta^2 + O(\Theta^3),$$

a thermal collision frequency

$$\nu_t = m\Gamma/p_t^3 = nq^4 m\log\Lambda/4\pi\epsilon_0^2 p_t^3,$$

and a collision frequency normalized to the speed of light

$$\nu_c = \Gamma/m^2c^3 = nq^4\log\Lambda/4\pi\epsilon_0^2 m^2c^3.$$

Note that these frequencies differ by a factor of 2 from those used in earlier publications.[1,9,5,6,8,2] Specifically, we have $\nu_t = \nu_0/2$ and $\nu_c = \nu/2$. This means that all our normalized efficiencies are a factor of 2 smaller than in these earlier papers. (We made this change because the normalized Fokker–Planck equation in the high-energy limit now has a simpler form. This convention is also used by other workers in this field.)

For $p \gg p_t$, the indefinite limits in the integrals in Eq. (4) can be replaced by ∞, giving[20]

$$A(p) = \Gamma(v_t^2/v^3), \tag{6a}$$
$$B(p) = \Gamma(1/2v)(1 - v_t^2/v^2). \tag{6b}$$

Note that the frictional force $F(p)$ reaches a constant value as $p \to \infty$. This implies, for instance, that an electric field smaller than $\Gamma v_t^2/qTc^2$ cannot produce runaways.[21] On the other hand, the pitch-angle scattering frequency $B(p)/p^2$ continues to decay as $p \to \infty$. As the energy of the electron increases, its effective mass increases; it is then more difficult to deflect the heavier particle. In this limit, pitch-angle scattering is negligible compared with frictional slowing down. This is to be contrasted with the nonrelativistic case where the pitch-angle scattering frequency and the frictional slowing down rate decay as $1/p^2$, and the two processes are of comparable importance.

The implication for current drive is that the efficiency of parallel wave-induced fluxes, say by lower-hybrid waves, approaches a constant. This can be seen as follows: Nonrelativistically, the efficiency increases as p^2. Relativistic electrons, however, slow down faster because they are heavier, and they also do not carry more current when pushed in the parallel direction. Each of these effects reduces the efficiency by $\gamma \sim p$; hence the approach to a constant.

The other term we shall need is $C(f_m, \mu f_1)$. This term is rather harder to compute. We define $f_1(p) = f_m(p)\chi_1(p)$ and write $C(f_m, \mu f_m\chi_1) = \mu f_m I(\chi_1)$. Again, we reduce (this time after much algebra) the integrals in Eq. (1b) to 1-D ones to give

$$\begin{aligned}I(\chi_1) = \frac{4\pi\Gamma}{n}\Biggl\{&\frac{mf_m(p)\chi_1(p)}{\gamma}\\ &+ \frac{1}{5}\int_0^p p'^2 f_m(p')\chi_1(p')\frac{m}{T}\left[\frac{\gamma}{p^2}\frac{v'}{\gamma^3}\left(\frac{T}{mc^2}(4\gamma'^2+6) - \frac{1}{3}(4\gamma'^3 - 9\gamma')\right) + \frac{\gamma^2}{p^2}\frac{v'}{\gamma^3}\left(\frac{mv'^2}{T}\gamma^3 - \frac{1}{3}(4\gamma'^2+6)\right)\right]dp'\\ &+ \frac{1}{5}\int_p^\infty p'^2 f_m(p')\chi_1(p')\frac{m}{T}\left[\frac{\gamma'}{p'^2}\frac{v}{\gamma^3}\left(\frac{T}{mc^2}(4\gamma^2+6) - \frac{1}{3}(4\gamma^3 - 9\gamma)\right) + \frac{\gamma'^2}{p'^2}\frac{v}{\gamma^3}\left(\frac{mv^2}{T}\gamma^3 - \frac{1}{3}(4\gamma^2+6)\right)\right]dp'\Biggr\}.\end{aligned} \tag{7}$$

The term is square brackets in the last integral matches that in the first integral except for the interchange of the primed and unprimed variables. The simplification of Eq. (7) was achieved, in part, with the help of the symbolic manipulation program, MACSYMA.[22]

Equations (4) and (7) are now in a computationally convenient form. Their evaluation involves the determination of a number of indefinite integrals (the unprimed variables should be factored out of the integrals for this step), and the multiplication of these integrals by various functions of p. If the distribution functions are known on a grid of N points, then the computational cost is just $O(N)$. Furthermore, the calculation can be arranged so that nearly all the computations vectorize.[23] The general solution of the linearized electron–electron collision operator $C(f,f_m) + C(f_m,f) = 0$ is

$$f = (a + \mathbf{b}\cdot\mathbf{p} + c\mathscr{E}) f_m,$$

where a, $\mathbf{b}$, and c are arbitrary constants. With $a = c = 0$ and $\mathbf{b} = \hat{\mathbf{p}}_\parallel$, this provides a useful check on Eqs. (4) and (7) and their computational realizations.

IV. FORMULATION

We now turn to the calculation of the rf efficiency. There are three steps involved: the specification of the rf current-drive problem, the identification of the Spitzer–Härm function as the Green's function for the rf-driven current, and the solution of the Spitzer–Härm problem.

We begin with the specification of the problem. This is just a standard application of the Chapman–Enskog procedure.[24] The most important assumption is that the collisional time scale is much shorter than the transport time scale (the time scale for heating the plasma by the rf). This places some restrictions on the rf drive. However, these are usually not severe ones in the case of fast-wave current drive since, even if the rf is strong, there are so few resonant particles that the heating rate is small.

The effect of the rf is to induce an electron flux

$$\mathbf{S} = -\mathsf{D}\cdot\frac{\partial f}{\partial \mathbf{p}} \tag{8}$$

in momentum space, where D is the quasilinear diffusion tensor.[25] In the Chapman–Enskog ordering this is taken to be of first order. The zeroth-order electron distribution is given by setting the collision term $C(f,f) + C(f,f_i)$ equal to zero. The general solution is a Maxwellian equation (5) with n and T arbitrary functions of time and position. For simplicity we ignore the spatial variations. Since the rf drive is particle conserving we may take n to be a constant. A drifting Maxwellian does not solve the zeroth-order system since the ions are taken to be stationary.

The first-order equation is given by substituting $f = f_m(1+\psi)$ with ψ ordered small to give

$$C(f_m\,\psi) = \frac{\partial}{\partial \mathbf{p}}\cdot\mathbf{S} + \frac{(\mathscr{E} - \langle\mathscr{E}\rangle)}{T} f_m \frac{d}{dt}\log T, \tag{9}$$

where

$$C(f) = C(f,f_m) + C(f_m,f) + C(f,f_i) \tag{10}$$

is the linearized collision operator, and $\langle\mathscr{E}\rangle$ is the mean energy per particle[19]

$$\langle\mathscr{E}\rangle = \frac{1}{n}\int \mathscr{E} f_m(p)\, d^3\mathbf{p} = mc^2\left(\frac{K_1(\Theta^{-1})}{K_2(\Theta^{-1})} + 3\Theta\right).$$

The last term in Eq. (9) represents the heating of the Maxwellian. The equation for the time evolution of T is given by the solubility condition for Eq. (9) which is obtained by taking its energy moment. Since the linearized collision operator is energy conserving (recall that we take the limit $m_i \to \infty$, so that there is no energy exchange between electron and ions), this gives

$$n\frac{d\langle\mathscr{E}\rangle}{dt} = P,$$

where P is the power dissipated per unit volume by the rf,

$$P = \int \mathbf{S}\cdot\mathbf{v}\, d^3\mathbf{p}. \tag{11}$$

[There is another solubility condition given by the density moment of Eq. (9). This is automatically satisfied by taking $dn/dt = 0$.] The solution to Eq. (9) is made unique by demanding that $f_m\psi$ have zero density and energy.

In the nonrelativistic limit, Eq. (9) is the equation solved numerically by Cordey *et al.*[10] However, since we are usually interested primarily in the current density generated by the rf

$$J = q\int v_\parallel f_m\, \psi\, d^3\mathbf{p}, \tag{12}$$

and the efficiency of current generation defined by the ratio J/P, we usually do not need to know the full solution for ψ.

The method for determining the current without solving for ψ was given by Hirshman[12] and Taguchi[13] for neutral-beam-driven currents and was introduced into the study of rf-driven currents by Antonsen and Chu[3] and Taguchi.[11] The key is to define an "adjoint" problem

$$C(f_m\chi) = -qv_\parallel f_m. \tag{13}$$

Again $f_m\chi$ is required to have zero density and energy. This is the Spitzer–Härm problem for the perturbed electron distribution function caused by an electric field $\mathbf{E} = T\hat{\mathbf{p}}_\parallel$. Using the self-adjoint property of the linearized collision operator $\int \psi C(f_m\chi)\, d^3\mathbf{p} = \int \chi C(f_m\psi)\, d^3\mathbf{p}$, it is readily found that

$$J = \int \mathbf{S}\cdot\frac{\partial}{\partial \mathbf{p}}\chi(\mathbf{p})\, d^3\mathbf{p}\,. \tag{14}$$

In this equation χ plays the role of a Green's function for the rf-driven current. The ratio of Eqs. (14) and (11) gives the efficiency

$$\frac{J}{P} = \frac{\int \mathbf{S}\cdot(\partial/\partial\mathbf{p})\chi(\mathbf{p})\, d^3\mathbf{p}}{\int \mathbf{S}\cdot\mathbf{v}\, d^3\mathbf{p}}. \tag{15}$$

An important special case is when the rf excitation is localized. Then it is only necessary to know the position and direction of the excitation to determine the efficiency

$$\frac{J}{P} = \frac{\hat{\mathbf{S}}\cdot(\partial/\partial\mathbf{p})\chi(\mathbf{p})}{\hat{\mathbf{S}}\cdot\mathbf{v}}, \tag{16}$$

where all quantities are now evaluated at the position of the excitation. If we compare this method with the Langevin method of Fisch,[2] we see that χ is the mean integrated current caused by a group of electrons released at $\mathbf{p}$ at $t = 0$,

$$\chi(\mathbf{p}) = q\int_0^\infty \langle v_\parallel\rangle\, dt.$$

The power of these results is that the calculation of J/P does not require a solution of Eq. (9) for the rf distribution ψ. On the other hand, Eq. (13) must be solved for the Spitzer–Härm function χ. This reduces to the solution of a 1-D integro-differential equation which may be accurately computed. Furthermore, in the nonrelativistic limit, it has been tabulated.[14] This method also substantially reduces the parameter space to be investigated numerically. The solution of Eq. (13) depends on two parameters only, Z and Θ. In contrast, the solution of Eq. (9) depends on various parameters specifying the nature of the rf excitation (for instance, the direction of S, the minimum and maximum phase velocities, etc.) as well as Z and Θ.

In order to determine the rf current-drive efficiency using Eq. (15) or (16), we must solve the Spitzer–Härm problem, Eq. (13). The solution χ consists of only the first Legendre harmonic, so we substitute $\chi(\mathbf{p}) = \mu\chi_1(p)$ into Eq. (13) giving

$$\frac{1}{p^2}\frac{\partial}{\partial p}p^2 A(p)\frac{\partial \chi_1}{\partial p} - \frac{vA(p)}{T}\frac{\partial \chi_1}{\partial p} - \frac{2B(p) + \Gamma Z/v}{p^2}\chi_1 + I(\chi_1) + qv = 0, \tag{17}$$

where $A(p)$ and $B(p)$ are given by Eq. (4), the electron–ion term is given by Eq. (3), and $I(\chi_1)$ is given by Eq. (7). The fact that the solution of Eq. (13) consists of only a single Legendre component constitutes an additional advantage to this method of determining current-drive efficiencies. The solution of the full rf problem given in Eq. (9) consists, in general, of many Legendre components. Often some truncation is performed in computing these numerically.

Equation (17) may be solved by approximate analytic methods either by expressing χ as a sum of Sonine polynomials[24,26] or by formulating the equation as a variational problem.[12] These methods have the disadvantage that they generally fail to reproduce the correct asymptotic (large $\mathbf{p}$) form for χ. This failing does not affect the calculation of the electrical conductivity significantly since in that case χ is integrated with a weighting factor proportional to f_m. However, it rules out such methods for the study of rf current drive, since the efficiency may depend on the local value of χ.

This leaves us either with asymptotic methods, which we apply in Sec. VI, or with numerical methods. Numerical solutions to Eq. (17) have been given in the nonrelativistic case in Refs. 14 and 27. Here we use a simpler method which avoids most of the problems with the application of boundary conditions. We cast Eq. (17) as a 1-D diffusion equation by setting the left-hand side to $\partial\chi_1/\partial t$ and solve this diffusion equation until a steady state is reached. (The initial conditions may be chosen arbitrarily.) The integration is carried out in the domain $0 < p < p_{\max}$ and the boundary conditions $\chi_1(0) = 0$ and $\chi_1''(p_{\max}) = 0$ are imposed. The diffusion equation describes the physical problem of the evolution of the perturbed electron distribution in the presence of an electric field and is therefore guaranteed to give the correct solution of Eq. (13) without having to worry about spurious solutions which diverge at $p = 0$ or $p = \infty$. Since this is a 1-D diffusion equation, it may be solved by treating the differential operator fully implicitly (the time step may be taken to be large). The integral operator $I(\chi_1)$ is treated explicitly and is recomputed after every time step. In the calculations shown here, the momentum step size was taken to be $p_t/50$, the time step was taken to be $1000/\nu_t$, and the process converged (i.e., the relative change in χ_1 per step was less than 1 part in 10^{10}) after about 50 steps.

In the following sections we will also need the function $G(p) = \chi_1(p)/p$ so that $\chi(\mathbf{p}) = p_\parallel G(p)$. In terms of G, the gradient of χ is

$$\frac{\partial}{\partial \mathbf{p}}\chi(\mathbf{p}) = G(p)\hat{\mathbf{p}}_\parallel + p_\parallel G_p(p)\hat{\mathbf{p}},$$

where $G_p(p) = dG(p)/dp$.

V. NUMERICAL RESULTS

The solution for χ is given as a contour plot in Fig. 1 for $Z = 1$ and $\Theta = 0$ and 0.01. From these and a knowledge of S, the direction of the rf-induced current can be determined. In the nonrelativistic case, Fig. 1(a), χ rises ever more steeply as p is increased giving the favorable p^2 scaling for the current-drive efficiency.[9] On the other hand in a hot plasma, Fig. 1(b), the slope reaches a constant (the contour levels are

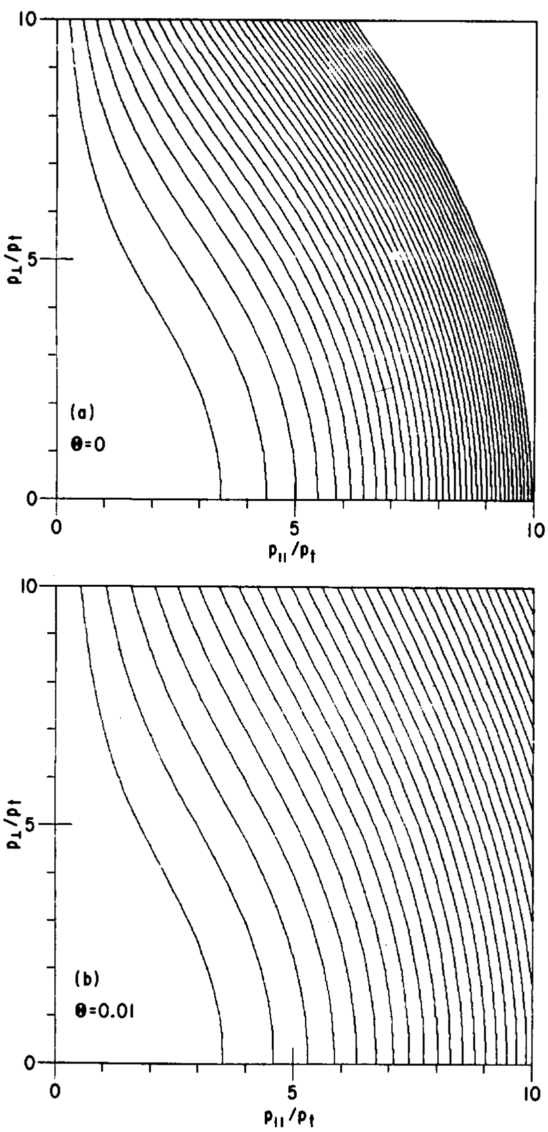

FIG. 1. Contour plots of $\chi(\mathbf{p})$ for $Z = 1$ and (a) $\Theta = 0$ and (b) $\Theta = 0.01$. The contour levels are evenly spaced with increments of 50 $qp_t/m\nu_t$. The higher levels are on the right [i.e., $\partial\chi(\mathbf{p})/\partial p_\parallel > 0$].

equally spaced) leading to a limit in the efficiency of the current drive.[2]

Figure 1 also shows that the contours become vertical for $p_{\parallel}$ small. This indicates that pushing electrons with small $p_{\parallel}$ in the perpendicular direction (as with cyclotron-damped waves) is not effective in generating current. Pushing electrons in the parallel direction is effective, especially for small $p_{\parallel}$, since the denominator in Eq. (16) can be small. In general, when the contours of constant energy (p = const) cross contours of constant χ, the efficiency can be very large.

Turning now to the numerical results for the efficiency, we begin with the case of localized spectrum, Eq. (16). Although this situation may not be realized in practice, it is important because it can help us to determine the best current drive schemes by showing where in velocity space to induce the flux. There are two major classes of fast waves that have been considered for current drive, namely Landau-damped waves (e.g., lower-hybrid waves) for which $\hat{\mathbf{S}} = \hat{\mathbf{p}}_{\parallel}$ and cyclotron-damped waves for which $\hat{\mathbf{S}} = \hat{\mathbf{p}}_{\perp}$. Taking the limit $p_{\perp} \to 0$, we have

$$J/P = [G(p) + pG_p(p)]/v, \tag{18a}$$

$$J/P = pG_p(p)/v, \tag{18b}$$

for Landau-damped and cyclotron-damped waves, respectively. The efficiencies are plotted in Fig. 2 for $Z = 1$ and $\Theta = 0$, 0.02, 0.05, 0.1, and 0.2 (these correspond to $T = 0$, 10, 26, 51 and 102 keV). The curves for $\Theta = 0$ in the two cases are given analytically from Eq. (24); they agree with the results of Ref. 2. This confirms the earlier analysis and shows that it is exact in the limit of $T \ll mc^2$ and $p^2 \gg mT$.

We next consider current drive by a narrow spectrum of Landau-damped waves. In this case all particles satisfying the Landau resonance condition $\omega - k_{\parallel} v_{\parallel} = 0$ interact with the wave and the quasilinear diffusion tensor is

$$\mathsf{D} \propto \delta(\omega - k_{\parallel} v_{\parallel})\, \hat{\mathbf{p}}_{\parallel} \hat{\mathbf{p}}_{\parallel} \propto (\gamma \delta(p_{\parallel} - m v_p \gamma)\, \hat{\mathbf{p}}_{\parallel} \hat{\mathbf{p}}_{\parallel},$$

where $v_p = \omega / k_{\parallel}$ is the parallel wave phase velocity. Assuming that the electron distribution is weakly perturbed, we can take $f = f_m$ in Eq. (8) to give

$$\mathbf{S} \propto \gamma v_{\parallel} f_m \delta(p_{\parallel} - m v_p \gamma) \hat{\mathbf{p}}_{\parallel}.$$

When we substitute this expression into Eq. (15), we obtain

$$\frac{J}{P} = \frac{1}{v_p} \frac{\int_{p_0}^{\infty} \{G(p) + [(m\gamma v_p)^2/p]\, G_p(p)\}\, \gamma f_m(p) p\, dp}{\int_{p_0}^{\infty} \gamma f_m(p) p\, dp}, \tag{19}$$

where $p_0 = m v_p/(1 - v_p^2/c^2)^{1/2}$ is the minimum resonant momentum. This efficiency is plotted in Fig. 3. In the limit $v_p \to 0$, the efficiency becomes large. This demonstrates that current may be efficiently driven by low phase velocity waves as was proposed by Wort.[4]

A similar analysis can be performed for a narrow spectrum of cyclotron-damped waves. The situation is more complicated here because the electron-cyclotron frequency depends relativistically on the momentum[28] and because relativistic effects distort the diffusion paths.[8] In addition, the variation of the diffusion coefficient with $p_{\perp}$ depends on the harmonic number. This means that the efficiency depends on three wave parameters $\omega/k_{\parallel}$, $\Omega/k_{\parallel}$ (Ω is the rest-mass cyclotron frequency), and the harmonic number. We therefore will treat this case only in the nonrelativistic limit.

In the nonrelativistic limit ($\Theta \to 0$, $p/mc \to 0$), the efficiencies for both kinds of waves have been calculated by Cor-

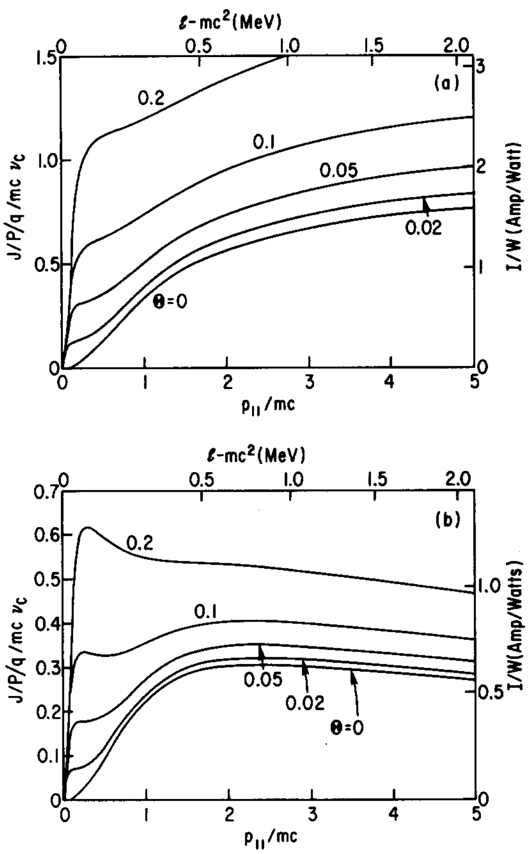

FIG. 2. Efficiencies for localized excitation for (a) Landau-damped waves (parallel diffusion) Eq. (18a), and (b) cyclotron-damped waves (perpendicular diffusion) Eq. (18b). The different curves show the efficiencies for various values of the temperature Θ as indicated. In all cases $Z = 1$. The top scale gives the kinetic energy of the electrons. The right scale gives the efficiency for a plasma with $n = 10^{20}\ \mathrm{m}^{-3}$, $\log \Lambda = 15$, and $R = 1$ m.

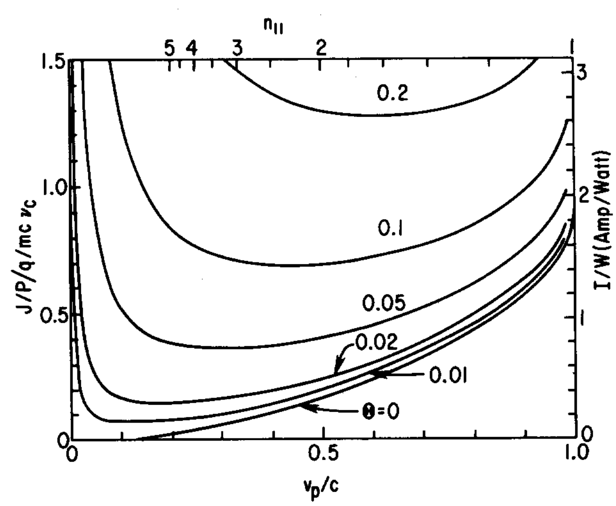

FIG. 3. Efficiencies for narrow Landau spectrum equation (19) as a function of the phase velocity v_p. The curves correspond to the various values of Θ. In all cases $Z = 1$. The top scale gives the parallel index of refraction $n_{\parallel} = c/v_p$. The right scale gives the efficiency for the same conditions as in Fig. 2.

dey *et al.*[10] and Taguchi.[11] They considered a narrow spectrum of Landau-damped waves for which the efficiency is given by the nonrelativistic limit of Eq. (19) and a narrow spectrum of cyclotron-damped waves for which the diffusion coefficient is

$$\mathsf{D} \propto v_\perp^{2(l-1)} \delta(v_\parallel - v_p) \hat{\mathbf{p}}_\perp \hat{\mathbf{p}}_\perp ,$$

where l is the harmonic number and $v_p = (\omega - l\Omega)/k_\parallel$. Assuming that $f = f_m$ in Eq. (8), the efficiency for cyclotron-damped waves is

$$\frac{J}{P} = m^2 v_p \frac{\int_{p_0}^{\infty} (p^2 - p_0^2)^l f_m(p) G_p(p)\, dp}{\int_{p_0}^{\infty} (p^2 - p_0^2)^l f_m(p) p\, dp}, \tag{20}$$

where $p_0 = m v_p$. (Here we consider only the fundamental cyclotron resonance $l = 1$). In Fig. 4, we plot these efficiencies normalized to the thermal quantities together with the asymptotic results, Eqs. (31) and (32a). For $m v_p \gg p_t$, the efficiencies scale as v_p^2 as predicted by Fisch and Boozer.[9] The $1/v_p$ scaling seen in the Landau-damping case for $m v_p \ll p_t$ is obtained by taking the limit $v_p \to 0$ in Eq. (19) to give

$$\frac{J}{P} = \frac{1}{v_p} \frac{\int D(p_\perp) f_m(p_\perp) G(p_\perp) p_\perp\, dp_\perp}{\int D(p_\perp) f_m(p_\perp) p_\perp\, dp_\perp}. \tag{21}$$

Here we have included an arbitrary dependence of D on $p_\perp$. In Ref. 6, three different types of low-phase-velocity current drive were identified, namely by Landau damping, transit-time magnetic pumping, and Alfvén waves. These methods differ in the forms for $D(p_\perp)$,

$$D(p_\perp) = \begin{cases} 1 & \text{(Landau damping)}, \\ (p_\perp / p_t)^4 & \text{(transit-time magnetic pumping)}, \\ [2 - (p_\perp / p_t)^2]^2 & \text{(Alfvén waves)}. \end{cases}$$

The case plotted in Fig. 4 is the first one (Landau damping). Evaluating the integrals in these cases gives

$$\frac{J}{P} = \begin{Bmatrix} C_L \\ C_M \\ C_A \end{Bmatrix} \frac{q}{m v_p \nu_t},$$

where the coefficients C are given in Table I. The coefficients

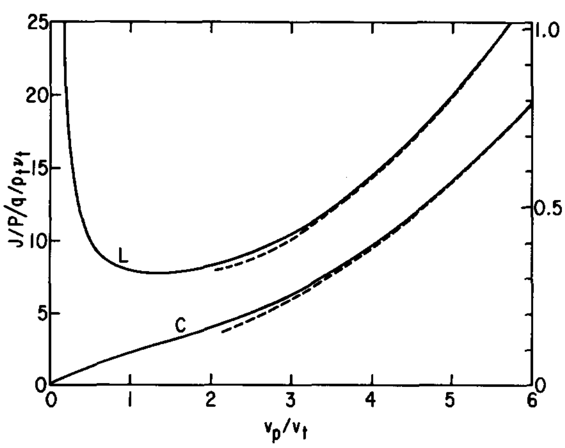

FIG. 4. Efficiencies for narrow spectra of Landau-damped (L) waves and cyclotron-damped (C) waves ($l = 1$) for the nonrelativistic case $\Theta \to 0$ and $Z = 1$. Also shown as dashed lines are the asymptotic results, Eqs. (31) and (32a). The right scale gives the efficiency for a plasma with $n = 10^{20}\ \mathrm{m}^{-3}$, $T = 10$ keV, $\log \Lambda = 15$, and $R = 1$ m.

TABLE I. The coefficients for the efficiency for the three types of current drive by low-frequency waves.

Z	C_L	C_M	C_A
1	3.76	8.49	8.09
2	1.88	5.17	5.07
5	0.75	2.55	2.60
10	0.38	1.42	1.48

for $Z = 1$ should be compared with the (less exact) results of Ref. 6 obtained by a numerical solution of the 2-D Fokker–Planck equation where the constants of proportionality are given as 4, 6.5, and 6.5, respectively. The coefficient C_L has been determined analytically by Cordey *et al.*[10] to be

$$C_L = \tfrac{3}{2}\sqrt{2\pi}/Z .$$

The dependence on Z indicates that the current is unaffected by electron–electron collisions. This result may be derived by taking the momentum moment of Eq. (13). The electron–electron collision term then drops out (from momentum conservation) and the electron–ion term is proportional to the numerator in Eq. (21).

The last numerical example is one in which we relax the condition that $f = f_m$ in Eq. (8). This allows us to find the flux S that develops in the presence of high rf power. In order to determine S, we numerically solve the 2-D Fokker–Planck equation

$$\frac{\partial f}{\partial t} = C_{\mathrm{num}}(f) + \frac{\partial}{\partial \mathbf{p}} \cdot \mathsf{D} \cdot \frac{\partial f}{\partial \mathbf{p}}, \tag{22}$$

until a steady state is reached. The numerical collision operator is defined as

$$C_{\mathrm{num}}(f) = C(f, f_m) + C(f_m, \mu f_1) + C(f, f_i),$$

where μf_1 is the first Legendre harmonic of f. The electron–ion term $C(f, f_i)$ is calculated using Eq. (3).

In order to justify our handling of the electron–electron collisions, let us consider the linearized electron–electron operator $C(f, f_m) + C(f_m, f)$. The first term describes the relaxation of the tail particles on the bulk and the second describes the concomitant heating of the bulk. The linearization is justified even with strong rf, as long as $f(\mathbf{p}) \approx f_m(p)$ for $\mathscr{E} \sim T$. The linearized electron–electron operator conserves energy, and if this were used in Eq. (22), there would be nothing to balance the power input by the rf (there is no transfer of energy to the ions in the limit $m_i \to \infty$), and so a steady-state solution to Eq. (22) would not be possible. In Eq. (9) this is handled by allowing the temperature of the Maxwellian to increase slowly with time. In the numerical code we adopt a different approach, namely to modify the collision operator so that energy is lost in an innocuous way. The term responsible for the bulk heating is the second term $C(f_m, f)$. Let us write f in this term as a Legendre harmonic expansion

$$f(p) = \sum_{k=0}^{\infty} P_k(\mu) f_k(p).$$

Of the terms in this series, only one, the $k = 0$ term, contributes to the bulk heating. (The energy moments of the other terms vanish.) Thus in order to lose energy we drop the term $C(f_m, f_0)$. Of the remaining terms in the series, only the first, the $m = 1$ term, is of importance—it is responsible for ensuring conservation of momentum. Thus we retain only this term and approximate $C(f_m, f)$ by $C(f_m, \mu f_1)$ to give the collision operator $C_{\rm num}$.

The collision operator $C_{\rm num}$ has the following properties: energy is not conserved (thus allowing a steady state to be reached); momentum is conserved; and quantities such as the Spitzer–Härm conductivity, which are given solely in terms of the first Legendre harmonic, are correctly given. To justify the way in which energy conservation is handled, we may check that the results are insensitive to the details of how this is done. One such check is given below where we compare the efficiency given by the numerical solution of Eq. (22), in which energy is lost, and that given by Eq. (15), where energy is conserved.

We assume that the rf diffusion term in Eq. (22) is caused by high-power lower-hybrid waves whose phase velocities lie between v_1 and v_2. Thus we take

$$\mathsf{D} = \begin{cases} D(\mathbf{p})\,\hat{\mathbf{p}}_\parallel \hat{\mathbf{p}}_\parallel, & \text{for } v_1 < p_\parallel/(m\gamma) < v_2, \\ 0, & \text{otherwise} \end{cases}$$

where $D(\mathbf{p})$ is chosen to be large enough to plateau f. [Here we choose $D(\mathbf{p}) = 10\nu_t\, p_t^2/(1 + p/p_t)$.]

Figure 5 shows the steady-state solution of Eq. (22) for $Z = 1$, $\Theta = 0.01$ ($T \approx 5$ keV), $v_1 = 0.4c = 4p_t/m$, and $v_2 = 0.7c = 7p_t/m$ (the parallel refractive index satisfies $1.43 < n_\parallel < 2.5$). Using the numerical solution for $f(\mathbf{p})$ and $\mathbf{S}(\mathbf{p})$, and the definitions (11) and (12), we obtain $J = 3.74 \times 10^{-4} qnc$, $P = 1.28 \times 10^{-3} mnc^2\nu_c$, and $J/P = 0.293 q/mc\nu_c$.

This is to be compared with the result given by Eq. (14) with the numerically determined flux $\mathbf{S}(\mathbf{p})$, namely $J = 3.77 \times 10^{-4} qnc$ and $J/P = 0.296 q/mc\nu_c$. (The figure for P remains unchanged since this depends on $\mathbf{S}$ alone.) These two sets of figures are within 1% of each other. The excellent agreement illustrates two points: the approximations made in the numerical collision operator, namely, the neglect of the heating term $C(f_m, f_0)$, has little effect on the results for the current-drive efficiencies (discretization effects are probably a greater source of error in these results); and the analytic result, Eq. (15) can be used to obtain reliable figures for the efficiency for cases of strong rf. What is needed in the latter instance is an estimate for the rf flux $\mathbf{S}$. This may be found from a numerical solution of 2-D Fokker–Planck equation (as here) or from an approximate analytical solution. Some saving may be possible using this method in conjunction with a numerical code; since $\mathbf{S}$ reaches a steady state sooner than f, it may not be necessary to run the code so long in order to obtain a reasonably accurate estimate for the efficiency.

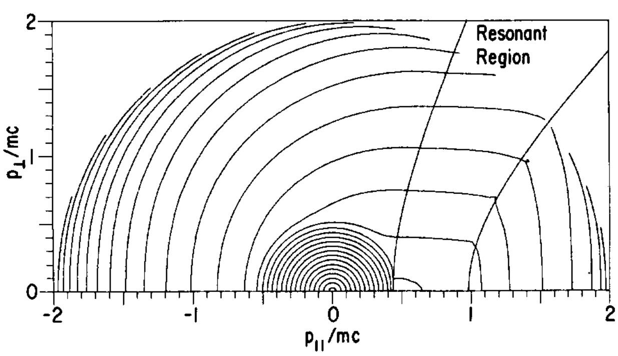

FIG. 5. Contour plot of the steady-state distribution f obtained by numerically integrating Eq. (22). Here $Z = 1$, $\Theta = 0.01$, $v_1 = 0.4c$, $v_2 = 0.7c$. The resonant region is indicated. The contour levels are chosen so that for a Maxwellian they would be equally spaced with $\Delta p = mc/30$.

VI. ASYMPTOTIC ANALYSIS

We have seen that the efficiency of current drive may be expressed in terms of the solution of the Spitzer–Härm problem Eq. (17). This equation may be approximately solved in the limit $p \gg p_t$. We will begin with the relativistic case and later treat the nonrelativistic limit. We start by writing down the normalized form of Eq. (17) in the limit $p \gg p_t$. We chose normalizations based upon q, m, c, and ν_c. Thus momenta are normalized to mc, χ_1 to qc/ν_c, J/P to $q/mc\nu_c$, etc. We use the same symbols to represent the normalized and unnormalized quantities. The coefficients $A(p)$ and $B(p)$ are given by Eqs. (6), suitably normalized. The integral term may be evaluated by replacing the indefinite limits in Eq. (7) by ∞, giving when normalized

$$I(\chi_1) \approx \Theta^{3/2}[H_a(\Theta,Z)/vp + H_b(\Theta,Z)/v^2],$$

where H_a and H_b are definite integrals of χ_1 (and thus independent of momentum) which must be determined numerically. In the limit $\Theta \to 0$, both H_a and H_b are finite. In normalized form with $p^2 \gg \Theta$, Eq. (17) reads

$$\begin{aligned}\frac{\Theta V_t^2}{v^3}&\left[\frac{\partial^2\chi_1}{\partial p^2} - \left(\frac{v}{\Theta} + \frac{3}{v\gamma^3} - \frac{2}{p}\right)\frac{\partial\chi_1}{\partial p}\right] \\ &- \frac{1}{vp^2}\left(1 + Z - \frac{\Theta V_t^2}{v^2}\right)\chi_1 \\ &+ \Theta^{3/2}\left(\frac{H_a}{vp} + \frac{H_b}{v^2}\right) + v = 0. \end{aligned} \tag{23}$$

The error in this equation is exponentially small.

We now make a subsidiary expansion in small Θ. In the limit $\Theta \to 0$, several terms in Eq. (23) drop out, leaving

$$-\frac{1}{v^2}\frac{\partial\chi_1}{\partial p} - \frac{1+Z}{vp^2}\chi_1 + v = 0.$$

This may be solved with the boundary condition $\chi_1(p = 0) = 0$ to give

$$\chi_1 = \left(\frac{\gamma+1}{\gamma-1}\right)^{(1+Z)/2}\int_0^p \left(\frac{\gamma'-1}{\gamma'+1}\right)^{(1+Z)/2} v'^3\, dp'. \tag{24}$$

This is the result derived using the Langevin equations by Fisch.[2] For integer values of Z, the integral may be expressed in terms of elementary functions. In particular for $Z = 1$ we have

$$\chi_1 = [(\gamma+1)/(\gamma-1)](vp - 2\log\gamma).$$

Of particular interest is the efficiency for large p since this gives a limit to the efficiency of current drive by fast waves. If we let $p \gg 1$, the integral may be approximately evaluated to give

$\chi_1 \to p - (1+Z)\log p.$

If we not take Θ to be finite, Eq. (23) cannot be easily solved. However, we may solve it in the limit $p \gg 1$. We achieve this by writing

$$\chi_1 \approx \alpha p + \beta \log p \tag{25}$$

in analogy to the situation with $\Theta = 0$. Substituting this form of χ_1 into Eq. (23) and balancing terms of equal order in p gives

$$\alpha = (1 + \Theta^{3/2} H_b) V_t^2 \tag{26a}$$

from the $O(p^0)$ terms and

$$\beta = -[(1 + Z - 3\Theta V_t^2)\alpha - \Theta^{3/2} H_a]/V_t^2 \tag{26b}$$

from the $O(p^{-1})$ terms. When the rf excitation is localized the current-drive efficiency is given by Eqs. (18) which, with χ_1 given by Eq. (25), read

$$J/P \approx \alpha + \beta/p, \tag{27a}$$

$$J/P \approx \beta(1 - \log p)/p, \tag{27b}$$

for current drive by Landau-damped and cyclotron-damped waves, respectively. [The factor of $1/v$ in Eqs. (18) is replaced by unity in the limit $p \to \infty$.] Equation (27a) (with p replaced by p_0) also applies for current drive by a narrow spectrum as given by Eq. (19). In the limit of $p \to \infty$, the efficiency of cyclotron-damped current drive vanishes, while for current drive by Landau-damped waves $J/P \to \alpha$. In order to determine this limiting efficiency, either Eq. (26a) may be evaluated using the numerically found value of $H_b(\Theta, Z)$ (see Table II) or else the equation may be expanded as a series in Θ to give for $p \to \infty$,

$$J/P \approx 1 + \tfrac{5}{2}\Theta + H_b(0,Z)\Theta^{3/2}. \tag{28}$$

Here $H_b(0,Z)$ is tabulated in Table III.

We now turn to the solution of Eq. (17) in the nonrelativistic limit $\Theta \to 0$. We shall still consider only the limit $p \gg p_t$. The limits here are nonuniform. Equation (23) was obtained by taking $p \gg p_t$ followed by $\Theta \to 0$. Here we will take the limits in the opposite order. To do this, it is convenient to renormalize Eq. (17) using q, m, p_t, and v_t as the system of units. In this case J/P is normalized to $q/p_t v_t$. Making this change of normalization and taking the limit $\Theta \to 0$ is equivalent to formally replacing Θ by unity and substituting $v = p$, $\gamma = 1$, and $V_t^2 = 1$ in Eq. (23) to give

$$\frac{1}{p^3}\left[\frac{\partial^2 \chi_1}{\partial p^2} - \left(p + \frac{1}{p}\right)\frac{\partial \chi_1}{\partial p}\right] - \frac{1}{p^3}\left(1 + Z - \frac{1}{p^2}\right)\chi_1 + \frac{H}{p^2} + p = 0, \tag{29}$$

TABLE II. Table of efficiencies J/P for Landau-damped waves in the limit $v_p \to c$. The efficiencies are normalized to q/mcv_c.

Θ	$Z=1$	$Z=2$	$Z=5$	$Z=10$
0.01	1.04	1.03	1.03	1.03
0.02	1.09	1.07	1.06	1.06
0.05	1.25	1.20	1.17	1.15
0.1	1.55	1.44	1.34	1.30
0.2	2.19	1.91	1.70	1.61

TABLE III. The coefficients $H_a(0,Z)$ and $H(Z)$.

Z	H_a	H
1	13.69	21.12
2	9.13	13.51
5	4.94	7.01
10	2.88	4.01

where $H(Z) = H_a(0,Z) + H_b(0,Z)$ (this is tabulated in Table III). For $p \gg 1$ (in this normalization this means $p \gg p_t$), we may develop an asymptotic expression for χ_1 as a series in powers of p. Balancing the terms in Eq. (29) from $O(p)$ (the

$$\chi_1 = \frac{p^4}{5+Z} + \frac{9p^2}{(5+Z)(3+Z)} + \frac{Hp}{2+Z} + \frac{9}{(5+Z)(3+Z)(1+Z)} + O(p^{-2}).$$

For localized excitation, Eq. (18) becomes

$$\frac{J}{P} = \frac{4p^2}{5+Z} + \frac{18}{(5+Z)(3+Z)} + \frac{Hp^{-1}}{2+Z} + O(p^{-4}), \tag{30a}$$

$$\frac{J}{P} = \frac{3p^2}{5+Z} + \frac{9}{(5+Z)(3+Z)} - \frac{9p^{-2}}{(5+Z)(3+Z)(1+Z)} + O(p^{-4}) \tag{30b}$$

for Landau-damped waves and cyclotron-damped waves, respectively. The leading order terms here (those proportional to p^2) are exactly those derived by Fisch and Boozer.[9]

In order to compute the efficiencies for current drive by a narrow spectrum of waves, it is necessary to carry out the integrations in Eqs. (19) and (20). The following asymptotic series is useful for this purpose:

$$\int_x^\infty \exp\left(-\frac{1}{2}y^2\right) y^{n+1}\,dy = \exp(-\tfrac{1}{2}x^2)[x^n + nx^{n-2} + n(n-2)x^{n-4} + \cdots].$$

For n even, the series terminates and is exact. The efficiency for current drive by a narrow spectrum of Landau-damped waves, Eq. (19), becomes

$$\frac{J}{P} = \frac{4v_p^2}{5+Z} + \frac{6(6+Z)}{(5+Z)(3+Z)} + \frac{Hv_p^{-1}}{2+Z} + O(v_p^{-2}). \tag{31}$$

For a narrow spectrum of cyclotron-damped waves, Eq. (20) gives

$$\frac{J}{P} = \frac{3v_p^2}{5+Z} + \frac{3(9+2Z)}{(5+Z)(3+Z)} + O(v_p^{-2}), \tag{32a}$$

$$\frac{J}{P} = \frac{3v_p^2}{5+Z} + \frac{9(4+Z)}{(5+Z)(3+Z)} + O(v_p^{-2}), \tag{32b}$$

for $l = 1$ and $l = 2$, respectively. The effect of the integrations is to change only the higher-order $O(v_p^0)$ corrections to the efficiencies. The leading-order terms are the same as for the localized excitation equations (30). Equations (31) and (32a) are plotted in Fig. 4. These closely approximate the exact results for $v_p > 2v_t$.

VII. HIGH-ENERGY LIMIT OF COLLISION OPERATOR

In the previous section we derived finite temperature corrections to the efficiency limit found in Ref. 2. However, the collision operator in the Landau form, Eqs. (1), was derived by assuming that the background electrons are only weakly relativistic or that $\Theta \ll 1$. We must check, therefore, that the finite Θ corrections to the Landau operator do not affect the formula for the efficiency limit Eq. (28).

The linearized collision operator Eq. (10) consists of three collision terms. Since in all practical cases the ions are nonrelativistic, the ion term $C(f,f_i)$ needs no correction. The term $C(f_m, f)$ contributes to the integral term $I(\chi_1)$ in Eq. (17). However, this resulted in a $O(\Theta^{3/2})$ contribution to efficiency limit Eq. (28), so that corrections to this term will be of still higher order.

Therefore we need consider only collisions off a Maxwellian electron background $C(f,f_m)$. Furthermore, if Θ is small and if $p \gg p_t$, we may take $v' \ll v$ in the full collision kernel equation (2) and approximate U by its Taylor expansion about $\mathbf{v}' = 0$. By retaining terms up to second order in $\mathbf{v}'$, we obtain

$$C(f,f_m) = \frac{\Gamma}{2}\frac{\partial}{\partial p_j}\left(U_{jk}^{(0)}\frac{\partial f}{\partial p_k} + \frac{\partial U_{jk}^{(0)}}{\partial v_k'}\frac{v_t^2}{T}f + \frac{1}{2}\frac{\partial^2 U_{jk}^{(0)}}{\partial v_m' \partial v_m'} v_t^2 \frac{\partial f}{\partial p_k}\right),$$

where summation over repeated indices is implied and the superscript (0) is used to indicate that U and its derivatives are evaluated at $\mathbf{v}' = 0$. Evaluating these coefficients gives

$$U_{jk}^{(0)} = (v^2\delta_{jk} - v_j v_k)/v^3,$$

$$\frac{\partial U_{jk}^{(0)}}{\partial v_k'} = \frac{2v_j}{v^3},$$

$$\frac{1}{2}\frac{\partial^2 U_{jk}^{(0)}}{\partial v_m' \partial v_m'} = \frac{2v_j v_k}{v^5} - (1-\beta^4)\frac{U_{jk}^{(0)}}{v^2}.$$

(This calculation was carried out using MACSYMA.[22]) If we compare these with the equivalent expressions using U from Eq. (1c), we find that only the term proportional to β^4 is new. The high-energy form of $C(f,f_m)$ is given by Eq. (4a) with $A(p)$ given by Eq. (6a), $F(p) = (v/T)A(p)$, and

$$B(p) = \Gamma\frac{1}{2v}\left[1 - \frac{v_t^2}{v^2}\left(1 - \frac{v^4}{c^4}\right)\right].$$

In other words, in the high-energy limit the electromagnetic correction only changes the pitch-angle scattering term. The new term has no effect on the asymptotic form for the efficiencies equations (27), because it is smaller by β^4 than another term in B which had no effect.

Connor and Hastie[21] also give an expression for collisions of high-energy particles off a fixed background. The corrections to Eq. (6) that they obtain differ from ours. This is possibly because the background distribution that they treat is only approximately Maxwellian.

VIII. CONCLUSIONS

We have considered the problem of current drive by fast waves in a relativistic plasma. Let us briefly review the approximations made. The major one is the reduction of the full collision operator to Landau form. We show in Sec. II that this holds if the background temperature is small $T \ll mc^2$. The corrections to the Landau operator in the high-energy limit are derived in Sec. VII and are shown to be small. The second important approximation is the linearization of the electron–electron collision operator. This is accurate provided the rf strongly affects only electrons on the tail of the distribution. The subsequent analysis leading to the formula for the current-drive efficiency equation (15) is exact. In order to apply this formula, it is necessary to determine the rf-induced flux S from Eq.(8) and the Spitzer–Härm function χ from Eq. (13).

We consider two methods for computing S: either by assuming that $f = f_m$ in Eq. (8) (corresponding to linear damping) or by solving the 2-D Fokker–Planck equation, Eq. (22), numerically. The latter method may be necessary in the case of high rf powers and wide spectra. Note that the efficiency can be accurately calculated even if the S is known only approximately since Eq. (15), being an integral operator on S, is insensitive to small errors in S. Often useful information can be extracted from Eq. (15) even with very limited information about S. If the rf spectrum is known, we can make some estimates (based on either numerical or approximate analytical solutions to the Fokker–Planck equation) of where in momentum space the flux is largest. We can then use Eq. (16) to give the efficiency.

The Spitzer–Härm function χ can be determined by solving Eq. (17) numerically as in Sec. V. Since this equation is just a 1-D equation, there is little difficulty in obtaining arbitrarily accurate results in this way. This method can be regarded as exact. Alternatively, we found asymptotic forms for χ in Sec. VI. From this we can write down analytical expressions for the current-drive efficiency in various cases as given in Eqs. (27), (28), (30)–(32).

The primary application of this work is, of course, to maintain a steady-state toroidal current in a tokamak reactor. The viability of this scheme depends upon the amount of circulating power that is required. Thus, an accurate calculation of the current-drive efficiency, as well as an assessment of the best possible efficiency, are of crucial importance.

When applying these results to the study of steady-state current drive in a tokamak, it is useful to convert the efficiency J/P to I/W where $I = AJ$ is the total current, $W = 2\pi RAP$ is the total rf power, A is effective poloidal cross-sectional area, and R is the tokamak major radius. This gives

$$\frac{I}{W} = \frac{1}{2\pi R}\frac{J}{P} = 2.08\,\frac{J/P}{q/mc\nu_c}\,\frac{10^{20}\,\mathrm{m}^{-3}}{n}\,\frac{1\,\mathrm{m}}{R}\,\frac{15}{\log\Lambda}\ \mathrm{A/W}$$

$$= 40.7\times 10^{-3}\,\frac{J/P}{q/p_t\nu_t}\,\frac{10^{20}\,\mathrm{m}^{-3}}{n} \times \frac{T}{10\,\mathrm{keV}}\,\frac{1\,\mathrm{m}}{R}\,\frac{15}{\log\Lambda}\ \mathrm{A/W}.$$

The last two equalities give the conversion from the normalized efficiencies given in the figures and in Sec. VI to practical units. Figures 2–4 contain scales in these units.

The present work calculates the efficiency that can be expected from an arbitrary wave-induced flux. It is possible, therefore, to come to some very general conclusions about the best possible efficiency that can be obtained by driving currents with different waves. In particular, there is a limit, given by Eq. (28), to the efficiency of current drive with fast waves, such as lower-hybrid waves, that interact through a Landau resonance with relativistic electrons. These waves are, perhaps, the most likely candidate for current drive in a reactor.

The present calculations also apply to other types of current drive, for example, relativistic electron beams.[29] Here the efficiencies will be similar to those of Landau-damped waves. Care must be taken, however, in interpreting experiments on relativistic electron beams because the assumption of a steady state is generally inapplicable.

The equations developed here apply to other forms of rf current drive. Some of these may be very efficient, more so than lower-hybrid wave-induced fluxes. For example, if low-phase-velocity waves interact through a cyclotron resonance with fast electrons, the rf flux may be nearly parallel to the constant energy contours, at the same time that the collisionality of the resonant electrons is small. This gives very high efficiency, but, in practice, these waves are much more difficult to generate than are lower-hybrid waves.

Settling the question of the highest attainable current-drive efficiency with fast waves we hope should enable tokamak reactor designers to assess the practicality of using waves to drive steady-state currents. There may, of course, be other effects that present difficulties, such as the accessibility of the waves or nonlinear effects. On the other hand, there may be effects, such as the bootstrap current, which could be helpful.

Finally, we hope that the form that we derived here for the relativistic collision operator, which enabled us to solve for the relativistic Spitzer–Härm function, will be of use in other numerical problems dealing with collisions in hot plasmas.

ACKNOWLEDGMENT

This work was supported by the United States Department of Energy under Contract No. DE-AC02-76-CHO-3073.

[1] N. J. Fisch, Phys. Rev. Lett. **41**, 873 (1978).
[2] N. J. Fisch, Phys. Rev. A **24**, 3245 (1981).
[3] T. M. Antonsen and K. R. Chu, Phys. Fluids **25**, 1295 (1982).
[4] D. J. H. Wort, Plasma Phys. **13**, 258 (1971).
[5] C. F. F. Karney and N. J. Fisch, Phys. Fluids **22**, 1817 (1979).
[6] N. J. Fisch and C. F. F. Karney, Phys. Fluids **24**, 27 (1981).
[7] R. W. Harvey, K. D. Marx, and M. G. McCoy, Nucl. Fusion **21**, 153 (1981).
[8] C. F. F. Karney and N. J. Fisch, Nucl. Fusion **21**, 1549 (1981).
[9] N. J. Fisch and A. H. Boozer, Phys. Rev. Lett. **45**, 720 (1980).
[10] J. G. Cordey, T. Edlington, and D. F. H. Start, Plasma Phys. **24**, 73 (1982).
[11] M. Taguchi, J. Phys. Soc. Jpn. **52**, 2035 (1983).
[12] S. P. Hirshman, Phys. Fluids **23**, 1238 (1980).
[13] M. Taguchi, J. Phys. Soc. Jpn. **51**, 1975 (1982).
[14] L. Spitzer and R Härm, Phys. Rev. **89**, 977 (1953).
[15] K. Hizanidis and A. Bers, Phys. Fluids **27**, 2669 (1984).
[16] S. T. Beliaev and G. I. Budker, Sov. Phys. Dokl. **1**, 218 (1956).
[17] L. D. Landau, Phys. Z. Sowjetunion **10**, 154 (1936).
[18] M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. **107**, 1 (1957).
[19] S. R. de Groot, W. A. van Leeuwen, and Ch. G.van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980).
[20] D. Mosher, Phys. Fluids **18**, 846 (1975).
[21] J. W. Connor and R. J. Hastie, Nucl. Fusion **15**, 415 (1975).
[22] The Mathlab Group, MACSYMA *Reference Manual*, Version 10, Laboratory for Computer Science, Massachusetts Institute of Technology, 1983.
[23] M. G. McCoy, A. A. Mirin, and J. Kileen, Computer Phys. Comm. **24**, 37 (1981).
[24] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-uniform Gases*, 3rd edition (Cambridge U.P., Cambridge, 1970).
[25] C. F. Kennel and F. Engelmann, Phys. Fluids **9**, 2377 (1966).
[26] S. I. Braginskii, in *Reviews and Plasma Physics* (Consultants Bureau, New York, 1965), Vol. 1, p. 205.
[27] J. G. Cordey, E. M. Jones, D. F. H. Start, A. R. Curtis, and I. P. Jones, Nucl. Fusion **19**, 249 (1979).
[28] R. A. Cairns, J. Owen, and C. N. Lashmore-Davies, Phys. Fluids **26**, 3475 (1983).
[29] V. L. Bailey, J. M. Creedon, B. M. Ecker, and H. I. Helava, J. Appl. Phys. **54**, 1656 (1983).