Transport in driven plasmas

N. J. Fisch

Plasma Physics Laboratory, Princeton University, P. O. Box 451, Princeton, New Jersey 08544

A plasma in contact with an external source of power, especially a source that interacts specifically with high-velocity electrons, exhibits transport properties, such as conductivity, different from those of an isolated plasma near thermal equilibrium. This is true even when the bulk of the particles in the driven plasma is near thermal equilibrium. To describe the driven plasma, we derive an adjoint equation to the inhomogeneous, linearized, dynamic Boltzmann equation. The Green's functions for a variety of plasma responses can then be generated. It is possible to modify the Chapman–Enskog [*Mathematical Theory of Nonuniform Gases*, 3rd ed., (Cambridge U.P., Cambridge, MA, 1970)] expansion in order to incorporate the response functions derived here.

I. INTRODUCTION

The expansion developed by Chapman and Enskog[1] provides a generally useful description of transport in a plasma. This expansion, which applies when the mean free path of particles is small compared to other lengths of interest, has been the starting point for describing plasmas near thermal equilibrium. At the heart of this method is a reduced description of each plasma species in terms of macroscopic fluid variables, namely, the density, temperature, and drift. It is these properties of the plasma that are transported in configuration space in order to describe the evolving plasma.

The transport numbers, such as the conductivity or the coefficients governing particle and heat diffusion, are calculated by considering small perturbations from thermal equilibrium. It is well known, however, that certain phenomena are not described well by this method, namely, those processes involving the highly energetic particles, particularly electrons, that collide infrequently. For example, the plasma conductivity, the so-called Spitzer–Härm conductivity,[2] is strictly derived only in the limit of the dc electric field E vanishing. For E finite, important corrections occur because of the high-velocity electrons, which tend to run away. If many of these electrons run away, the Spitzer–Härm conductivity may no longer provide a useful description. The regime in which only a few electrons run away, where conductivity is a good description, is the small-mean-free-path limit, which arises for E small enough.

In the case of driven plasmas, however, the Spitzer–Härm conductivity may not be a useful approximation even for $E \to 0$. These plasmas are not near thermal equilibrium; they are actively heated by an external source of radiation or particles. One example of a driven plasma that is not described well by the Chapman–Enskog (C–E) expansion, and hence not by Spitzer–Härm conductivity, is a plasma in contact with external radiation that interacts specifically with the fast, collisionless electrons. This is a situation that commonly arises in rf-heated plasmas, particularly when the purpose is to generate toroidal current. The deviation from Spitzer–Härm conductivity is because of the larger number of fast electrons. These electrons have long mean free paths, and even though small in number compared to the bulk electrons, they may play a large part in carrying current.[3]

In this report, we develop a description suitable for driven plasma. We consider those plasmas for which it is valid to apply the fundamental Chapman–Enskog approximation, that collisions are large enough to create a nearly Maxwellian distribution of particle velocities. We use this approximation to linearize the Boltzmann collision operator, and as a result obtain the dynamic linearized, inhomogeneous Boltzmann equation. Plasma responses to arbitrary wave excitation affecting the fast electrons are then determined by solving an adjoint equation. The result is an improved characterization of the relatively collisionless phenomena associated with the external excitation of the plasma. A modification of the C–E expansion then allows a dual characterization of driven plasmas; a slowly evolving fluid description of the bulk properties and a dynamic response, linear in the excitation, for the driven, relatively collisionless processes.

In order to put the present work in perspective, we must place it both in the context of the problem of current drive and in the context of techniques used to describe transport in plasmas. First, let us briefly review some of the approaches taken to this problem. Response functions for the current generated per power dissipated, J/P_d, were central to our early understanding of both beam-driven currents,[4] and wave-driven currents with low[5] and high[6] phase velocity waves. The early work, however, was characterized by a crude one-dimensional model of velocity space, and was unable to predict, for example, current drive by electron-cyclotron (ECRH) waves. Fisch and Boozer[7] used slowing-down equations to formulate the problem in two dimensions in velocity space. This gave the first precise response function for the current-drive efficiency and predicted the ECRH current-drive effect.

An improved derivation of the 2-D Fisch–Boozer response function was provided by Antonsen and Chu.[8] Drawing upon neoclassical techniques, these authors extended the Fisch–Boozer efficiency calculation to toroidal geometry by writing an adjoint equation to the steady-state linearized Boltzmann equation. Hirshman[9] and Taguchi[10,11] employed a similar technique to solve similar equations. Using the adjoint equation, Antonsen and Yoshioka[12] have calculated rf-induced particle transport across field lines. The Antonsen

and Chu approach was also generalized to include a small dc electric field in order to evaluate the conductivity of a driven plasma.[3]

In order to interpret the recent high efficiency PLT ramp-up experiments,[13] response functions had to be identified and evaluated in the presence of a strong dc electric field. The most straightforward approach, adopted by Fisch and Karney,[14] generalized the Fisch–Boozer slowing-down equations to include the strong field. In addition to the current-drive efficiency, this work also identified the Green's function for runaway production. A comparison by Karney et al.[15] of this theory and the experiment showed very good agreement.

A better approach to the strong-field case is to build upon the Antonsen–Chu work.[8] The present work seeks to do this. Here, an adjoint equation is written for the dynamic, linearized Boltzmann equation, allowing for the presence of a strong dc electric field. Also, in addition to a response function for the current, other response functions are identified and evaluated. It is interesting to note that were this adjoint equation solved in the high-velocity limit by the method of characteristics, the characteristic equations would be the Fisch–Boozer slowing-down equations.

The second context in which to place the present work is techniques in transport theory,[16–19] which exploit, too, the self-adjoint property of the collision integral. However, these previous works treat transport associated with the bulk of the plasma, rather than with a select few tail, high-energy, rarely colliding electrons. Therefore, it is possible to neglect both the inertial term ($\partial/\partial t$) and the acceleration term ($E\,\partial/\partial v_\parallel$), and the effect of the electric field is taken into account only as a driving term ($E\,\partial f_m/\partial v_\parallel$) to the simplified Boltzmann equation. Hence, Spitzer conductivity is obtained by this formalism, but the dynamics of runaway electrons cannot be described. To treat these electrons, both the inertial term and the acceleration term must be retained.

Retaining these terms results in a considerably more complicated adjoint equation, and different simplifications are sought. Particular phenomena associated with rf-driven plasmas, including the rf-induced runaways, the rf-induced current, and the rf-induced radiation, can then be identified and described.

The paper is organized as follows. In Sec. II we discuss the validity and the utility of the linearization of the Boltzmann equation. In Sec. III we derive the adjoint equation and we write the general Green's function response. This is our principal result. In Sec. IV we give several examples of the utility of this general equation. A modified C–E expansion, that includes the derived response functions in addition to the normal fluid terms, is given in Sec. V. A summary of our findings is given in Sec. VI.

II. LINEARIZED BOLTZMANN EQUATION

The Boltzmann equation provides an excellent description of the evolving plasma, but it is too complicated to apply directly to most problems of interest. Certain approximations, however, allow us to extract information from this equation reliably and efficiently. The most helpful simplifications that are unique to the wave-driven plasma rely on the localization in velocity space of the wave-driven fluxes and the fact that these fluxes often involve only very fast electrons.

To be specific, let us consider the electron distribution function $f(r,v,t)$ driven by some wave-induced flux $\Gamma(r,v,t)$, evolving as described by the inhomogeneous Boltzmann equation

$$\frac{df}{dt} = C(f,f) + C(f,f_i) - \frac{\partial}{\partial v}\cdot\Gamma, \tag{1}$$

where $C(f,f)$ represents the self-collisions of electrons, $C(f,f_i)$ represents the scattering of electrons off ion distribution f_i, and the total time derivative, d/dt, is defined by

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v\cdot\frac{\partial}{\partial r} + F\cdot\frac{\partial}{\partial v}, \tag{2}$$

where F is the acceleration resulting from the background dc electric field E and magnetic field B, i.e.,

$$F = (e/m)(E + v\times B). \tag{3}$$

To be sure, we generally expect that the magnetic field will be so large that the conventional expansions leading, for example, to the drift kinetic equation are appropriate. To solve Eq. (1), we will make use of any of these expansions, which are useful whether or not the electrons are subject to wave-induced fluxes. However, here, we shall focus only on approximations that are specific to the problem at hand.

The main effect of the injection of intense rf-waves at high-phase velocities parallel to the magnetic field B is to distort the electron distribution in the resonant region of velocity space. Waves at frequency ω are injected with a range of parallel wavenumbers $k_\parallel$ such that $v_1 < \omega/k_\parallel < v_2$. The region of the electron velocity space resonant with the waves is $v_1 < v_\parallel < v_2$, where $v_\parallel$ is the electron velocity parallel to B. In the resonant region the waves act to diffuse electrons, the so-called "quasilinear diffusion," and if the injected power is intense enough, then a plateau of electrons forms. The plateau represents a distribution function $f(v_\parallel,v_\perp)$ of electrons such that $\partial f/\partial v_\parallel \simeq 0$ in the resonant region, with contours of f assuming levels in the parallel direction that are parametrized by $v_\perp$, the electron velocity perpendicular to the magnetic field.

For many problems of interest and, in particular, in the instance of the efficient generation of current, the wave-induced fluxes create a plateau of energetic (superthermal) electrons, while the bulk of the electron velocity distribution remains Maxwellian. This observation leads us to two helpful approximations. Firstly, most plateau electrons collide with bulk electrons rather than with each other, since there are so many more bulk electrons. Hence, the collision integral may be linearized by letting $f = f_m + \tilde{f}$ and approximating

$$C(f,f) \simeq C(f_m,\tilde{f}) + C(\tilde{f},f_m), \tag{4}$$

where we used $C(f_m,f_m) = 0$, since a Maxwellian distribution has no further to relax by collisions.

The second helpful approximation arises from the inference that Γ is usually quite localized in velocity space, even though Γ itself may depend on f. More precisely, while Γ

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may, in general, depend on $v_\parallel$ and $v_\perp$ in both direction and magnitude, much information may be gleaned from the resonance condition $\omega - k_\parallel v_\parallel - n\Omega_e = 0$, where n is an integer and Ω_e is the electron-cyclotron velocity. For $n = 0$, the Landau resonance, Γ must be in the parallel direction. This is a consequence of energy and momentum conservation between the wave and the particle. Similarly, for $n \neq 0$, we can infer the direction of Γ, which for most applications, such as heating or current drive with electron-cyclotron waves, will be nearly in the perpendicular direction. If the spectrum of waves is narrow in $k_\parallel$ space, then only electrons within a narrow band in $v_\parallel$ space will be resonant, such as those electrons with $v_\parallel \simeq v_{res}$. If this region is narrow and $v_{res} > v_{Te}$, where v_{Te} is the electron thermal velocity, then even if the electron distribution is distorted by the waves, the wave-induced flux Γ is of known direction and finite only for $v_\parallel \simeq v_{res}$, $v_\perp \simeq v_{Te}$, or in other words, $\mathbf{v} \simeq v_{res} \hat{i}_\parallel$. Only the magnitude of Γ, which for Landau damping depends on $\partial f/\partial v_\parallel$ in the resonant region, remains unknown.

Knowledge of all but the magnitude of Γ, together with the license to linearize Eq. (1), allows us to solve easily an important class of problems. Suppose that a certain power P_d is absorbed from the waves by the resonant electrons. As a result, there may appear in the plasma a wave-induced current J. The quantity J/P_d, which is a measure of the efficiency of the current drive, is independent of the magnitude of Γ, since both J and P_d are proportional to it. The ratio, however, does depend on both the direction and velocity locale of Γ, which could be inferred approximately, as outlined above, without knowledge of the magnitude of Γ. Other responses, such as the incremental bremmstrahlung radiation per power adsorbed, can be computed in the same manner. The quantity J/P_d, and other quantities that are similarly inferred, describe the response of the plasma to the absorption of wave energy by resonant electrons, and provide insight as to how to inject these waves in order to maximize a desired response.

The magnitude of Γ is also an important quantity, since all plasma responses will be proportional to it. This quantity, however, may sometimes be found by more crude means, e.g., by linear theory,[20] by a one-dimensional quasilinear calculation,[6] or by experimental observation of any one of the responses dependent on it.

III. ADJOINT EQUATION

The linearization of the Boltzmann equation enables us to find responses to arbitrary wave-induced fluxes. Instead of solving the Boltzmann equation, we solve an adjoint equation for the Green's function. In this section, we derive the adjoint equation. For simplicity, we will restrict our attention to the homogeneous case, $\partial /\partial \mathbf{r} = 0$. However, we will solve the problem in a finite velocity domain, which facilitates a numerical implementation.

There is a certain amount of arbitrariness in imposing a specific linearization. We will adopt a linearization that will enable us to generalize easily the C–E expansion. We utilize the fact that the heating of the bulk of f occurs on a time scale long compared to the time scales for other processes of interest, while collisions, predominant in the bulk of f, as-

sure for thermal velocities a nearly Maxwellian distribution. Thus, we linearize

$$f = f_m (1 + \phi) , \qquad (5)$$

with

$$f_m \equiv n(2\pi T/m)^{-3/2} \exp(-\epsilon/T) , \qquad (6)$$

where density n and temperature T may be slow functions of time (and would be of space, too, in a more general consideration of the problem), and we defined the kinetic energy $\epsilon \equiv mv^2/2$. For simplicity of presentation, we have made one further simplification in the C–E approach by linearizing about a stationary rather than the more general drifting Maxwellian distribution. To find a unique solution with this procedure necessitates two compatibility conditions; we choose n and T such that f_m (integrated over velocity space) contains the same number of electrons and the same amount of energy as does f. Accordingly, $\phi(\mathbf{v},\mathbf{r},t)$ is orthogonal both to f_m and to ϵf_m, namely

$$\int_V f_m \phi \, d^3v = \int_V \epsilon f_m \phi \, d^3v = 0 , \qquad (7)$$

where V is the velocity space domain under consideration, which necessarily is finite in a numerical implementation of the method.

Substituting Eq. (5) into Eq. (1), and specializing to the homogeneous case, $\partial /\partial \mathbf{r} = 0$, we obtain

$$L\phi \equiv \frac{\partial}{\partial t} f_m \phi + \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{v}} f_m \phi - C(\phi)$$

$$= -\frac{\partial}{\partial \mathbf{v}} \cdot \Gamma - \frac{\partial}{\partial t} f_m - \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{v}} f_m , \qquad (8)$$

where ϕ obeys a homogeneous initial condition and we use

$$\frac{\partial f_m}{\partial t} = \frac{\partial n}{\partial t} \frac{\partial f_m}{\partial n} + \frac{\partial T}{\partial t} \frac{\partial f_m}{\partial T}$$

$$\equiv \frac{\dot{n}}{n} f_m + \left(\frac{\epsilon}{T} - \frac{3}{2} \right) \frac{\dot{T}}{T} f_m \qquad (9)$$

although f_m is assumed to evolve on a time scale larger than the scale t on which ϕ evolves. In Eq. (9) we also adopted the convenient notation

$$C(\phi) \equiv C(f_m, f_m \phi) + C(f_m \phi, f_m)$$

$$+ C(f_m \phi, f_i) . \qquad (10)$$

In order to assure that Eq. (7) continues to be obeyed as ϕ evolves in time, two solubility, or compatibility, conditions are applied to Eq. (8). To assure the orthogonality of ϕ to f_m, we integrate Eq. (8) over V to obtain

$$\frac{\partial}{\partial t} \int_V f_m \, \phi \, d^3v = -\int_\Sigma f_m \, \phi \mathbf{F} \cdot d\mathbf{a} + \int_V C(\phi) \, d^3v$$

$$- \int_\Sigma \Gamma \cdot d\mathbf{a} - \int_\Sigma f_m \mathbf{F} \cdot d\mathbf{a} - \dot{n} , \qquad (11)$$

where Σ is the surface in velocity space bounding V, and $\dot{n}$ is chosen so that the right-hand side of Eq. (11) vanishes. In practice, we wish to choose V large enough such that the boundary terms do not introduce severe particle losses, and such that particles that do leave through the boundary can be accounted for in some other simple way (such as free-

streaming). The physical implication of decreasing n as a result of runaway production is that the bulk population is somewhat depleted of electrons and, hence, serves somewhat less effectively as a scatterer. Very fast electrons, i.e., those outside the domain V, are ineffective in contributing to the collisions of other electrons. It is often sufficient to account for these electrons merely by allowing them to stream freely from their point of exit on Σ. One fine point here is that we assume that the Maxwellian background is not confined to the bounded domain. This is consistent with the other approximations made so far, and it is a convenience.

The second compatibility condition gives us $\dot{T}$, the temperature evolution of the background electrons. Multiplying Eq. (8) by ϵ, integrating over the domain V, and applying Eq. (7), we obtain

$$\left(\frac{3}{2}\right) n \dot{T} = - \int_V \epsilon \left(\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{\Gamma} + \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{v}} f_m \right.$$
$$\left. + \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{v}} f_m \, \phi - C(\phi) \right) d^3 v , \tag{12}$$

where the first right-hand-side term above represents the energy input of the wave-driven flux, the second term vanishes, the third term contains the joule heating effect of the rf-driven current, and the last term tends to zero as $V \to \infty$.

Equation (8), together with Eq. (9) and the expression for $\dot{n}$ and $\dot{T}$, comprises an inhomogeneous, linear, integrodifferential equation for ϕ. This equation, difficult to treat analytically, may be solved numerically. The evolving solution $\phi(t)$ automatically obeys the orthogonality conditions, so that the linearization, Eq. (5), continues to hold, allowing the collision integral to retain its form. Rather than solve Eq. (8) for each excitation Γ, however, we consider an adjoint equation instead.

To find an operator adjoint to L, we first define a commutative operation on the two functions $\phi(\mathbf{v},t')$ and $\psi(\mathbf{v},t')$ by

$$[\phi, \psi]_t \equiv \int_V d^3 v \int_0^t \phi(\mathbf{v},t-\tau)\psi(\mathbf{v},\tau) \, d\tau , \tag{13}$$

where the operation [ ] is parametrized by t. Define a linear operator D_t operating on $\psi(\mathbf{v},t')$ by

$$D_t \psi(\mathbf{v},t') \equiv \left(f_m \frac{\partial}{\partial t'} \psi - f_m \mathbf{F}(\mathbf{v},t-t') \cdot \frac{\partial}{\partial \mathbf{v}} \psi - C(\psi) \right) . \tag{14}$$

For functions ϕ and ψ with homogeneous initial and boundary conditions, it may be shown that the operator D_t is adjoint to the operator L with respect to the inner product [ ]$_t$, i.e.,

$$[\phi, D_t \psi]_t = [\psi, L\phi]_t . \tag{15}$$

Suppose that $\phi(\mathbf{v},t')$ and $\psi(\mathbf{v},t')$ are restricted to functions orthogonal both to f_m and to ϵf_m over the domain V. Suppose further that $\psi(\mathbf{v},t')$ obeys the evolution equation

$$D_t \psi(\mathbf{v},t') = q_1 f_m + q_2 \epsilon f_m , \tag{16}$$

where the constants q_1 and q_2 are chosen to ensure that the orthogonality conditions on ψ are obeyed subsequently, given that they are obeyed initially. The quantities q_1 and q_2 are independent of $\mathbf{v}$, but are linear functionals of ψ. We now

supplement Eq. (16) with inhomogeneous initial and boundary conditions on ψ, namely

$$\psi(\mathbf{v},t'=0) = \psi_0(\mathbf{v}) , \tag{17a}$$

$$\mathbf{S}_\psi(\mathbf{v},t') = \mathbf{S}_B(\mathbf{v},t') \quad \text{on } \Sigma , \tag{17b}$$

where Σ is the boundary in velocity space to domain V, and $\mathbf{S}_\psi$ is the flux in velocity space associated with the operator D_t, i.e.,

$$\mathbf{S}_\psi(\psi) \equiv -f_m \, \mathbf{F}(\mathbf{v},t-\tau)\psi(\mathbf{v},\tau) + \mathbf{S}_c(\psi) , \tag{18}$$

where $\mathbf{S}_c(\psi)$ is related to $C(\psi)$ by

$$C(\psi) = - \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{S}_c(\psi) . \tag{19}$$

For the collision operator C, we have the property

$$\int_V [\phi C(\psi) - \psi C(\phi)] \, d^3 v$$
$$= - \int_\Sigma [\phi \, \mathbf{S}_c(\psi) - \psi \mathbf{S}_c(\phi)] \cdot d\mathbf{a} . \tag{20}$$

For $V \to \infty$, the surface terms vanish and we have the well-known self-adjoint property of the collision operator. For our present needs, however, we shall keep the boundary terms.

We take the inner product of ϕ and $D_t \psi$, where ϕ obeys Eq. (8) with homogeneous initial condition and ψ obeys Eqs. (16) and (17). We need to specify a boundary condition on Eq. (8), too; for our present purposes let us assume we know $S_c(\phi)$ on the boundary [but, we shall, in Eq. (23), give a more useful, less general, condition]. Exploiting the orthogonality properties of both ϕ and ψ, we obtain

$$\int_V d^3 v f_m \phi(\mathbf{v},t)\psi_0(\mathbf{v}) + \int_0^t d\tau \int_\Sigma \phi(\mathbf{v},\tau)\mathbf{S}_B(\mathbf{v},t-\tau) \cdot d\mathbf{a}$$

$$= \int_V d^3 v \int_0^t d\tau \mathbf{\Gamma}_*(\mathbf{v},t-\tau) \cdot \frac{\partial}{\partial \mathbf{v}} \psi(\mathbf{v},\tau)$$

$$+ \int_0^t d\tau \int_\Sigma \psi(\mathbf{v},t-\tau)[\mathbf{\Gamma}_* + \mathbf{S}_c(\phi)] \cdot d\mathbf{a} . \tag{21}$$

The driven flux

$$\mathbf{\Gamma}_*(v,t) \equiv \mathbf{\Gamma}(v,t) + \mathbf{F}(v,t) f_m \tag{22}$$

is the sum of the wave-induced flux Γ and the flux induced by the background fields that tend to accelerate the background Maxwellian distribution. The latter flux leads to the Ohmic current.

Equation (21) is a principal result of this work. Both ψ_0 and $\mathbf{S}_B$ are arbitrary functions, and they may be used to express any moment of ϕ either in the interior of V or on its bounding surface Σ. Depending on the desired moment of ϕ, a different Green's function, ψ, is solved for by Eqs. (16) and (17). Once ψ is known, the right-hand side of Eq. (21) may be evaluated for arbitrary wave-induced fluxes Γ.

Note that in generalizing the steady-state adjoint equations to include time dependence, several significant steps were taken. The inner product, with respect to which the adjoint property is defined, was changed to include a convolution over time. Also note that the orthogonality conditions are imposed here to ensure the continued validity of lineari-

zation in Eq. (5), and of the resulting simplification of the collision operator. These conditions reduce to the compatibility conditions of the steady-state theory in the limit $\partial/\partial t \to 0$. However, while the imposition of these conditions in the dynamic case results in considerable mathematical simplification, it is not necessary here for the existence of solutions.

IV. EXAMPLES

In this section, we give several examples showing how Eq. (21) may be used to find interesting plasma responses. For example, in addition to the current, the rf-induced runaway rate may be computed. A second example shows how contributions to the rf-induced plasma current (or for that matter radiation, etc.) can be separated into runaway and nonrunaway contributions. A third example shows how analytic progress can be made, using Eq. (21), to find the rf-induced conductivity in certain limiting cases.

To calculate how rf waves affect the runaway production rate, we must first be precise in defining a runaway electron. If the parallel electric field always points in one direction (i.e., electrons are pushed unidirectionally), then electrons can be accelerated by this field to high enough energy that they overcome the dynamic frictional force of collisions. If the domain V is large, then an electron accelerated from the bulk of the distribution to the boundary Σ may be deemed a runaway, for frictional forces are extremely unlikely to return this electron to the domain V. It is convenient to separate the boundary Σ into two parts; an electron appearing on Σ_{out} leaves the domain V, while an electron appearing on Σ_{in} is forced into the domain V, and Σ is the union of Σ_{out} and Σ_{in}. For large V, it is primarily the electric field that determines this dichotomy. A more precise separation of Σ takes into account the small, but finite, inwardly (i.e., into V) pointing collisional force in addition to the force of the electric field, so that Σ_{out} is slightly smaller than Σ_{in}. All runaway electrons appear at some time on Σ_{out}, and all electrons that appear on Σ_{out} are runaways, freely accelerating out of the domain V.

Equivalently, we can imagine solving a problem in which collisions operate within the domain V, but the collision operator vanishes outside this domain, allowing electrons to stream freely. Where the collision operator vanishes, the governing equation reverts from parabolic to hyperbolic. It is then correct to impose a boundary condition on just Σ_{in}, rather than on the full boundary Σ. The adjoint equation, on the other hand, is well-posed when a boundary condition is imposed only on Σ_{out}.

Define a flux $\mathbf{S}_\phi(\phi) \equiv \phi f_m \mathbf{F} + \mathbf{S}_c(\phi)$. Then, on Σ_{in}, we specify $\mathbf{S}_\phi(\phi)$, the incoming flux into domain V. For the runaway problem, and most other problems of interest, this flux is zero. For the adjoint equation, we require that $\mathbf{S}_c(\psi) = 0$ on Σ_{in}. On Σ_{out}, the opposite holds. Here we specify $\mathbf{S}_c(\phi) = 0$, indicating that runaway electrons eventually stream freely. We are then left with the freedom to choose $\mathbf{S}_\psi(\psi)$ in order to pose the adjoint equation.

Recognizing the vanishing of $\mathbf{S}$ on Σ_{in} and the vanishing of $\mathbf{S}_c$ on Σ_{out}, we can manipulate Eq. (21) into the more useful form

$$\int_V d^3v \, f_m \phi(\mathbf{v},t)\psi_0(\mathbf{v})$$

$$+ \int_0^t d\tau \int_{\Sigma_{\text{out}}} \phi(v,\tau) \mathbf{S}_B(\mathbf{v},t-\tau) \cdot d\mathbf{a}$$

$$= \int_V d^3v \int_0^t d\tau \, \mathbf{\Gamma}_*(\mathbf{v},t-\tau) \cdot \frac{\partial}{\partial \mathbf{v}} \psi(\mathbf{v},\tau)$$

$$+ \int_0^t d\tau \int_{\Sigma} \psi(v,t-\tau) \mathbf{\Gamma}_*(\mathbf{v},\tau) \cdot d\mathbf{a}. \tag{23}$$

The above formulation also allows a slight refinement. The collisional slowing down may be distinguished from the collisional induced diffusion. Requiring that only the collisional diffusion vanish at the boundary gives a more accurate calculation of both the total flux at the boundary and the demarcation between Σ_{in} and Σ_{out}. For example, for the runaway problem, the flux at the boundary may be expressed as the sum $\hat{i}_\parallel eE_\parallel/m - v_s\mathbf{v}$, where $v_s\mathbf{v}$ represents the dynamical friction resulting from collisions. The boundary Σ_{out} is then defined to exist where this sum points outward from the domain V.

Let us now find the response function for the rf-induced runaway rate. The number of rf-induced runaway electrons, N_R, appearing on Σ_{out} between time $\tau = 0$ and time $\tau = t$ may be written as

$$N_R = \int_0^t d\tau \int_{\Sigma_{\text{out}}} \mathbf{S} \cdot d\mathbf{a}, \tag{24}$$

where $\mathbf{S} = (\hat{i}_\parallel eE_\parallel/m - v_s\mathbf{v})\phi f_m$ on Σ_{out} is the more exact posing of the problem. The less exact posing takes $v_s \to 0$. In either event, the response function satisfies Eq. (16), $D_t\psi = Q_\psi$, where Q_ψ is chosen to ensure orthogonality, with initial condition

$$\psi(0) = 0, \tag{25a}$$

and boundary conditions

$$\mathbf{S}_B = f_m (\hat{i}_\parallel eE_\parallel/m - v_s\mathbf{v}) \quad \text{on } \Sigma_{\text{out}}, \tag{25b}$$

$$\mathbf{S}_B = 0, \quad \text{on } \Sigma_{\text{in}}. \tag{25c}$$

Solving Eqs. (24) and (25) for ψ, and using Eq. (23), we obtain (neglecting non-rf-induced contributions)

$$N_R = \int_V d^3v \int_0^t d\tau \, \mathbf{\Gamma}(\mathbf{v},t-\tau) \cdot \frac{\partial\psi(\mathbf{v},\tau)}{\partial\mathbf{v}}. \tag{26}$$

Note that $\psi(\mathbf{v},t)$ may be interpreted here as the probability with which a particle appears on Σ_{out} by time t given initial coordinate $\mathbf{v}$. For finite temperatures, eventually all particles would run away, since even those that slow down into the bulk distribution will ultimately scatter out of the bulk and run away. It is convenient, however, to label such electrons as stopped electrons, since they would run away on a much longer time scale. Suppose that the collision frequency were made artificially large as $v \to 0$ (or particles appearing near $v \to 0$ were frozen and not allowed to run away). Then particles running away without passing through the bulk would be termed runaways, and the others would be termed stopped. The probability of a particle running away given initial coordinate $\mathbf{v}$ may then be expressed as

$$R(\mathbf{v}) = \psi(\mathbf{v}, t \to \infty). \tag{27}$$

For some runaway speed v_R, collisional deceleration and dc

field acceleration balance. For $v \ll v_R$, $R \to 0$; for $v_\parallel < -v_R$, $R = 1$; and for $v_\parallel \to \infty$, $R \to 1$. This runaway rate has been obtained previously through an equivalent formulation of the problem using Langevin equations.[14] The numerical evaluation of the runaway rate was carried out by a Monte Carlo method. Solving Eqs. (24) and (25) directly, however, would be much easier.

It is often helpful to distinguish the contributions of distinct groups of electrons to a given effect. For example, both runaway and stopped electrons contribute to the current. By identifying the separate contributions, one can determine if there is anything to be gained by somehow increasing or decreasing the number of runaways, possibly by controlling their confinement time. In Ref. 15, distinguishing these contributions facilitated a comparison between theory and experiment.

In order to deduce the contribution to an effect, say the current, resulting solely from stopped electrons, we make use of the function $R(\mathbf{v})$ defined in Eq. (27). We write the parallel current as $J = J_R + J_S$, where J_R is the runaway current and J_S is the current resulting from stopped electrons. We can write J_S as

$$J_S = \int_V d^3v f_m \, \phi(\mathbf{v},t)\{v_\parallel [1 - R(\mathbf{v})] + c_1 + \epsilon c_2\}, \qquad (28)$$

where c_1 and c_2 are constants to be determined. Evidently, the appropriate Green's function ψ_s for J_S solves Eq. (16) with homogeneous boundary conditions and initial condition

$$\psi_s(\mathbf{v},t = 0) = v_\parallel [1 - R(\mathbf{v})] + c_1 + \epsilon c_2, \qquad (29)$$

with c_1 and c_2 determined by the orthogonality properties of ψ_s. Using Eq. (21), we then have $J_S = -[\psi_s, (\partial/\partial \mathbf{v}) \cdot \Gamma]$.

The last example in this section illustrates the ease with which analytic solutions can sometimes be found for the response functions. Analytic expressions are often available when the collision operator can be written in the large-velocity limit. For example, the incremental conductivity caused by the heating of superthermal electrons by rf waves can be calculated analytically. This is the so-called "hot conductivity." The simplification of the collision operator in the large-velocity limit is valid, because only the superthermal electron contribution to the current is important. In contrast, the Spitzer–Härm conductivity is not available analytically. Both slow and fast electrons contribute importantly to the Spitzer–Härm current and a drastic simplification of the collision operator is not possible.

In the high-velocity limit, the collision operator simplifies to

$$C(\phi) = -\frac{v_0 f_m}{2u^3}\left[-u \frac{\partial}{\partial u}\phi \right.$$
$$\left. + \left(\frac{1+Z}{2}\right)\frac{\partial}{\partial \mu}(1 - \mu^2)\frac{\partial}{\partial \mu}\phi \right], \qquad (30)$$

where $v_0 = \omega_p^4 \ln \Lambda / 2\pi n v_T^3$ is a collision frequency, $u \equiv v/v_T$, $\mu \equiv v_\parallel/v$, $v_T \equiv (T/m)^{1/2}$, and Z is the ion charge state. In order to find the hot conductivity, we need to solve Eq. (16) in the limit $\mathbf{F} = \hat{i}_\parallel eE_\parallel/m \to 0$. Also, since we wish to find the steady-state conductivity rather than the tran-

sient effects, it is necessary only to calculate the time integral of ψ rather than ψ itself.

Accordingly, define

$$\chi(\mathbf{v}) = \int_0^\infty dt \, \psi(\mathbf{v},\tau), \qquad (31)$$

where $\psi(\mathbf{v},\tau)$ solves Eq. (16) with homogeneous boundary conditions and initial condition $\psi(\mathbf{v},\tau = 0) = ev_T u\mu$. Integrate Eq. (16) from $\tau = 0$ to $\tau = \infty$ to obtain

$$-f_m\left(\mathbf{F} \cdot \frac{\partial}{\partial \mathbf{v}}\right)\chi - C(\chi) = f_m ev_T u\mu, \qquad (32)$$

where we noted that $\psi(\mathbf{v},\tau) \to 0$ as $\tau \to 0$ (for finite conductivity), and that the constants q_1 and q_2 in Eq. (16) are, in this case, zero. Expanding χ in a power series in $E_\parallel$, i.e., $\chi = \chi_0 + E_\parallel \chi_1 + \cdots$, allows us to solve Eq. (32) analytically. Using $F = eE_\parallel/m$ in Eq. (32), we obtain

$$\chi_0 = [2/(5 + Z_i)](e/mv_T v_0)\mu u^4 + O(u^2), \qquad (33a)$$

$$\chi_1 = \frac{1}{3v_0}\left(\frac{e}{mv_T}\right)^2 \frac{4u^6}{5 + Z_i}\left(1 + \frac{3\mu^2 - 1}{3 + Z_i}\right) + O(u^4), \qquad (33b)$$

and the current is found from

$$J = \int_V d^3v \, \Gamma \cdot \frac{\partial}{\partial \mathbf{v}}(\chi_0 + E_\parallel \chi_1 + \cdots), \qquad (33c)$$

where the first term gives the current in the absence of an electric field and the second term gives the current that is bilinear in both the dc electric field on the rf power absorbed.

We can express quantities of interest more conveniently by noting that the absorbed power is given by

$$P_d = \int_V d^3v \, \Gamma \cdot \frac{\partial \epsilon}{\partial \mathbf{v}}. \qquad (34)$$

Expanding $J = J_0 + \sigma_1 E_\parallel + \sigma_2 E_\parallel^2 + \cdots$ and comparing to Eqs. (33), we can write

$$\frac{J_0}{P_d} = \frac{\int_V d^3v \, \Gamma \cdot (\partial \chi_0/\partial \mathbf{v})}{\int_V d^3v \, \Gamma \cdot (\partial \epsilon/\partial \mathbf{v})}, \qquad (35)$$

and

$$\sigma_1 = \frac{P_d \int_V d^3v \, \Gamma \cdot (\partial \chi_1/\partial \mathbf{v})}{\int_V \Gamma \cdot (\partial \epsilon/\partial \mathbf{v})}. \qquad (36)$$

In Eq. (35), we have exactly the Fisch–Boozer current-drive efficiency.[7] In Eq. (36), we have exactly the "hot" conductivity, found previously[3] in a more circuitous fashion. Correction terms to both quantities are available by considering higher-order terms in $1/u$ in Eqs. (33).

V. MODIFIED C–E EXPANSION

The Chapman and Enskog expansion cannot address the phenomena calculated here, although it is useful for describing bulk transport. The question is how we can modify this expansion in order to incorporate effects associated with the fast, collisionless electrons [especially those effects that can be described by Eq. (18) and its associated formalism] in addition to the bulk transport. We seek an expansion that would reduce to the C–E expansion in the limit $\Gamma \to 0$, and would, for finite Γ, describe adequately the dynamics of the fast electrons. In this expansion, the heating of the bulk elec-

trons by the fast electrons, as well as other effects that link the two distributions, should play a role in the evolution of the C–E variables $(n,T,\mathbf{c})$ for $\Gamma \neq 0$. (Here we use $\mathbf{c}$ to describe the drift of a Maxwellian distribution, with the understanding that all these variables, n, T, $\mathbf{c}$, must be specified separately for each species.) In this section, we describe an expansion that meets these criteria. We remark, however, that this expansion, like the C–E expansion itself, is not a unique choice, although it does possess some desirable features.

We begin with Eqs. (1) and (2), and expand f using the formal expansion parameters θ and δ (assumed small but eventually set equal to one), i.e., we expand

$$f = f_0 + \sum \theta^j f_j + \sum \delta^k h_k , \quad j,k \geqslant 1 , \tag{37}$$

where the f_j represent corrections primarily to the bulk electron distribution and the h_k represent the wave-driven corrections. Accordingly, the collisions among the f_j are assumed to take place on a much shorter time scale than collisions either between the f_j and the h_k, or among the h_k. We shall order

$$C(f_j, f_k) \sim 1/\theta , \quad j,k \geqslant 0 , \tag{38a}$$

$$C(f_j, h_k) \sim 1 , \quad j \geqslant 0, k \geqslant 1 , \tag{38b}$$

$$C(h_j, h_k) \sim 1 , \quad j,k \geqslant 1 . \tag{38c}$$

The $1/\theta$ Eq. (38a) ordering is inherited from the C–E approach, whereas the other ordering, as we shall see, allows us to describe better the dynamics of fast electrons.

In the C–E approach, the use of multiple time scales serves as a means of convenient bookkeeping. Here, in addition to the C–E time scales, we introduce a parallel scaling in δ, i.e.,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau_0} + \theta \frac{\partial}{\partial \tau_1} + \theta^2 \frac{\partial}{\partial \tau_2}$$
$$+ \cdots + \delta \frac{\partial}{\partial t_1} + \delta^2 \frac{\partial}{\partial t_2} + \cdots , \tag{39}$$

which facilitates the formal expansion. We assume the functional dependencies

$$f_0 = f_0(\tau_0, \tau_1, \tau_2, \ldots, t_1, t_2, \ldots, \mathbf{v}, \mathbf{r}) , \tag{40a}$$

$$f_j = f_j(\tau_0, \tau_1, \tau_2, \ldots, \mathbf{v}, \mathbf{r}) , \quad j \geqslant 1 , \tag{40b}$$

$$h_j = h_j(\tau_0, t_1, t_2, \ldots, \mathbf{v}, \mathbf{r}) , \quad j \geqslant 1 . \tag{40c}$$

Consider terms in the expansion of order θ^k, which represent the C–E terms. To order $1/\theta$, we have

$$C(f_0, f_0) = 0 , \tag{41}$$

which has the drifting Maxwellian solution

$$f_0 = f_m(n,T,\mathbf{c}) , \tag{42}$$

with n, T, and $\mathbf{c}$ functions of time and space. To higher order in θ, we have

$$\theta^0: \quad C(f_1) = \frac{d}{d\tau_0} f_0 , \tag{43a}$$

$$\theta^1: \quad C(f_2) = \frac{d}{d\tau_0} f_1 + \frac{\partial}{\partial \tau_1} f_0 - C(f_1, f_1) , \tag{43b}$$

where

$$\frac{d}{d\tau_0} \equiv \frac{\partial}{\partial \tau_0} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{v}} . \tag{44}$$

Equations (43) represent the first terms of the C–E expansion. The f_j, $j \geqslant 1$ are chosen to contain no particles, current, or energy. Compatibility is ensured, order by order, by using

$$\frac{\partial f_0}{\partial \tau_j} = \frac{\partial n}{\partial \tau_j} \frac{\partial f_m}{\partial n} + \frac{\partial T}{\partial \tau_j} \frac{\partial f_m}{\partial T} + \frac{\partial \mathbf{c}}{\partial \tau_j} \cdot \frac{\partial f_m}{\partial \mathbf{c}} , \tag{45}$$

and solving for $\partial n/\partial \tau_j$, $\partial T/\partial \tau_j$, $\partial \mathbf{c}/\partial \tau_j$ to meet the requirements on the f_j. The C–E quantities n, T, $\mathbf{c}$ then evolve according to, e.g.,

$$\frac{dn}{dt} = \sum_{j=0} \theta^j \frac{\partial n}{\partial \tau_j} + \sum_{j=1} \delta^j \frac{\partial n}{\partial t_j} . \tag{46}$$

In the absence of contributions arising from $\partial/\partial t_j$, the terms f_0, f_1, f_2 are identical to the first three terms of the classic C–E expansion.

Matching terms of order δ^j allows us to describe additionally the effects of the wave-driven fluxes. For example, the first two equations in the expansion are

$$Lh_1 \equiv \frac{d}{d\tau_0} h_1 - C(h_1) = - \frac{\partial}{\partial t_1} f_0 - \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{\Gamma} , \tag{47a}$$

$$Lh_2 = - \frac{\partial}{\partial t_1} h_1 - \frac{\partial}{\partial t_2} f_0 + C(h_1, h_1) . \tag{47b}$$

Note that Eq. (47a) is identical to Eq. (8) for $f_m \phi$ except that f_0 is slightly more general here, that h_1 can be spatially dependent, and that the inhomogeneous driving term associated with the Spitzer–Härm current has been accounted for in the C–E part of the expansion. The last difference means that, consistent with our assumption of the low mean-free-path limit, we ignore any runaway electrons except those created by the waves. Those electrons could have been described had we ordered the term $\mathbf{F} \cdot \partial f_0/\partial \mathbf{v} \sim \delta$; however, that would take us further than we wish away from the C–E approach.

The differences between Eq. (47a) and Eq. (8) do not affect the transferability of the technique to solve these equations. Boundary and initial conditions are imposed similarly. The adjoint method used to solve Eq. (8) may be employed both in Eqs. (47a) and (47b) and higher-order equations in the δ expansion. The effect of the wave-driven fluxes on the bulk transport properties is then accounted for through the evolution equations for n, T, and $\mathbf{c}$, as in Eq. (46).

While the above procedure accurately treats to first order both the wave-driven fluxes and the bulk transport within the spirit of Chapman and Enskog, a question arises as to higher-order terms in the expansion that are hybrid, i.e., bilinear in h_j and f_k. There is a certain amount of arbitrariness in treating these hybrid terms; their main effect would be to correct somewhat the properties of the bulk plasma, rather than to change the dynamics of the fast electrons. Accordingly, it is most expedient to account for these terms by allowing them to drive the f_k. To incorporate them, we assume $\delta \sim \theta$. So, for example, the lowest-order hybrid terms, which are bilinear in f_1 and h_1 [which are computable from Eqs. (43a) and (43b)], drive the third-order C–E term f_3,

i.e.,

$$C(f_3) = \frac{d}{d\tau_0}f_2 + \frac{\partial}{\partial\tau_1}f_1 + \frac{\partial}{\partial\tau_2}f_0 - C(f_2,f_1) - C(f_1,f_2)$$
$$- C(f_1,h_1) - C(h_1,f_1) . \qquad (48)$$

In such a manner, we may solve order by order both for f_j and h_j, evolving n, T, c [e.g., through Eq. (46)] as we do so. Finally, we set the formal expansion parameters θ, $\delta = 1$. In practice, however, one may contemplate that only the first few terms of such an expansion are important.

VI. SUMMARY AND CONCLUSIONS

This paper concerns itself with the description of effects associated with fast, relatively collisionless electrons, which have a mean free path that may be long compared to other lengths of interest. It is possible that these few, fast electrons will contribute importantly to plasma transport, particularly in the case of an intensely rf-driven plasma. Intense rf waves interacting primarily with fast electrons produce particle fluxes in velocity space that can substantially alter the distribution of electrons. The result may be a large change in the plasma current, synchrotron radiation, bremsstrahlung, runaway production, or other quantities associated with the transport of fast electrons.

In order to calculate the plasma response to wave-induced fluxes in velocity space, we linearized the inhomogeneous dynamic Boltzmann equation. Although the linearized equation is not much more easily solved numerically than is the original equation, it is possible, because of the linearization, to find the Green's functions for quantities of interest. In Sec. III we identified an adjoint operator, defined over a suitable inner product, that allowed us to generate adjoint equations for calculating these quantities. By imposing initial and boundary conditions on the adjoint equation, plasma responses may be calculated, via Eq. (23), for arbitrary wave-induced fluxes Γ.

An important simplification in the practical application of Eq. (23) arises from our ability to infer information about Γ from the resonance condition, as discussed in Sec. II. This allows us to compare, for example, likely plasma responses per power dissipated as the phasing of the waves is varied.

We gave several examples in the use of these response functions in Sec. IV. These examples showed how useful adjoint equations may be defined, and how, in certain limits, analytic solutions are available.

The responses associated with these fast electrons are not fluid effects, and are not naturally found in small mean-free-path expansions of the Boltzmann equation, such as the expansion of Chapman and Enskog. In Sec. V, however, we showed how the C–E expansion may be modified to account for the wave-driven fluxes. An expansion, formally in two small parameters, was proposed: the frequently colliding bulk particles are treated as fluid-like, exactly as in the C–E expansion, while the fast, long-mean-free-path electrons evolve subject to external forces and subject to collisions predominantly with bulk electrons and ions. The description of the fast electrons is facilitated by the use of adjoint equations, and the effect of these electrons on the bulk transport processes appears in the higher-order terms describing the bulk particles.

Although the response functions derived here can be directly implemented in existing transport codes (see, e.g., Ref. 21) that simulate completely the evolving plasma, we speculate that a more advantageous use of this work is through the development of a set of correlations from which plasma behavior might be inferred.

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