AS551 General Plasma Physics 1
November 11, 1996

Problem Set #6 (due Nov. 18)

At the end of the last class, I showed how the fluid momentum equation could be summed over species and simplified in certain limits to obtain the MHD momentum equation. For a plasma in equilibrium ($\partial/\partial t = 0$), it reduces to the MHD force balance relation:

$$0 = -\nabla \cdot \vec{\vec{P}} + \frac{\vec{j} \times \vec{B}}{c}$$

where $\vec{\vec{P}}$ is the pressure tensor. (Actually, this is usually called the "MHD" force balance equation only if the pressure is isotropic.) In the following problems, you will explore some of the properties of plasmas implied by this equation. For the first 2 problems below, assume that the pressure tensor is isotropic, $\vec{\vec{P}} = p\vec{\vec{1}}$.

1. General cylindrical pinch equilibria. Consider a plasma with cylindrical symmetry ($\partial/\partial z = 0$, $\partial/\partial \theta = 0$), with specified radial profiles for the pressure $p(r)$, azimuthal field $B_\theta(r)$, and axial field $B_z(r)$. From the MHD equation for equilibria, derive the equation:

$$p(r) + \frac{B^2(r)}{8\pi} = p(0) + \frac{B^2(0)}{8\pi} - \frac{1}{4\pi}\int_0^r dr \frac{B_\theta^2}{r}$$

Assume the plasma extends to $r = a$, and is surrounded by a vacuum (with a magnetic field) outside that radius. For the case of uniform current distribution $j_z =$ constant for $r < a$, and $j_z = 0$ for $r > a$, show that the plasma is paramagnetic, i.e. $B_z(0) > B_z(a)$, if $p(0) < B_\theta^2(a)/(4\pi)$. Is this result dependent, or not, on the current and pressure distributions in the plasma?

2. Bennet pinch condition. Consider a standard z-pinch equilibrium configuration (current in the z-direction, magnetic field only in the θ direction). The plasma is uniform in z and θ, and is confined to $r < a$ by the magnetic field. Calculate the volume integrated plasma energy (per length in z):

$$W = \int_0^a dr 2\pi r \frac{3}{2} p$$

Integrate this by parts (assuming $p = 0$ at $r = a$) to express this in terms of $\partial p/\partial r$, and use the MHD force balance equation to show that W can be expressed in terms of the total current carried by the plasma. (This is known as the Bennett pinch condition.)

3. CGL pressure balance along field lines. Consider the CGL form of the pressure tensor which allows for anisotropic in pressure parallel and perpendicular to the field line. Consider the parallel component of the force balance equation at the top of this page. Prove that it leads to the condition:

$$\hat{b} \cdot \nabla p_\parallel = -\frac{(p_\perp - p_\parallel)}{B} \hat{b} \cdot \nabla B$$

Give a physical interpretation of this expression.