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Resonance Broadening for Wave-Particle and Wave-Wave Turbulence*

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We propose an approximate but simple and general procedure for treating resonance broadening in weak-turbulence interactions. The resonance-broadening corrections to the wave-particle, wave-wave, nonlinear-Landau-damping, and four-wave weak-turbulence equations are computed as examples. The procedure may be used to predict at the outset whether resonance broadening occurs in a specific higher-order process.

The major modification to the conventional weak-turbulence equations demanded by renormalized plasma turbulence theory is a broadening of the resonances.[1-4] A particularly fruitful and powerful approach is to postulate this result and then determine from simple physical considerations the extent of the broadening. In other words, it is decided *a priori* that it is unphysical and incorrect to allow the evolution of the plasma to be determined by quantities describing a "granulation" of wave or particle coordinate space that is finer than a certain resonance-broadening width. It then follows that the weak-turbulence equations are improved through the use of smoothed-over driving quantities

$$\bar{f}_0(v) \equiv \frac{1}{2\delta v}\int_{v-\delta v}^{v+\delta v} f_0(v')\,dv',\qquad \bar{N}(k) \equiv \frac{1}{2\delta k}\int_{k-\delta k}^{k+\delta k} N(k')\,dk', \tag{1}$$

where δv and δk are resonance-broadening widths and reflect the level of turbulence present. (In this Letter we work, for simplicity, in one dimension only.) As an illustrative example, the modified nonlinear-Landau-damping equations could then be put in the form

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left(D\frac{\partial \bar{f}_0}{\partial v}\right),\quad \frac{\partial N_i}{\partial t} = \gamma_i \bar{N}_i, \tag{2}$$

$$D = \frac{1}{nm}\iint \bar{N}_1\bar{N}_2 H\delta[\omega_1-\omega_2-(k_1-k_2)v]\,dk_1\,dk_2, \tag{3}$$

$$\gamma_i = \int \frac{dk_j \bar{N}_j H}{k_i-k_j}\frac{\partial \bar{f}_0}{\partial v}\bigg|_{v=(\omega_i-\omega_j)/(k_i-k_j)}, \tag{4}$$

where H is a coupling coefficient. Other weak-turbulence equations are similarly modified—obviously, in all cases the modified equations conserve the same quantities as the unmodified equations such as energy, momentum, and particles.[5]

It remains, of course, to calculate the resonance widths for particular interactions. It is expected that the broadening occurs when the interaction is especially sensitive to properties of the interacting modes (e.g., particle velocity or wave phase or group velocity) which may be altered because of the turbulence. For example, the quasilinear wave-particle interaction is especially sensitive to the velocity of the resonant particles. Hence, it may be argued that it makes no sense to granulate wave phase velocity or resonant-particle velocity space finer than

$$\delta v = \delta(\omega/k) = |q(16\pi\mathcal{E}_k\delta k)^{1/2}/mk|^{1/2}, \tag{5}$$

which is the trapping width of a single wave with amplitude based on the spectral energy density $\mathcal{E}_k$ times the width δk consistent with δv. In other words, the waves in a width δk do not get out of phase with each other in a bounce time $\delta(\omega/k)/k$ in the frame of a resonant particle and hence act as one wave in scattering the resonant particle over a self-consistent width $\delta v = \delta(\omega/k)$. Note that $\delta(\omega/k) = |v_g - v|\delta k/k$ so that (5) may be solved to obtain the well-known result[1]

$$\delta\left(\frac{\omega}{k}\right) = \left|16\pi\frac{q^2}{m^2k}\frac{\mathcal{E}_k}{|v_g - v|}\right|^{1/3} = \left|\frac{D^r}{k\pi}\right|^{1/3}, \tag{6}$$

where D^r is the quasilinear diffusion coefficient for resonant particles. Note that the resonance width is proportional to $\mathcal{E}_k^{1/3}$ whereas simple detuning effects are proportional to $\mathcal{E}_k$ and hence are much smaller.[6]

The same considerations apply in finding the resonance widths for the nonlinear-Landau-damping interaction. Here an equation analogous to (5) may be written,

$$\delta v = \delta\left(\frac{\omega_1 - \omega_2}{k_1 - k_2}\right) = \left[\frac{qV}{m(k_1 - k_2)}(16\pi\mathcal{E}_{k1}\delta k_1)^{1/2}(16\pi\mathcal{E}_{k2}\delta k_2)^{1/2}\right]^{1/2}, \tag{7}$$

where V is a coupling coefficient[7] related to H in (3) and (4), and δv is the trapping width of the beat wave based self-consistently on the amount of energy in the widths δk_1 and δk_2 of the driving spectra. Note that the resonance widths are related to each other by

$$\delta\left(\frac{\omega_1 - \omega_2}{k_1 - k_2}\right) = \frac{\delta k_1|v_{g1} - v| + \delta k_2|v_{g2} - v|}{|k_1 - k_2|}, \tag{8}$$

and since a nonlinearly resonant particle must be allowed to see each wave packet of width δk_i for the same amount of time,

$$\delta k_1/|v_{g2} - v| = \delta k_2/|v_{g1} - v|. \tag{9}$$

From (7)–(9) the resonance widths are obtained, e.g.,

$$\delta v = (16\pi qV/m)[\mathcal{E}_{k1}\mathcal{E}_{k2}/|(v_{g1} - v)(v_{g2} - v)|]^{1/2}. \tag{10}$$

Note that for this interaction the resonance widths are proportional to $\mathcal{E}_k$ whereas detuning effects are proportional to $\mathcal{E}_k^2$. Higher-order wave-particle interactions do not, however, exhibit resonance broadening. For example, by analogy with (7), for a three-wave–particle interaction

$$\delta v \sim [(\mathcal{E}_{k1}\delta k_1)^{1/2}(\mathcal{E}_{k2}\delta k_2)^{1/2}(\mathcal{E}_{k3}\delta k_3)^{1/2}]^{1/2}. \tag{11}$$

From (11), δv scales as $\mathcal{E}_k^3$ which is of the same order as detuning effects.

For wave-particle interactions, fluctuations in the particle modes (v, the particle velocity) lead to resonance broadening when the turbulent interaction is sensitive to these fluctuations, i.e., when $v \sim \omega/k$. In a similar manner, fluctuations in the wave modes (v_g, the group velocity) lead to resonance-broadening effects in wave-wave interactions when the interaction is sensitive to these fluctuations. In general, however, a sufficiently accurate description of wave-wave interactions may be based on a granulation of wave coordinate space as fine as detuning effects allow.[8] The exception to this rule occurs when waves within the same packet may interact in a manner resulting in the cascading of energy within the wave packet. For this type of interaction we may assume that the two high-frequency modes in a three-wave interaction belong to the same wave packet and the diffusion approximation[9] can be made, i.e., $k_1 \gg \Delta k_1 \gg k_2$ and $N_2 \gg N_1$, where N_i is wave action density and k_1 and k_3 belong to the same wave packet subscripted 1. The weak-turbulence equations describing this interaction are analogous to the quasilinear wave-particle equations, i.e.,

$$\begin{gathered}\frac{\partial N_1}{\partial t} = \frac{\partial}{\partial v_{g1}}\left(D\frac{\partial N_1}{\partial v_{g1}}\right), \quad \frac{\partial N_2}{\partial t} = \gamma N_2, \\ \gamma = 8\pi|V|^2\partial N_1/\partial v_{g1}|_{v_{g1} = \omega_2/k_2}, \\ D = \frac{8\pi|V|^2N_2k_2^2|\partial v_{g1}/\partial k_1|^2}{|v_{g1} - v_{g2}|}\bigg|_{k_2 = \omega_2/v_{g1}}.\end{gathered} \tag{12}$$

This analogy extends to the resonance-broadening correction to (12) in v_g space which is also $\sim D^{1/3}$. This is because a coherent long-wavelength wave (ω_2, k_2) can "trap" the short-wavelength wave packet by stimulating decay of high-group-velocity waves to low-group-velocity waves thus decelerating the packet or vice versa for accelerating the packet. One decay process takes place at one wall of the trough whereas the opposite decay takes place at the other wall thus trapping the packet. This wave-wave-packet interaction leads to orbit secularities in v_{g1} space, resulting in a resonance broadening much in the same way as the analogous wave-particle interaction. A rigorous and detailed description of this process is given elsewhere.[10,11] Here it is sufficient to note that it is possible to obtain a continuity equation for the action density,

$$\mathrm{Re}\left(\frac{\partial}{\partial t}+v_{g1}\frac{\partial}{\partial x}+2\frac{\partial v_{g1}}{\partial k_1}\frac{\partial}{\partial v_{g1}}k_2VA_2\exp[i(k_2x-\omega_2t)]\right)|A_1|^2=0, \tag{13}$$

by averaging the equations for the coherent three-wave interaction over the fast variations of wave packet 1 while retaining the phase information associated with the long-wavelength wave packet 2. In obtaining (13) use is also made of the expansion $|A_1(k_1)|^2=|A_1(k_3)|^2+k_2(\partial v_{g1}/\partial k_1)\partial|A_1|^2/\partial v_{g1}$. It may now be recognized that (13) is a "Vlasov" equation for $|A_1|^2$ in (x, v_{g1}, t) space and enjoys the same mathematical properties as the usual Vlasov equation and an analogous physical interpretation. In particular, for constant A_2, (13) must describe wave-packet trapping analogous to particle trapping with a trapping width $\delta v_{g1}=(VA_2\,\partial v_{g1}/\partial k_1)^{1/2}$ which for a wide spectrum 2 may be cast into the form of (5) from which

$$\delta v_{g1}=|D/\pi k_2|^{1/3} \tag{14}$$

is obtained. The resonance width δk_2 is related to δv_{g1} by $\delta k_2=k_2\delta v_{g1}/|v_{g2}-\omega_2/k_2|$.

For four-wave interactions, it is expected that resonance broadening does not occur unless two waves (say 1 and 3) belong to the same spectrum in such a way that the diffusion approximation can be made. It then follows that the interaction, which involves the cascading of energy in wave packet 1, becomes analogous to nonlinear Landau damping. It can then be argued that

$$\delta v_{g1}\sim[N_2N_4/|(v_{g2}-v_{g1})(v_{g4}-v_{g1})|]^{1/2}. \tag{15}$$

It may also be argued by analogy to the wave-particle interaction that resonance broadening does not occur in wave interactions with five or more waves even if two waves belong to the same wave packet. The exception to this rule occurs when wave-wave interactions exhibit a singularity in the resonance widths analogous to that for wave-particle interactions when the resonant-particle velocity is equal to the group velocity of one of the wave packets. When these singularities are present, assumptions in the derivation of the resonance widths (i.e., that they be small) are violated. For example, in the case of nonlinear Landau damping suppose that the group velocity of wave 2 is equal to the resonant-particle velocity. Then from (9) it follows that δk_2 is infinite, so that in (7), the quantity $\mathcal{E}_{k2}\delta k_2$ includes all the energy in wave packet 2. Thus, (7) may be written as $\delta v\sim E_2^{1/2}(\mathcal{E}_{k1}\delta k_1)^{1/4}$ which implies $\delta v\sim E_2^{2/3}\mathcal{E}_{k1}^{1/3}$ where E_2 is now the rms amplitude associated with spectrum 2. This physically corresponds to the particle being "quasilinearly resonant" with wave packet 1 in a time-varying plasma due to wave 2. It may be observed that the multiple equality of group velocities in wave interactions reduces the order of the resonance width whereas the number of equalities does not. For example, in the nonlinear-Landau-damping interaction, $v_{g1}=v_{g2}$ does not affect the order of the resonance width. In any case, it is clear from the preceding that in all cases considered here, there are only two types of resonance-broadening widths, those proportional to $\mathcal{E}_k^{1/3}$ and those proportional to $\mathcal{E}_k$, and resonance widths in interactions exhibiting singularities reduce to one of these types.

In conclusion, it should be emphasized that the results presented here lead only to the identification of modes that must be allowed to interact with each other because of the finite level of turbulence—the exact details of this interaction remain uncalculated. The philosophy adopted here involves recognizing that in the limit of zero turbulence the details of this interaction are known and given by the weak-turbulence equations. Thus, a reasonable correction is to "expand" about this limit, allowing modes close to exact resonance to interact in much the same way. Because of the simplified form obtained, a direct comparison with other attempts[2-4] in the field is, in general, difficult. For the special case of wave-particle resonance broadening, however, Dupree[1] obtained a simple expression for the resonance width which is in agreement with the result here.[12]

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Four-Particle Exchange in Solid ^{3}He

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We report calculations which suggest that there is a physically important four-atom exchange process in bcc ^{3}He and thus an important four-spin term in the exchange Hamiltonian. A simple, mean-field analysis of this Hamiltonian appears to account for a number of the perplexing properties of bcc ^{3}He. An understanding of other properties may require treatment of the exact four-spin term. It is our hope to stimulate such effort by this Letter.

We report the results of calculations which suggest there is a physically important four-atom exchange process in bcc solid ^{3}He. The process gives rise to a *four-spin* term in the effective spin or exchange Hamiltonian with an exchange energy comparable to the nearest-neighbor two-spin term. A simple-minded mean-field treatment suggests that this four-spin term could lead to a temperature-dependent exchange frequency which offers partial insight to the several perplexing properties of bcc solid ^{3}He.[1]

To facilitate discussion we define the exchange Hamiltonian including pair, triple, and the important cyclic quadruple exchange[2]:

$$H_{\text{ex}} = -2[J_1 - 6J_{112} + 3J_{1111,23}] \sum_{i<j}^{(1)} \vec{I}_i \cdot \vec{I}_j - 2[J_2 - 4J_{112} + J_{1111,23}] \sum_{i<j}^{(2)} \vec{I}_i \cdot \vec{I}_j - 4J_{1111,23} \sum_{i<j<k<l} [(\vec{I}_i\cdot\vec{I}_j)(\vec{I}_k\cdot\vec{I}_l) + (\vec{I}_j\cdot\vec{I}_k)(\vec{I}_l\cdot\vec{I}_i) - (\vec{I}_i\cdot\vec{I}_k)(\vec{I}_j\cdot\vec{I}_l)]. \quad (1)$$

The first two-spin term involves nearest-neighbor spins [the (1) over the sum], while the second involves next-nearest-neighbor spins [the (2)]. Finally the one four-spin term involves four atoms located at the corners of the rhombus, lying in the (110) plane, whose sides $\overline{ij}$, $\overline{jk}$, $\overline{kl}$, and $\overline{li}$ are first-neighbor distances (the subscript 1111 in $J_{1111,23}$) and whose diagonals $\overline{ik}$ and $\overline{jl}$ are second- and third-nearest neighbors (the 23).