Conductivity of rf-heated plasma

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The electron velocity distribution of rf-heated plasma may be so far from Maxwellian that Spitzer conductivity no longer holds. A new conductivity for such plasmas is derived, and the result can be put in a general form. The new expression should be of practical value in examining schemes for current ramp-up in tokamaks by means of lower-hybrid or other waves.

I. INTRODUCTION

The conductivity of fully ionized plasma has been determined under the usually excellent assumption that the velocity distribution of electrons is nearly Maxwellian.[1] An important exception may occur when intense power is injected into a tokamak to heat the plasma to thermonuclear ignition temperatures or to generate substantial toroidal current.

For example, there is present interest in producing toroidal current in a tokamak by means of intense lower-hybrid waves.[2] As the rf current is turned on, however, a toroidal dc electric field is induced, producing a countercurrent in order to oppose any abrupt change in the flux linkage to the plasma. In effect, it is difficult to turn on the total current in less than an L/R time, where L and R are the plasma inductance and resistance. The resistance is that of the heated plasma, so to find the turn-on time, we must first determine the conductivity of the heated plasma.

The total current resulting from the application of rf power in the presence of a dc electric field E may be written as $J_{\text{tot}} = J_{\text{OH}} + J_{\text{rf}} + \sigma_1 E + \sigma_2 E^2 + \dots$, where $J_{\text{OH}} = \sigma_{sp} E$ is the ohmic current that results in the absence of rf power, J_{rf} is the rf current that results in the absence of a dc electric field, and $\sigma_1 E$ is the additional current, linear in E, that results when both rf power and the dc field are present (with the rf input power held constant). There are also terms present that are higher order in E, as well as exponentially small runaway terms that do not have an expansion in E; they will not be calculated here. Our problem is to find the conductivity σ_1, which is a well-defined transport coefficient in the limit of $E \to 0$.

At first glance, it may appear that in order to find the conductivity, one must first calculate the velocity distribution function of the rf-heated plasma. Such a program can certainly be carried out numerically, but analytic progress would be difficult. An alternative approach, however, is quite powerful. The self-adjoint property of the Fokker–Planck collision operator has been exploited to great advantage by Hirshman[3] in calculating beam-driven currents and by Antonsen and Chu[4] in formulating the general case of current drive in toroidal geometry. Taguchi[5,6] has independently come to similar conclusions. These authors recognized a relation between nonohmic currents and the Spitzer–Härm[1] function. Here, by making repeated use of the self-adjoint property and by constructing certain new functions, we shall find the conductivity, with both accuracy and generality, without troubling ourselves to find an explicit solution for the distribution.

II. EXPLICIT CONSTRUCTION OF THE CONDUCTIVITY

The electron distribution function f evolves according to

$$\frac{\partial f}{\partial t} + \frac{eE}{m}\frac{\partial f}{\partial v_\parallel} = C(f,f) + C(f,f_i) + \frac{\partial}{\partial \mathbf{v}}\cdot\boldsymbol{\Gamma}, \tag{1}$$

where e/m is the electron charge/mass ratio, $C(f,f)$ represents the self-collisions of electrons, $C(f,f_i)$ represents the scattering of electrons off ion distribution f_i, $\boldsymbol{\Gamma} \equiv \mathsf{D}_{QL}\cdot\partial f/\partial \mathbf{v}$ is the wave-induced flux of electrons in velocity space, and $v_\parallel$ is the electron velocity parallel to magnetic field $\mathbf{B}$. It is generally a good approximation to assume that the bulk heating of f occurs on a time scale long compared to a bulk collision time, so that at least for thermal velocities, f is nearly Maxwellian, i.e., $f \simeq f_m \equiv n(2\pi T/m)^{-3/2}\exp(-\epsilon/T)$, where density n and temperature T may be slow functions of time, and we define energy $\epsilon \equiv mv^2/2$. Using $f = f_m(1+\phi)$, we linearize Eq. (1), where we may note too that $\partial/\partial t$ is evidently of order ϕ. We can define $\phi \equiv \phi_0 + E(g + \phi_1)$, where g is the Spitzer function from which we can find ϕ_{sp}, so we can write

$$EC(g) = -(eE/T)v_\parallel f_m, \tag{2a}$$

$$C(\phi_0) = -\frac{\partial}{\partial \mathbf{v}}\cdot\boldsymbol{\Gamma} + \left(\frac{\epsilon}{T} - \frac{3}{2}\right) f_m \dot{H}_0, \tag{2b}$$

$$EC(\phi_1) = \frac{eE}{m}\frac{\partial}{\partial v_\parallel} f_m \phi_0 + \left(\frac{\epsilon}{T} - \frac{3}{2}\right) f_m \dot{H}_1 + O(E^2), \tag{2c}$$

where $C(\phi)$ is the linearized Fokker–Planck collision operator on perturbation ϕ taking into account background Maxwellian electrons at temperature T and background infinite mass ions. The heating terms are determined by the solubility constraints obtained by multiplying Eq. (1) by ϵ, integrating over velocity, and noting that $C(\phi)$ conserves energy under the assumption of massive ions. We get

$$\frac{3}{2} n T \dot{H}_0 = \int d^3v\, \boldsymbol{\Gamma}\cdot\frac{\partial \epsilon}{\partial \mathbf{v}}, \tag{3a}$$

$$\tfrac{3}{2} n T \dot{H}_1 = E J_{\text{rf}}. \tag{3b}$$

Note that $\dot{H}_1$ and $\dot{H}_2$ are not restricted to be positive. In fact, during ramp-up, it is expected that $EJ_{\text{rf}} < 0$, as energy flows from the plasma to the poloidal field rather than the reverse, as in ohmic heating.

The solutions to Eqs. (2) are unique once we demand that the perturbation quantities each contain zero particles and zero energy. For example, the integrals over velocity space of $\phi_0 f_m$ and $\epsilon\phi_0 f_m$ vanish. Satisfaction of this demand

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determines the homogeneous solution to Eq. (2b), which is of the form $c_1 + c_2\epsilon$.

As noted above, g is the Spitzer function that has been exploited previously.[3–6] Following Antonsen and Chu,[4] we can find $J_{\rm rf}$, the rf current in the absence of an electric field, by writing

$$J_{\rm rf} = \int d^3v\, ev_\parallel f_m \phi_0 = -T\int d^3v\, \phi_0 C(g)$$
$$= -T\int d^3v\, gC(\phi_0) = -T\int d^3v\, \boldsymbol{\Gamma}\cdot\frac{\partial g}{\partial \mathbf{v}}, \qquad (4)$$

where the third equality relies upon the self-adjoint property of C, and the last equality used, Eq. (2b), employed an integration by parts and exploited the orthogonality of g to both f_m and ϵf_m.

The current J_1, proportional to both E and Γ, may be found by writing

$$J_1 = E\int d^3v\, ev_\parallel f_m \phi_1 = -ET\int d^3v\, \phi_1 C(g)$$
$$= -ET\int d^3v\, gC(\phi_1) = -T\int g\left[\frac{eE}{m}\frac{\partial}{\partial v_\parallel} f_m\phi_0 + \left(\frac{\epsilon}{T} - \frac{3}{2}\right) f_m \dot{H}_1\right] d^3v = \frac{eTE}{m}\int d^3v\, \phi_0 f_m \frac{\partial}{\partial v_\parallel} g, \qquad (5)$$

where the reasoning here is analogous to that employed in writing Eq. (4). The trouble is, of course, that while the right-hand side of Eq. (4) contains, in principle, known quantities, here ϕ_0 is not a known quantity.

Define a function ψ satisfying

$$C(\psi) = \frac{e}{m} f_m \frac{\partial}{\partial v_\parallel} g + f_m \dot{N}_\psi + \left(\frac{\epsilon}{T} - \frac{3}{2}\right) f_m \dot{H}_\psi, \qquad (6)$$

where $\dot{N}_\psi$ and $\dot{H}_\psi$ are constants determined by solubility constraints, and ψ is uniquely defined by the additional requirement that it be orthogonal both to f_m and ϵf_m. Using Eq. (6) in Eq. (5), we can write

$$J_1 = ET\int d^3v\, \phi_0\left[C(\psi) - f_m \dot{N}_\psi - \left(\frac{\epsilon}{T} - \frac{3}{2}\right) f_m \dot{H}_\psi\right]$$
$$= ET\int d^3v\, \psi C(\phi_0) = ET\int \boldsymbol{\Gamma}\cdot\frac{\partial}{\partial \mathbf{v}}\psi, \qquad (7)$$

where the second equality above exploited the orthogonality of ϕ_0 to f_m and to ϵf_m and made use of the self-adjoint property of C. The last equality utilized a similar orthogonality condition on ψ and employed integration by parts. Equation (7) explicitly describes J_1; it remains now only to calculate g and ψ. It turns out that analytic solution is possible in the limit $v \gg v_T$, where $v_T \equiv (T/m)^{1/2}$, which includes the important case of efficient lower-hybrid current drive.

III. HIGH-VELOCITY LIMIT

The Fokker–Planck collision operator in the high-velocity limit may be written as

$$C(g) = \frac{\nu_0 f_m}{2u^3}\left(-u\frac{\partial g}{\partial u} + \frac{1+Z_i}{2}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu} g\right), \qquad (8)$$

where $\nu_0 \equiv \omega_p^4 \ln\Lambda / 2\pi n v_T^3$, Z_i is the ion charge state, and where we normalized $\mathbf{u} = \mathbf{v}/v_T$ and $w = v_\parallel / v_T$ with $\mu \equiv w/u$. Terms ignored in Eq. (8) are of order u^{-2} times terms kept. Employing Eq. (8) in Eq. (2a) readily gives

$$g = [2/(5+Z_i)](e/mv_T\nu_0)\mu u^4 + O(u^2), \qquad (9)$$

and using Eq. (4), we get $J_{\rm rf}$. If we note that the power dissipated by the rf waves, P, is the integral of $\boldsymbol{\Gamma}\cdot\partial\epsilon/\partial\mathbf{v}$, we can form the efficiency ratio $J_{\rm rf}/P$ obtained previously.[7] Now using Eq. (9) in Eq. (6), which separates conveniently into Legendre components, we readily obtain

$$\psi = \frac{1}{3\nu_0}\left(\frac{e}{mv_T}\right)^2 \frac{4u^6}{5+Z_i}\left(1 + \frac{3\mu^2 - 1}{3+Z_i}\right) + O(u^4). \qquad (10)$$

By Eq. (7), this solves for $\sigma_1 = J_1/E$ in terms of $\boldsymbol{\Gamma}$. This is the principal result of this work.

For weak rf waves, it may be possible to deduce $\boldsymbol{\Gamma}$ directly from the interaction of the rf waves with the Spitzer–Härm function. For strong rf waves, but a weak dc electric field, numerical studies[8–10] conducted in the absence of an electric field could be used to determine $\boldsymbol{\Gamma}$.

Even if $\boldsymbol{\Gamma}$ is not known explicitly, it is often somewhat localized in velocity space, in which case it suffices to know only the input power, and σ_1 can be put into the form

$$\sigma_1 = \left(\frac{e^2 n}{m\nu_0}\right)\frac{4P_N}{3(Z_i+5)}\,\frac{\boldsymbol{\Gamma}\cdot\partial/\partial\mathbf{u}\, u^6[1 + (3\mu^2 - 1)/(Z_i+3)]}{\boldsymbol{\Gamma}\cdot\mathbf{u}}, \qquad (11)$$

where $P_N = P/\nu_0 n m v_T^2$ is the normalized injected rf power, and μ and u are the coordinates of resonant electrons. Note that if power were introduced into the electron distribution at several different resonant velocities, the conductivities would be additive. Conversely, if power were removed from high-energy electrons, say by synchrotron emission in extremely hot plasma, then the decrease in conductivity is also described by Eq. (11).

It is expected that the present results are valid only if resonant electrons do not run away. Other electrons may, however, run away; that would affect σ_{sp} but not σ_1. To the extent that the waves enhance the runaways, a treatment such as given in Ref. 11 is necessary. The results here apply only when $\sigma_1 E \ll J_{\rm rf}$, but may apply when $\sigma_1 E \gg J_{\rm OH}$ (unless, of course, $J_{\rm OH} \sim J_{\rm rf}$, such as would occur immediately upon turning on the rf current).

An important special case occurs for $\boldsymbol{\Gamma}$ parallel to $\mathbf{B}$ and for $w \simeq u$, which characterizes lower-hybrid current drive, where we have

$$\sigma_1 = (e^2 n/m\nu_0)[1 + 2/(Z_i+3)]J_N w^2. \qquad (12)$$

Here $J_N = J_{\rm rf}/env_T$ is the normalized rf-driven current and, to write Eq. (12) we made use of the efficiency ratio J_N/P_N.[7] Noting that $\sigma_{sp} \simeq 32\,(2/\pi)^{1/2} ne^2/m\nu_0(Z_i + 0.72)$, which is accurate at $Z_i = 1$ and $Z_i \to \infty$ and a fair approximation in between, we can write

$$\frac{\sigma_1}{\sigma_{sp}} = \left(1 + \frac{2}{3+Z_i}\right)\frac{Z_i + 0.72}{32(2/\pi)^{1/2}} J_N w^2 \equiv \alpha(Z_i) J_N w^2. \qquad (13)$$

For $Z_i = 1$, $\alpha = 0.10$; for $Z_i = 5$, $\alpha = 0.28$. As $Z_i \to \infty$, $\alpha \sim Z_i$, so that for a given rf current, the relative importance of the enhanced conductivity grows with Z_i.

IV. DISCUSSION

To get a feel for the implications of Eq. (13), consider a few simple thoughts: we can write

$$J_N \equiv \frac{I_{\rm rf}}{\pi a^2 nev_T} \simeq \frac{I_1}{50}\,\frac{1}{n_{13} T_1^{1/2} a_1^2}, \qquad (14)$$

where n_{13} is the plasma density normalized to 10^{13}/cc, T_1 is temperature in keV, a_1 is the minor radius in meters, and where I_1 is the rf-driven current I_{rf} in mega-amperes. Note that driving 10 MA of current in a tokamak with $n_{13} = T_1 = a_1 = 1$ (characteristic TFTR or TFCX start-up parameters) gives $J_N = 0.2$; doing so with $w = 7, Z_i = 1$ implies that the conductivity is doubled. Other consequences may be explored by writing the plasma circuit equations with an rf-current source.[12] For example, the larger the conductivity, the longer it will take the current to turn on in a tokamak. To turn the current on quickly (known as current ramp-up), it is necessary to overdrive it, i.e., employ $J_{rf} \gg J_{tot}$ $(t \to \infty)$. To date, there have been several experiments which have approached this regime.[13–17] Under overdrive conditions we have

$$\frac{dI}{dt} = \frac{I_{rf} - I}{L/R} \simeq \frac{I_{rf}}{L/R} = \frac{I_{rf}}{L/R_{sp}} \left(\frac{1}{1 + \alpha J_N w^2} \right), \quad (15)$$

where I is the total current and L/R_{sp} is the L/R time in the absence of rf power. Using Eq. (14), we can write a limitation on amperes increased per L/R_{sp} time, namely,

$$\frac{dI}{dt} < \frac{50\ \text{MA}}{L/R_{sp}} (n_{13} T_1^{1/2} a_1^2) \frac{J_N}{1 + \alpha J_N w^2} \leqslant \frac{2\ \text{MA}}{L/R_{sp}} \frac{n_{13} T_1^{1/2} a_1^2}{w_5^2 \alpha}, \quad (16)$$

where $w_5 \equiv w/5$ and the equality above obtains in the large overdrive limit, $I_{rf} \to \infty$. Note, however, that our result for the current does not hold in the overdrive limit if also $\sigma_1 E > J_{OH}$. Although this implies that the second inequality is not strictly derived, it would appear to be correct anyway, since the neglected terms are likely only to enhance the back-current.

Doing away with the normalizations in Eq. (16) yields an interesting form of the ramp-up limit, i.e.,

$$\frac{dI}{dt} \leqslant 76 \left(\frac{Z_i + 3}{Z_i + 5} \right) \left(\frac{c^2/v_\parallel^2}{\ln R/a} \right) n_{13}\ \text{kA/sec}, \quad (17)$$

which exhibits more clearly the independence of this limit of T and its near independence of geometry ($\ln R/a \simeq 1$). The rather spectacular PLT experiment[16] (35 kA/sec, $n_{13} = 0.1$) approaches this limit. For $n_{13} \simeq 1$, $Z_i = 1$, and $v_\parallel \simeq c/2$, we find that at least 50 sec are required to produce 10 MA. Getting around this ramp-up limit is not easy. For example, lowering $v_\parallel^2$ or increasing n both cost in current-drive efficiency. The only alternative, is to impede, or to somehow remove, fast, but not resonant, electrons. Introducing toroidal ripple to remove high-$v_\perp$, low-$v_\parallel$ electrons is a speculative possibility.

It is important to bear in mind that the conductivity enhancement discussed here is solely caused by the production by rf power on non-Maxwellian tail electrons. There is also, of course, the conductivity that arises simply through the heating of bulk electrons. To maintain by rf a steady-state current I requires a power of

$$P = \frac{150}{J_N/P_N} I_1 \left(\frac{R_3 n_{13}}{T_1} \right) \text{MW} \simeq 4 I_1 \left(\frac{R_3 n_{13}}{w_5^2 T_1} \right) \text{MW}, \quad (18)$$

where R_3 is the major radius normalized to three meters. Taking $R_3 = n_{13} = w_5 = T_1 = 1$, we would find 30 MW necessary to maintain 10 MA. The assumptions leading to $T_1 = 1$ would then be suspect. In equilibrium, in fact, we have

$$T_1 \simeq 5(P_1 \tau_E / n_{13} a_1^2 R_3) \simeq 4.5 (I_1 \tau_E / a_1 w_5)^{1/2}, \quad (19)$$

where P_1 is the power in megawatts and where τ_E is the energy confinement time in seconds. The second approximate equality emphasizes that a very small confinement time τ_E will be necessary if $T_1 \simeq 1$ is to be maintained.

V. CONCLUSIONS

In summary, what we have found here is an important transport coefficient, the conductivity of a non-Maxwellian plasma, such as occurs not in isolated equilibrium, but in equilibrium with an outside power source, e.g., rf waves that transmit power to high-energy electrons. The conductivity uncovered is rigorously derived only in the limit $E \to 0$ (as is the Spitzer–Härm conductivity). Herein lies the principal caveat, namely, that the result may be inapplicable for E large, a parameter regime which, in fact, occurs in very quick ramp-up experiments.

When the hot conductivity is a valid description, i.e., when E is small, it may be expressed in terms of the location of the resonant electrons, the amount of power transmitted to them, and the nature of the resonant interaction. It is a simple matter, therefore, to incorporate this transport coefficient into transport codes that study plasma heating by lower-hybrid or other waves.

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[1] L. Spitzer, Jr. and R. Härm, Phys. Rev. **59**, 977 (1953).
[2] N. J. Fisch, Phys. Rev. Lett. **41**, 873 (1978).
[3] S. P. Hirshman, Phys. Fluids **23**, 1238 (1980).
[4] T. M. Antonsen, Jr. and K. R. Chu, Phys. Fluids **25**, 1295 (1982).
[5] M. Taguchi, J. Phys. Soc. Jpn. **51**, 1975 (1982).
[6] M. Taguchi, J. Phys. Soc. Jpn. **52**, 2035 (1983).
[7] N. J. Fisch and A. H. Boozer, Phys. Rev. Lett. **45**, 720 (1980).
[8] C. F. F. Karney and N. J. Fisch, Phys. Fluids **22**, 1817 (1979).
[9] R. W. Harvey, K. D. Marx, and M. G. McCoy, Nucl. Fusion **21**, 153 (1981).
[10] C. F. F. Karney and N. J. Fisch, Nucl. Fusion **21**, 1549 (1981).
[11] V. S. Chan and F. W. McClain, Phys. Fluids **26**, 1542 (1983).
[12] N. J. Fisch, in *Proceedings of the 3rd Joint Varenna-Grenoble International Symposium on Heating in Toroidal Plasmas* (Commission of the European Communities, Brussels, Belgium, 1982), Vol. III, p. 841.
[13] T. Yamamoto, T. Imai, M. Shimeda, N. Suzuki, M. Maeno, S. Konoshima, T. Fujii, K. Vehara, T. Nagashima, A. Funahashi, and N. Fujisawa, Phys. Rev. Lett. **45**, 716 (1980).
[14] T. Maekawa, T. Saito, M. Nakamura, T. Cho, S. Kubo, T. Shimozuma, Y. Terumichi, Y. Hamada, and S. Tanaka, Phys. Lett. A **85**, 339 (1981).
[15] S. C. Luckhardt, M. Porkolab, S. F. Knowlton, K.-I. Chen, A. S. Fischer, F. S. McDermott, and M. Mayberry, Phys. Rev. Lett. **48**, 152 (1982).
[16] F. Jobes, J. Stevens, R. Bell, S. Bernabei, A. Cavallo, T. K. Chu, S. Cohen, B. Denne, P. Efthimion, E. Hinnov, W. Hooke, J. Hosea, E. Mazzucato, R. McWilliams, R. Motley, S. Suckewer, G. Taylor, J. Timberlake, S. Von Goeler, and R. Wilson, Phys. Rev. Lett. **52**, 1005 (1984).
[17] M. Porkolab, J. J. Schuss, Y. Takase, S. Texter, B. Blackwell, C. Fiore, R. Gandy, R. S. Granetz, M. Greenwald, D. Gwinn, B. Lipschultz, E. S. Marmar, S. McCool, D. S. Pappas, R. R. Parker, P. Pribyl, J. E. Rice, J. L. Terry, R. Watterson, S. M. Wolfe, S. F. Knowlton, S. C. Luckhardt, G. Bekefi, K.-I. Chen, A. Fischer, R. Ruhatgi, A. Bers, P. I. Bonoli, B. Coppi, R. Englade, V. Krapchev, A. Ram, R. E. Slusher, C. M. Surko, J. S. Levine, M. Read, and V. L. Granatstein, in *Proceedings of the Ninth International Conference on Plasma Physics and Controlled Nuclear Fusion Research*, Baltimore, September 1982 (IAEA, Vienna, 1983), Vol. 1, p. 227.