Nonlinear relativistic interaction of an ultrashort laser pulse with a cold plasma

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The nonlinear, relativistic dynamics that results when intense (10^{18} W/cm^2 and above) and ultrashort (one plasma period or shorter) laser pulse travels through a cold underdense plasma is investigated. Using a Lagrangian analysis of the plasma response, it can be demonstrated that the nonlinear wake, the collective dissipation, the nonlinear Compton losses, and the harmonic generation, are all determined by a finite set of integrated scalar quantities. This result holds for one-dimensional, short pulses of arbitrary amplitude, shape, and polarization, so that these very short intense laser pulses in a plasma can be viewed essentially as a quasiparticle characterized by a small set of global parameters.

I. INTRODUCTION

That electrons or photons might be accelerated by high phase velocity nonlinear plasma waves[1] focuses attention on the possibility that such waves might be generated by means of intense ultrashort laser pulses in a cold plasma. Techniques of pulse compression[2] now make possible the exploration of the laser–plasma interaction at fluxes above 10^{18} W/cm^2. Thus there is a need to develop new theoretical tools to understand the intense relativistic pulse–plasma interaction regime.

The nonlinearity parameter of an electromagnetic transverse wave with vector potential a is $\eta = ea/mc$, where c is the velocity of light, $-e$ the electron charge, and m the electron mass.[3] When $\eta \sim 1$ ($\sim 10^{18}$ W/cm^2 for visible light), the quiver velocity becomes relativistic. When the fields are so strong that the nonlinearity parameter $\eta > 1$, the dipolar approximation is no longer valid, the Lorentz equations become nonlinear because of the occurrence of a space-dependent term in the driving wave phase, and the relativistic momentum is a nonlinear function of the velocity. Yet another source of nonlinearity, appearing in the Eulerian representation but avoided in a Lagrangian analysis, arises from the convective derivative of the velocity.

This paper addresses the problem of the nonlinear relativistic interaction of a given one-dimensional, very short, electromagnetic pulse of arbitrary amplitude, polarization and shape, with a cold underdense plasma. The plasma is considered cold, in that both the electron quiver velocity in the pulse, and the electron longitudinal velocity in the wake, are larger than the electron thermal velocity.

Although there are no general methods to deal with such a nonlinear, initial value problem, a Lagrangian analysis of the electrons plasma response is powerful mathematically, and affords a clear physical picture of the nonlinear processes. Here, we shall consider the effects of a given pulse on the plasma; the effects of the plasma on the pulse occur on a time scale longer than the time scales involved in the dynamics of the plasma responses. The slow evolution of the shape of the pulse will be considered in a forthcoming paper.

The consideration of an ultrashort pulse actually simplifies considerably the mathematical analysis. This is fortunate, since, in the relativistic regime, the dipolar expansion which assumes $\eta \ll 1$ cannot be used. Moreover, the usual formalism of parametric coupling of nonlinear laser–plasma interactions cannot be put at work efficiently, because, in the regime we consider, the plasma experiences a ballistic, very short, passing perturbation rather than a harmonic one.

Consider an ultrashort pulse with a broad spectra whose mean frequency, $\bar{\omega}$, is above the plasma frequency ω_p, and whose width $\delta\omega$ is larger than ω_p. The plasma is assumed to be underdense for the main spectral components of the pulse ($\bar{\omega} \pm \delta\omega > \omega_p$). Thus there are two small parameters, $\omega_p/\bar{\omega} < 1$, and $\omega_p/\delta\omega < 1$, and, since $\delta\omega < \bar{\omega}$, the second inequality ensures the first one.

The inequality $\omega_p/\delta\omega < 1$ means that the pulse duration is shorter than the time for the electrons to set up a collective response. In other words, the electron motion *inside* the pulse is dominated by the single particle response to the transverse wave packet, and the collective longitudinal response can be treated as a perturbation in front of the driving pulse forces. On the other hand, *behind* the pulse, the plasma reorganizes itself through a purely longitudinal collective response to the initial perturbation induced by the pulse, and this result in an electrostatic wake.

A Lagrangian analysis, efficient in analyzing the weakly relativistic regime of the beat wave problem,[4] is also suited particularly to the present parameter regime of a given arbitrary short pulse on a cold plasma.

This problem of intense relativistic pulse–plasma interaction has been recently addressed with a quasistatic reduction of the Eulerian representation,[5] which assumes an ordering of the same type. These reduced Eulerian equations account for both the effect of the pulse on the plasma and the the self-consistent reaction of the plasma on the pulse, result-

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ing in wave depletion through an adiabatic redshifting.

Using the present Lagrangian method, rather than the quasistatic reduction, it can be shown that the plasma responses can be calculated in terms of a small set of global quantities; moreover, this can be accomplished for arbitrary amplitude, shape and polarization, of the pulse. From this analysis, a given pulse appears as a quasiparticle characterized by a few scalar parameters, which we calculate. The Lagrangian analysis provided here can also deal with discreteness effects, such as nonlinear Compton losses which do not appear in the fluid representation.

Suppose a wave packet propagates in the z direction, with vector potential $\mathbf{a}(t-z/c)$. Suppose further that $\mathbf{a}(t-z/c<0)=\mathbf{a}(t-z/c>T)=\mathbf{0}$. The pulse is then characterized by two time scales: its mean frequency $\overline{\omega}$ and its total phase duration T, where $T\delta\omega\sim 1$. Our study is restricted to a pulse traveling at the velocity of light, which is a good approximation for waves in an underdense plasma, and an even better approximation for very intense waves.[6] In that case, the duration seen by a rest observer is, in fact, $T\sqrt{2}$.

It turns out that, for short pulses, many important phenomena do not depend on the details of the pulse shape; rather, certain global parameters play a key role. For example, consider the total energy content per unit surface, u. This quantity can be expressed in terms of the square of the electric field ($\dot{a}^2=\dot{\mathbf{a}}\cdot\dot{\mathbf{a}}$) as follows:

$$u\equiv\epsilon_0 U\equiv\epsilon_0\int_0^T \dot{a}^2(u)du. \tag{1}$$

As will be demonstrated in the forthcoming sections, a set of integrated quantities, or what we call "global parameters," of which U is a member, turns out to characterize the plasma dynamical responses.

The paper is organized as follows: In Secs. II and III, we review the exact relativistic orbit of an electron in an arbitrary electromagnetic pulse, and the Lagrangian theory of relativistic nonlinear plasma waves. In Sec. IV, we study the relativistic interaction of a short, intense pulse with an underdense plasma. The electron response inside the pulse is calculated through an expansion that exploits the exact calculation, discussed in Sec. II, of electron motion in vacuum fields. The net effect of the pulse on the plasma depends on two quantities that describe the electron as it leaves the pulse, the exit position H, and the exit velocity V. These quantities, like U above, can be expressed as weighted integrals over the pulse.

In Secs. V and VI, on the basis of H and V, the wake structure, and collective energy losses, are calculated. Then, in Secs. VII and VIII, nonlinear Compton losses and harmonic generation are studied, and these effects are shown to be determined by two new global parameters. A Lagrangian picture of photon acceleration is briefly analyzed in Sec. IX. In Sec. X, the various density regimes for nonlinear dissipation are explored. In Sec. XI, our results and conclusions are summarized.

To simplify the presentation, in the following, rather than the I.S. of units we shall use $m=c=e=1$. Thus the nonlinearity parameter η is in fact a, and the permittivity of free space ϵ_0 is the inverse of the clasical electron radius $1/4\pi r_e$.

II. EXACT RELATIVISTIC MOTION IN A LASER PULSE

The relativistic motion of an electron in an electromagnetic pulse of arbitrary polarization and shape is integrable.[7] In this section, we will briefly review this important result which is the underpinning of the Lagrangian analysis. Integrability is a consequence of the existence of a space-time symmetry associated with the phase of the wave: Since the system is invariant with respect to translation perpendicular to the phase direction in space time, Noether's theorem assures the existence of an additional invariant associated with this symmetry. The motion of the electron in the wave is described by the Lorentz equation,

$$\frac{d\mathbf{p}}{dt}=\dot{\mathbf{a}}+\mathbf{v}\times(\mathbf{n}\times\dot{\mathbf{a}}),\quad \frac{d\gamma}{dt}=\dot{\mathbf{a}}\cdot\mathbf{v}, \tag{2}$$

where the wave travels in the direction of the unitary vector $\mathbf{n}$, the electron momentum is denoted by $\mathbf{p}$, and the velocity by $\mathbf{v}$. The dot stands for differentiation with respect to the phase argument, $(t-z)$, and γ is the relativistic energy. Multiplying the first equation by $\mathbf{n}$, using $\mathbf{a}\cdot\mathbf{n}=0$, and subtracting the second equation, we find that the quantity $\gamma-\mathbf{n}\cdot\mathbf{p}$ is a constant of the motion. In a cold plasma ($T_e\ll 511$ keV), this constant is clearly $\gamma(-\infty)-\mathbf{n}\cdot\mathbf{p}(-\infty)=1$. Thus we have

$$\gamma-\mathbf{n}\cdot\mathbf{p}=1,\quad \tau=t-z, \tag{3}$$

where τ is the particle proper time, and the equation at right is simply a proper time integration (with a suitable choice of the integration constant) of the equation at left, i.e., $\gamma=dt/d\tau$, $\mathbf{n}\cdot\mathbf{p}=dz/d\tau$. The solution of Eq. (2) can then be expressed in terms of this proper time as follows:

$$z(\tau)=\frac{1}{2}\int_0^\tau a^2(u)du,\quad \gamma(\tau)=1+\frac{1}{2}a^2(\tau), \tag{4}$$

where we have used the conservation of the canonical transverse momentum, $\mathbf{n}\times\mathbf{p}=\mathbf{n}\times\mathbf{a}$. This implicit result, if not given explicitly, is, however, *exact to all orders in a*. Alternatively, we can express this implicit solution in the form

$$t(\tau)=\tau+\frac{1}{2}\int_0^\tau a^2(u)du \tag{5}$$

from where we can deduce the following physical interpretation: As the pulse passes the electron, no final exchange of energy or momentum between the pulse and the particle takes place, and the only effect of the wave packet on the electron after the packet passes the electron is a relativistic ponderomotive displacement of the electron, $\delta z=\int_0^T a^2(u)du/2$, which occurs precisely in the direction of the wave propagation.

Note that this relativistic ponderomotive displacement, Eq. (4), is different from the nonrelativistic one because the proper time, Eq. (5), is a nonlinear function of the time.

In a plasma, this displacement will induce electric forces resulting from perturbation of the charge density. The plasma will try to restore local charge neutrality. The competition between this plasma collective response and the pulse-induced ponderomotive displacement dominates the physics of nonlinear short pulse–plasma interaction inside the pulse. Note that the first-order plasma correction to the electron motion is of order $\omega_p^2/\delta\omega^2$ and will be studied in Sec. IV, but the plasma correction to the wave dynamics is of order ω_p^2/ω^2 and thus can be neglected. Clearly, however, from this picture we can anticipate certain excellent approximations, namely, that the main result of the pulse is a displacement, and subsequent dynamics can be considered without regard to the pulse.

III. EXACT RELATIVISTIC MOTION IN INTENSE PLASMA WAVES

We consider a cold plasma perturbed in the z direction. Each electron is described by its unperturbed position z_0 and by its Lagrangian displacement $h(t,z_0)$, so that the running Eulerian position is given by $z = z_0 + h$.[8] Assuming that the initial perturbation and the subsequent dynamics does not invert the initial z_0 ordering of the electrons (no overtaking), we can apply Gauss's theorem to find the relativistic motion along the z axis:

$$\frac{dp}{dt} = -\omega_p^2 h, \quad \frac{d\gamma}{dt} = -\omega_p^2 h \frac{dh}{dt}. \tag{6}$$

Integrating the equation at right with respect to time, we find that $\gamma + \omega_p^2 h^2/2 = 1 + \omega_p^2 h_M^2/2$ is an invariant, where h_M is the maximum elongation of the considered oscillation. This allows us to introduce the proper time τ, and to express the dynamics in the proper time representation:

$$\frac{d^2h}{d\tau^2} + \omega_p^2\left(1 + \frac{\omega_p^2 h_M^2}{2}\right)h - \frac{\omega_p^4}{2} h^3 = 0. \tag{7}$$

Thus, the proper time representation of the dynamics leads to a nonlinear oscillator equation, whose solution can be put analytically in terms of Jacobian elliptic functions:

$$h(\tau) = h_M \,\mathrm{sn}(\omega\tau,k),$$
$$t(\tau) = \tau - \frac{K(k)}{\omega} + \omega_p^2 \frac{h_M^2}{2}\int_{K(k)/\omega}^{\tau} \mathrm{cn}^2(\omega u,k)\,du, \tag{8}$$

where ω and k are, in fact, functions of the amplitude, given by

$$\omega^2 = \omega_p^2\left[1 + \omega_p^2(h_M^2/4)\right], \quad k^2 = \omega_p^4 h_M^2/4\omega^2. \tag{9}$$

The initial condition is $h(t=0) = h_M$, and K is the complete elliptic integral of the first kind. It is to be noted that the integral of cn^2 in Eqs. (8) can be expressed in terms of the elliptic integral of the second kind E, i.e., $k^2\int^u \mathrm{cn}^2[v,k]\,dv = E[\mathrm{am}(u),k] - (1-k^2)u$. As for the solution in the pulse, but without the plasma, described in the previous section, the solution here is fully relativistic, and *exact to all order in* h_M, but implicit.

The nonlinear oscillator described by Eq. (7) can also be approached through perturbative methods; the well-known result of such an analysis is relevant to the weakly relativistic regime and is given by

$$h = h_M \cos(\omega t), \quad \omega = \omega_p(1 - 3\omega_p^2 h_M^2/16). \tag{10}$$

We shall now address the problem of the competition between the motion described in Secs. II and III, the regime in which the electron is inside a relativistic wave packet that propagates in a cold underdense plasma. The result of this competition will be captured by the parameters H and V, the exit position and exit velocity from the pulse. Then, behind the pulse, the motion is described by Eqs. (8), with the initial conditions being H and V.

IV. PULSE–PLASMA INTERACTION

To address the problem of the motion of a plasma electron first inside a pulse, and then behind it, let us refer to the space-time diagrams in Figs. 1 and 2. The forward and backward fronts of the pulse travel along two light characteristics, and the length of the pulse is T. When an electron enters the pulse, it is deflected according to the equations of motion as given by Eq. (2). Figure 1 corresponds to the case of circular polarization and Fig. 2 corresponds to the case of linear polarization of the wave. This deflection causes a density perturbation inside the pulse, which leads to a longitudinal electric field. This collective electrostatic field tends to pull the particle back to its unperturbed position. Accordingly, we have

$$\frac{d^2h}{d\tau^2} = \mathbf{n}\cdot[\mathbf{a}\times(\mathbf{n}\times\dot{\mathbf{a}})] - \omega_p^2\gamma h = \frac{\dot{a}^2}{2} - \omega_p^2\gamma h. \tag{11}$$

The initial condition is $h = 0$. If the pulse is short enough, the second term on the right-hand side will remain smaller than the first term, so that we can expand about the exact result of Sec. II, namely, we can expand $h = h_0 + h_1 + h_2 + \cdots$. The other dynamical quantities of the problem can be expanded similarly, $\gamma = \gamma_0 + \gamma_1 + \gamma_2 + \cdots$ and so on. The first-order correction h_1, due to the presence of plasma, will be of order $\omega_p^2/\delta\omega^2$. As discussed in the ordering of Sec. I, this is, by assumption, a small parameter; similarly, the h_2 term scales as ω_p^4, etc. The system of equations to be solved is:

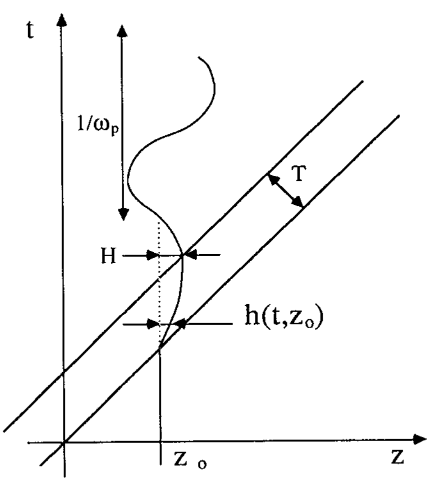

FIG. 1. Space-time diagram of the interaction of an electron initially at z_0 with a circularly polarized laser pulse. Within the pulse, the electron is deflected by the laser. Upon leaving the pulse, the electron oscillates in the self-consistent, nonlinear plasma wake.

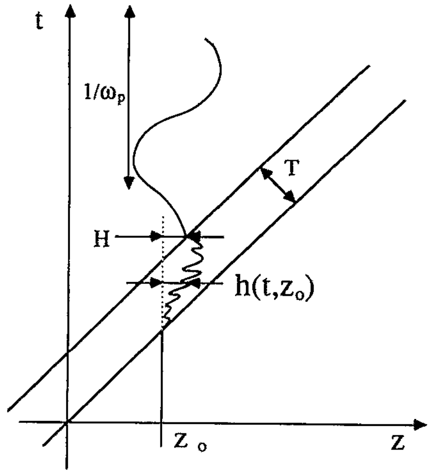

FIG. 2. Space-time diagram of the interaction of an electron initially at z_0 with a linearly polarized laser pulse. Within the pulse, the electron is deflected by the laser, where it oscillates at twice the mean pulse frequency. Upon leaving the pulse, the electron oscillates in the self-consistent, nonlinear plasma wake.

$$\frac{dp}{d\tau}=\frac{\dot{a}^2(t-h)}{2}-\omega_p^2 h\gamma, \tag{12}$$

$$\frac{d\gamma}{d\tau}=\frac{\dot{a}^2(t-h)}{2}-\omega_p^2 hp, \tag{13}$$

$$\frac{dh}{d\tau}=p, \tag{14}$$

$$\frac{dt}{d\tau}=\gamma. \tag{15}$$

The zeroth-order response is the one found in Sec. II. The invariant $\gamma_0-p_0=1$ allows to calculate the proper time $\tau=t_0-h_0$ and the zeroth-order energy and position

$$h_0=\frac{1}{2}\int_0^\tau a^2(u)du,\quad \gamma_0=1+\frac{1}{2}a^2(\tau). \tag{16}$$

The first plasma correction is governed by the system of equations

$$\frac{dp_1}{d\tau}=\frac{(t_1-h_1)\ddot{a}^2(t_0-h_0)}{2}-\omega_p^2 h_0\gamma_0, \tag{17}$$

$$\frac{d\gamma_1}{d\tau}=\frac{(t_1-h_1)\ddot{a}^2(t_0-h_0)}{2}-\omega_p^2 h_0 p_0, \tag{18}$$

$$\frac{dh_1}{d\tau}=p_1, \tag{19}$$

$$\frac{dt_1}{d\tau}=\gamma_1. \tag{20}$$

To solve this system of equations, we subtract the first equation from the second one to derive an expression for γ_1-p_1. Then, we can solve for the quantity t_1-h_1 using

$$\gamma_1-p_1=\omega_p^2\int_0^\tau h_0(s)ds,$$
$$t_1-h_1=\omega_p^2\int_0^\tau du\int_0^u h_0(s)ds. \tag{21}$$

With these expressions for the first-order quantities, Eq. (17) can be solved explicitly. After integrating twice by parts, we obtain

$$p_1=\frac{\omega_p^2}{4}\Big(\dot{a}^2\int_0^\tau du\int_0^u dv\int_0^v ds\, a^2(s)-a^2\int_0^\tau du \times\int_0^u dv\, a^2(v)-2\int_0^\tau du\int_0^u dv\, a^2(v)\Big). \tag{22}$$

After further algebra, the total effect of the pulse on an electron can be expressed as a net displacement H and as a small exit velocity V. The displacement, $H=h_0(T)+h_1(T)$, is illustrated schematically in Figs. 1 and 2. Note that the exit velocity

$$V=p_0(T)+p_1(T)/[\gamma_0(T)+\gamma_1(T)]=p_1(T)$$

is a pure plasma effect. These exit quantities can be written as

$$H=\frac{1}{2}\int_0^T a^2(u)du-\frac{1}{2}\omega_p^2\int_0^T ds\int_0^s du\int_0^u dv\, a^2(v) \times[1+a^2(s)], \tag{23}$$

$$V=-\frac{1}{2}\omega_p^2\int_0^T du\int_0^u dv\, a^2(v). \tag{24}$$

From this solution, we can define precisely the range of validity of the Lagrangian expansion. When an electron exits the pulse, its displacement, due to the plasma collective effect, must be smaller than the displacement due to the ponderomotive force described in Sec. II, i.e., $h_0(T)>h_1(T)$. In the weakly relativistic regime, $a\sim 1$, this condition is equivalent to $\omega_p^2<\delta\omega^2$. In the strongly relativistic regime, $a\gg 1$, this condition is equivalent to $\omega_p^2 a^2<\delta\omega^2$.

A complementary case, where the effect of a pulse on a plasma can be calculated analytically, is when the time ordering condition is relaxed, but the interaction is nonrelativistic, ($a\ll 1$). Admittedly, for the purpose of wake and harmonic generation, this nonrelativistic case is less important, but we present it here, for academic reasons, for the sake of completeness. Consider an electromagnetic pulse described by its vector potential $\mathbf{a}(t,z)\ll 1$. We do not assume the pulse length or the wave period to be smaller than the plasma wave period, nor do we assume the group velocity to be the light velocity. In this linear, nonrelativistic case, a is the small parameter. The Lagrangian coordinates of an electron inside the pulse are described by

$$\frac{d^2h}{dt^2}=-\mathbf{n}\cdot\Big[\mathbf{a}(t,z)\times\Big(\mathbf{n}\times\frac{\partial\mathbf{a}(t,z)}{\partial z}\Big)\Big]-\omega_p^2 h$$
$$=-\mathbf{a}(t,z)\frac{\partial\mathbf{a}(t,z)}{\partial z}-\omega_p^2 h. \tag{25}$$

This equation is valid provided that the velocity of the electron remains smaller than the velocity of light. This occurs if $a\ll 1$, otherwise proper time corrections are needed. We can then use a dipolar approximation for the first term on the right-hand side of Eq. (16), $\mathbf{a}(t,z)=\mathbf{a}(t,z_0)$, and $\partial\mathbf{a}(t,z)/\partial z=\partial\mathbf{a}(t,z_0)/\partial z$. Thus the equation to be solved becomes linear, and can be integrated directly. Let T be the pulse length. The sole effect of the pulse on the plasma, as in the previous case, is to disturb the electron position and velocity by an amount H and V given by

$$H=\frac{1}{2\omega_p}\int_0^T \sin[\omega_p(u-T)]\,\frac{\partial a^2(u,0)}{\partial z}\,du,$$

$$V=-\frac{1}{2}\int_0^T \cos[\omega_p(u-T)]\,\frac{\partial a^2(u,0)}{\partial z}\,du. \qquad (26)$$

Whereas from either a practical, or a fundamental, point of view, this nonrelativistic result may be less interesting than the relativistic one, this solution does arouse some interest. Note that, if $\omega_p T<1$, the scalings, with respect to ω_p, and a, of the incremental displacement and velocity, H and V, are different from the fully relativistic nonlinear case. This indicates that to reach perturbatively, from the nonrelativistic case, the relativistic case may require a large number of terms in the expansions that rely upon the smallness of ω_p or a.

V. NONLINEAR WAKE

Using the results of the previous sections, the structure of the nonlinear wake behind the pulse can be calculated easily. First, let us investigate the weakly relativistic response described by Eq. (10). The weakly relativistic Lagrangian displacement behind the pulse can be written as

$$h(t,z_0)=H\cos[\omega(t-z_0)]+(V/\omega)\sin[\omega(t-z_0)],$$
$$t>z_0, \qquad (27)$$

where z_0 is the initial unperturbed position of the electron. The nonlinear frequency is given by

$$\omega(H,V)=\omega_p(1-3\omega_p^2h_M^2/16),\quad h_M^2=H^2+V^2/\omega_p^2. \qquad (28)$$

It can be verified that Eq. (27) fulfills both the dynamical equation of Sec. III and the initial condition just behind the pulse calculated in Sec. IV. The Eulerian density perturbation can be expressed on the basis of the unperturbed density $n(z_0)$ and h, namely

$$n(z,t)=\int dz_0\, n(z_0)\delta[z-z_0-h(t,z_0)]$$
$$=\int dz_0\int\frac{dk}{2\pi}\,n(z_0)e^{-ik[z-z_0-h(t,z_0)]}, \qquad (29)$$

where δ is the Dirac function. The exponential of the oscillating Lagrangian position in Eq. (27) can be expanded in terms of Bessel functions of the first kind J_n. For $t>z$, we obtain, behind the pulse,

$$n=\int_{-\infty}^{t}dz_0\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\sum_{m=-\infty}^{m=+\infty}\sum_{n=-\infty}^{n=+\infty}i^m n(z_0)J_n\left(\frac{kV}{\omega}\right)$$
$$\times J_m(kH)e^{-ik(z-z_0)}e^{i(m+n)\omega(t-z_0)}. \qquad (30)$$

Note that after at least one plasma period, $h(z_0,t)$ is unaffected by the electrons just leaving the pulse. Thus the z_0 integral can be extended from $-\infty$ to $+\infty$. It then becomes apparent that Eq. (30) simplifies, because the integral over z_0 can be performed, giving the Fourier transform of the unperturbed density profile $n(z_0)$. Thus Eq. (30) can be put into a particularly convenient form to study the effect of inhomeneous density distributions, such as might arise, for example, in tapering the plasma in wake field accelerator schemes. Here, we shall restrict our attention to the case of uniform unperturbed density, so that the wake can be put in the form of a sum of harmonic waves, namely

$$\frac{n}{n_0}=\sum_m\sum_n i^nJ_{m-n}(mV)J_n[m\omega(H,V)H]$$
$$\times e^{im\omega(H,V)(t-z)}. \qquad (31)$$

The use of the weakly relativistic approximation for the electron oscillation, Eq. (10), means that Eq. (31) is valid where the sum over m is dominated by the small m.

Consider now the case of highly relativistic electrons described by Eq. (8). Here, it appears that the wake structure can be expressed as an implicit function of $(t-z)$. Rather than using the harmonic representation of the Dirac distribution in Eq. (29), we first make use of the representation

$$n(z,t)=\int dz_0\, n(z_0)\delta[z-z_0-h(t,z_0)]$$
$$=n_0\left(1+\frac{\partial h}{\partial z_0}\right)^{-1}. \qquad (32)$$

In order to simplify, we retain only zeroth-order plasma effects inside the pulse, namely, we use $H=h_0$ and $V=0$. The Lagrangian compression can then be calculated with the help of the chain rule for differentiation applied to Eq. (8). After some algebra, one obtains

$$\frac{n}{n_0}=\left(1-\frac{2\omega H\,\mathrm{cn}(\omega\tau,k)\mathrm{dn}(\omega\tau,k)}{2+\omega_p^2H^2\,\mathrm{cn}^2(\omega\tau,k)}\right)^{-1}, \qquad (33)$$

where τ is an implicit function of $t-z$ given by

$$t-z=\tau-K(k)/\omega-H\,\mathrm{sn}(\omega\tau,k)$$
$$+\frac{\omega_p^2H^2}{2}\int_{K(k)/\omega}^{\tau}\mathrm{cn}^2(\omega u,k)du. \qquad (34)$$

Overtaking between neigboring electrons occurs when the velocity of an oscillating electron reaches the phase velocity of the wave.[9] The overtaking in the Lagrangian picture leads to wave breaking in the Eulerian picture, which results in the production of fast electrons. Because we have assumed that the group velocity of the pulse is equal to the velocity of light, overtaking clearly is impossible. Thus the phenomenon of wave breaking cannot be addressed within the framework of the present model. In fact, in the strongly nonlinear regime, $a\gg1$, a precise and meaningful definition of the group velocity is still lacking.

We have shown in Sec. III that $\gamma+\omega_p^2h^2/2$ is an invariant, so that, behind the pulse, there is a maximum electric field E_M, and a maximum relativistic energy γ_M of an electron during its nonlinear oscillation in the plasma wave. These quantities are given by

$$E_M^2=2\omega_p^2(\gamma_M-1)=\omega_p^4H^2+\omega_p^2V^2. \qquad (35)$$

Note that, here, E_M is the peak electric field behind the pulse and should not be confused, in the literature, with the maximum electric field of an infinite nonlinear plasma wave,[9] arising from the wave-breaking limit. Although, as mentioned above, the wave-breaking limit is not considered in the present model, it so happens that even if the group velocity of the pulse were smaller than c, Eq. (35) remains valid.

This happens because the maximum values of these quantities are reached here just behind the pulse, where we solved only an initial value problem, prior to the completion of a plasma oscillation. On the other hand, the difficulties that arise in overtaking occur only upon the completion of one plasma oscillation, and there arise particular difficulties in studying the steady state of nonlinear plasma waves.

VI. NONLINEAR COLLECTIVE ENERGY LOSSES

If one views the pulse as a ballistic perturbation sweeping a one-dimensional array of nonlinear oscillators, one can imagine a continuous transfer of energy from this ballistic perturbation to the array of oscillators. The energy balance of such a system can be performed straightforwardly. Compare the plasma energy before and after the pulse passing; the difference is the work of the pulse on the plasma,

$$\frac{dU}{dt} = -\frac{\omega_p^4}{2} H^2 - \frac{\omega_p^2}{2} V^2. \tag{36}$$

The nonlinear character of this dissipation is obvious from the fact that it is proportional to the square of the electron density. A linear, resonant dissipative process would incur this loss of energy through a decrease in the amplitude of the wave at constant frequency. However, since the processes involved here in this energy exchange are nonresonant, the wave action is conserved, and that the interaction result in a slowing down of the pulse, i.e., a decrease in the mean frequency $\overline{\omega}$.

VII. NONLINEAR COMPTON SCATTERING

Competing with the losses calculated in Sec. VI, due to a coherent transfer of energy to longitudinal waves, is an incoherent transfer of energy to transverse waves. The latter transfer can dominate at low density. When an electron enters the pulse, it is accelerated in the transverse direction, in the process radiating part of the energy in the pulse. This spontaneous process is, in fact, nonlinear Compton scattering. The single particle dissipation is given by[10]

$$\frac{d\gamma}{d\tau} = -\frac{2r_e}{3} \gamma\left(\frac{d\mathbf{p}}{d\tau}\cdot\frac{d\mathbf{p}}{d\tau} - \frac{d\gamma}{d\tau}\cdot\frac{d\gamma}{d\tau}\right). \tag{37}$$

Neglecting both the plasma response, and the radiation reaction force, inside the pulse we can exploit the fact that $t - z$ is the proper time τ. The total energy loss S, resulting from one electron transit through the pulse, can then be put in the form

$$S = \frac{2r_e}{3}\int_0^T \dot{a}^2(u)\left[1 + \frac{a^2(u)}{2}\right]du. \tag{38}$$

The bracketed term in the integral above accounts for the nonlinear, relativistic modifications of the usual Thomson cross section. To obtain the pulse energy loss due to Compton scattering, we assume the existence of a decorrelation mechanism and sum the contributions from all electrons entering the pulse. The additional loss term, which would complete the coherent losses in Eq. (36), is then given by

$$\frac{dU}{dt} = -\omega_p^2 S. \tag{39}$$

As might be imagined, these incoherent losses tend, at very low density, to dominate dissipation due to collective effects.

VIII. APPLICATION TO RELATIVISTIC HARMONIC GENERATION

A short intense pulse produces transverse harmonic fields because of the nonlinear response of the plasma.[5] This occurs in addition to the effects considered above: nonlinear Compton scattering and the generation of a longitudinal wake. In harmonic generation, the polarization of the wave plays an important role. In a linearly polarized pulse, the quantity a^2 is characterized by two time scales, $2\overline{\omega}$, and $\delta\omega$; on the other hand, a circularly polarized pulse, in effect, characterizes a^2 with only one time scale, namely, the pulse width, $\delta\omega$. This difference between the circular and linear polarization is depicted schematically in Figs. 1 and 2. Linear polarization, because a^2 contains the time scale $2\overline{\omega}$, gives rise to harmonic generation at $3\overline{\omega}$.

The transverse current, due to the electrons response inside the pulse, is given by

$$\begin{aligned}\mathbf{j} &= -\epsilon_0\omega_p^2\int dz_0\,\delta(z - z_0 - h)\,\frac{\mathbf{a}}{\gamma}\\ &= -\epsilon_0\omega_p^2\,\frac{\mathbf{a}}{\gamma}\left(1 + \frac{\partial h}{\partial z_0}\right)^{-1}.\end{aligned} \tag{40}$$

The ponderomotive perturbation of the density inside the pulse, $(1 + \partial h/\partial z_0)$, can be calculated using the results of Sec. IV. Expanding in the density, the leading terms for the proper time of an electron initially at z_0 are given by

$$\tau = t - z_0 - h - \frac{\omega_p^2}{2}\int_0^{t-z_0-h} du \int_0^u dv \int_0^v ds\, a^2(s). \tag{41}$$

Now use the identity $\partial h/\partial z_0 = (dh/d\tau)(\partial\tau/\partial z_0) = p\partial\tau/\partial z_0$, to find, after some algebra, the density perturbation

$$\left(1 + \frac{\partial h}{\partial z_0}\right)^{-1} = \gamma\left(1 - \frac{\omega_p^2}{2}\int_0^\tau du \int_0^u dv\, a^2(v)\right). \tag{42}$$

Note that using Eq. (42), the current defined in Eq. (40) appears to be a sum of the usual linear reactive contribution and an active nonlinear contribution. In the case of linear polarization, the nonlinear contribution contains the third harmonic of the original pulse. Let $\mathbf{a}_1$ be the harmonic vector potential radiated by this nonlinear part of the current, where $\mathbf{a}_1$ obeys the inhomogeneous wave equation,

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right)\mathbf{a}_1 = -\mathbf{a}\,\frac{\omega_p^4}{2}\int_0^{t-z} du \int_0^u dv\, a^2(v), \tag{43}$$

which can be solved using the Green's function for the one-dimensional wave operator. The Green's function is 0 everywhere except in the backward light cone denoted, in Fig. 3, by C_+ and C_-. In this causal light cone, the Green's function takes the value 1/2. Thus, to solve Eq. (43), the integral over the Green's function is restricted to the intersection of this light cone with the support of j_1, shown as the shaded area in Fig. 3. Assume now that the interface between the vacuum and the plasma is located at $z = 0$, make a change of

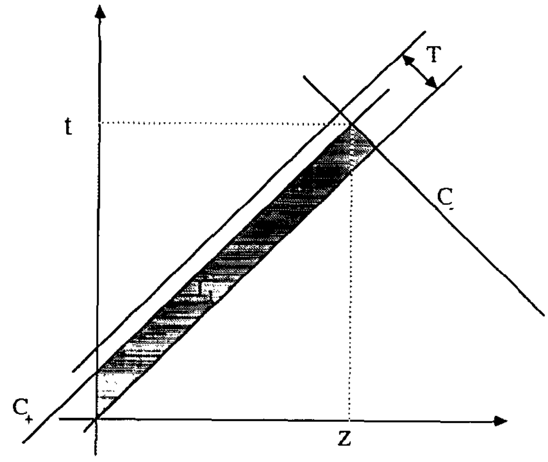

FIG. 3. Only events inside the causal cone (C_+, C_-) can influence the point (t,z). Inside the causal cone, only the shaded area, where the interaction between the incident pulse and the plasma takes place, can support the nonlinear current source j_1.

variables from (z,t) to $(z-t,z+t)$, whose Jacobian is 2, and find for the first-order plasma induced harmonic pulse

$$\mathbf{a}_1(t,z) = -\frac{\omega_p^4}{2}(z+t)\int_0^{t-z}\mathbf{a}(w)dw \times \int_0^w du \int_0^u dv\, a^2(v). \quad (44)$$

This result can be simplified if we write $z+t=2t-(t-z)$, and then note that the first-order harmonic response appears as the sum of a growing propagating term and a constant propagating one, i.e., function of $(t-z)$ only. After few plasma periods, the constant response is dominated by the growing term. The associated unstable electric field $-\partial(\mathbf{a}_1/\partial t)$ can then be written as

$$\frac{\partial \mathbf{a}_1}{\partial t} = \omega_p^4 t\mathbf{a}(t-z)\int_0^{t-z} du \int_0^u dv\, a^2(v). \quad (45)$$

To describe the power conversion due to this unstable term, we introduce the integrated quantities R, where

$$R = \int_0^T dl\, a^2(l)\left(\int_0^l dw \int_0^w dv\, a^2(v)\right)^2. \quad (46)$$

The energy, in the harmonic pulse grows as $\omega_p^8 t^2 R$, representing a power loss, competing with the terms in Eq. (36), which may be written as

$$\frac{dU}{dt} = -2\omega_p^8 tR. \quad (47)$$

The integral R captures the information relevant to the harmonic generation effect. Evidently, the harmonic power conversion scales as ω_p^8, or with the fourth power of the plasma density.

Note that, by varying the degree of polarization of the original pulse, the mean frequency of the harmonic unstable pulse can be tuned between $\overline{\omega}$ and $3\overline{\omega}$.

IX. APPLICATION TO RELATIVISTIC PHOTON ACCELERATION

In an optimally efficient scheme for relativistic photon acceleration, a leading pulse, depicted as "P1" in Fig. 4, delivers energy to the plasma wake, and this energy is entirely reabsorbed by a lagging or accelerated pulse, depicted as "P2." Complete pulse reabsorption, leaving no energy in the plasma, implies

$$H_1 = H_2, \quad V_1 = V_2, \quad (48)$$

where H_i and V_i are the position and velocity shifts following pulse i. Since the energy exchange process does not involve resonant processes, it follows that the pulse action is an adiabatic invariant. Hence, in absorbing the wake energy, the lagging pulse experiences a mean frequency upshift,[11] rather than a growth in amplitude.

The salient parameters of such a photon accelerator are the mean frequency upshift of the accelerated pulse and the delay between the two pulses. A precise calculation of the mean frequency upshift requires a study of the adiabatic transfer of energy from an oscillating electron to a pulse, which is beyond the scope of the present study. We can, however, calculate directly the optimally efficient time delay in an underdense plasma.

In the nonrelativistic regime this delay, is simply $n+1/2$ times the plasma period, where n is an integer. On the other hand, in the relativistic regime, which is particularly important because of the possibility of large density gradients, proper time corrections are needed to calculate this delay.

Consider an underdense plasma, so that the exit velocities V_i can be neglected ($V_i=0, H_i=H$). The condition for an efficient transfer of energy then becomes $h(D) = -H$, where D is the electron proper time delay between the two pulses. Using now Eq. (8), the condition can be put in the form

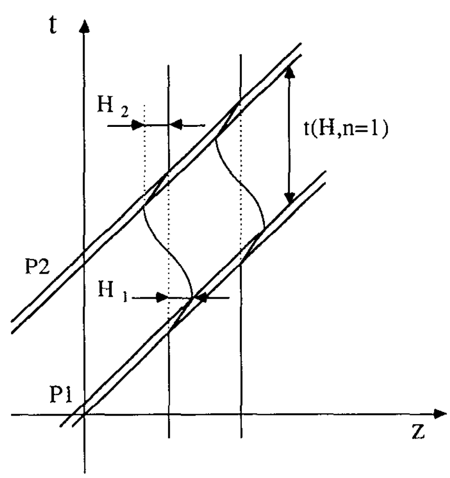

FIG. 4. Two pulses P_1 and P_2 such that all the energy transferred from pulse P_1 to the plasma longitudinal relativistic wake is reabsorbed by pulse P_2 after 1/2 a relativistic plasma period.

$$\omega D = 2nK(k), \quad \omega^2 = \omega_p^2 (1 + \omega_p^2 H^2/4),$$
$$k^2 = \omega_p^4 H^2/4\omega^2. \tag{49}$$

This condition on the proper time of the electron becomes a condition on the time delay between the two pulses $t(H,n)$, such that

$$\omega t(H,n) = 2nK(k) + \frac{\omega_p^2 H^2}{2} \int_0^{2nK(k)} \mathrm{cn}^2(u,k)du, \tag{50}$$

which gives the condition for the time delay that allows a photon accelerator, operating in an underdense plasma but in the relativistic wake regime, to be maximally efficient.

X. DENSITY REGIMES FOR DISSIPATION

The previous sections show that, to lowest order in ω_p, there are three main channels for nonlinear energy dissipation, and, for very short pulses ($V = 0$), the total dissipation is

$$\frac{dU}{dt} = -\frac{\omega_p^4}{2} H^2 - \omega_p^2 S - 2\omega_p^8 tR. \tag{51}$$

From Eq. (51), we discern several different density regimes of dissipation. To investigate the boundaries between these regimes, consider, for $a > 1$, the scaling of the parameters, H, S, and R:

$$H \sim a^2 \delta\omega^{-1}, \quad S \sim a^4 \delta\omega^{-1} \overline{\omega}^2 r_e, \quad R \sim a^6 \delta\omega^{-5}. \tag{52}$$

We expect, then, that incoherent spontaneous losses will dominate coherent longitudinal dissipation for densities as low enough that

$$r_e > \left(\frac{\omega_p}{\overline{\omega}}\right)^2 \left(\frac{c}{\delta\omega}\right), \tag{53}$$

where r_e is the classical electron radius. Harmonic generation may dominate wake generation at higher density and higher laser intensity, but here there is an important dependence upon the duration of the process or the length of the plasma. After a time t, the losses due to harmonic generation dominate if

$$\omega_p t > \left(\frac{\delta\omega}{\omega_p}\right)^3 \left(\frac{mc}{ea}\right)^2. \tag{54}$$

At the present, state of the art of short, intense laser pulse technology, losses incurred through wake generation always dominate, and losses due to harmonic generation are negligible.

XI. SUMMARY AND DISCUSSION

We have shown in the previous sections that many of the effects of an intense, short laser pulse on a cold, underdense plasma can be calculated by solving the equation of motion of each plasma electron, and then summing the effects of all these motions. The result, valid to all order in the intensity a, gives the collective transverse and longitudinal induced fields. The nonlinear energy losses are mainly due to the longitudinal-induced wake, which scales as ω_p^4. The losses due to the transverse-induced response, which happens at high frequency only in the case of non circular polarization, scales as ω_p^8. In the weakly relativistic regime, these losses are negligible compared to the longitudinal losses. Incoherent nonlinear Compton scattering becomes important only at very low density.

We have been mainly concerned by the effect of a given pulse on the plasma; what we have not considered in detail is the self-consistent problem that considers also the effect of the plasma on the pulse. We show, however, that these effects occur on a longer time scale than do the effects considered above, and so may be neglected.

Two time scales are associated with the modification of the pulse due to the plasma: one, a nonlinear time scale associated with the energy losses; and, two, a linear time scale associated with the dispersive dynamics of the pulse envelope. The time scale t_L associated with the linear dispersive evolution of the pulse shape arises because of the dependence of the group velocity on the frequency, $\delta\omega t_L\ \partial v_g/\partial\omega \sim c/\delta\omega$. Using the usual dispersion relation for a cold plasma, we find, $\omega_p t_L \sim (\overline{\omega}/\delta\omega)^2 (\overline{\omega}/\omega_p)$. This time scale is larger than the time scales associated with the processes studied above. To evaluate the nonlinear time scale t_N, associated with the longitudinal energy losses, we use the global energy balance, and we find $\omega_p t_N \sim (\overline{\omega}/\omega_p)^2 (\delta\omega/\omega_p)$. This time scale is also longer than those associated with the processes studied in this paper.

The above result can be compared with that of the quasistatic reduction of the Eulerian representation.[5] As far as harmonic generation is concerned, in this reduced description, the nonlinear current can be expressed as a function of the electrostatic potential ϕ so that the second term of Eq. (43) can be written in terms of this potential as $\omega_p^2 \mathbf{a}\phi$. Solving the quasistatic equation for the potential gives

$$\phi = -\frac{\omega_p^2}{2} \int_0^{t-z} du \int_0^u dv\, a^2(v),$$

so that Eq. (44) describing harmonic generation from a short pulse can be recovered within the quasistatic framework. As far as wake generation is concerned, the comparison is less straightforward because the quasistatic equation has been mainly solved for square pulses[5] and the quantities H and V have not yet been identified in this Eulerian framework. However results for the short square pulse case are in agreement with the more general results presented here.

In summary, on the basis of a fully relativistic Lagrangian density expansion, we have demonstrated that the effects of an ultrashort intense laser pulse on a cold plasma can be captured, in fact, by a small set of integral parameters, H, V, R, S, and U. The general formula for the nonlinear wake, nonlinear Compton losses, and harmonic generation have been obtained in term of these global quantities. In addition, what emerges from these new results, and from the Lagrangian method for obtaining them, is a more clear physical picture of the nonlinear processes involved in the ultrashort, pulse–plasma interaction.

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