Free energy in plasmas under wave-induced diffusion

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When waves propagate through a bounded plasma, the wave may be amplified or damped at the expense of the plasma kinetic energy. In many cases of interest, the primary effect of the wave is to cause plasma diffusion in velocity and configuration space. In the absence of collisions, the rearrangement of the plasma conserves entropy, as large-grain structures are mixed and fine-grain structures emerge. The maximum extractable energy by waves so diffusing the plasma is a quantity of fundamental interest; it can be defined, but it is difficult to calculate. Through the consideration of specific examples, certain strategies for maximizing energy extraction are identified.

I. INTRODUCTION

Consider the passage of a pulse of waves through a slab of plasma, in which the pulse first enters the otherwise undisturbed plasma slab, and then emerges from the slab, leaving the plasma entirely. The plasma is rearranged in response to the wave pulse, and the amount of energy in the plasma may change. Since the total energy is conserved, the wave may have either less or greater energy as it emerges from the slab. What is of interest in this paper is the maximum amount of energy, under certain practical constraints, that may be extracted from the plasma.

Under different circumstances, there would be different constraints governing the ways in which the plasma could be rearranged by the wave. If the plasma is collisionless, each species must conserve its volume in the six-dimensional phase space of velocity and configuration space. In such a case, the maximum extractable energy from the plasma may be obtained through the so-called "Gardner restacking algorithm,"[1,2] which we restate here.

Suppose, the six-dimensional phase space is chopped into N little bins of volume $V=\Delta x\Delta y\Delta z\Delta v_x\Delta v_y\Delta v_z$. Each bin has an average energy $\epsilon_i=m(v_{xi}^2+v_{yi}^2+v_{zi}^2)/2$, where i varies from 1 to N. Let us order the bins such that if $i>j$, then $\epsilon_i>\epsilon_j$. Each bin also has an initial density f_{0i}, so that, initially, the ith bin has energy $f_{0i}\epsilon_iV$. The total energy of the plasma is then found just by summing $f_{0i}\epsilon_iV$ over the repeated index. Now consider a rearranging of the plasma that respects only that each bin density be preserved throughout the flow in the six-dimensional phase space. Then, let us restack (reorder) the f_{0i} in order of decreasing density, i.e., such that the set of final densities, $\{f_{fi}\}$, is a permutation of the set of initial densities $\{f_{0j}\}$, such that if $i>j$, then $f_{fi}<f_{fj}$. The maximum extractable energy under this "Gardner restacking" is clearly

$$W_{\mathrm{G}}=\sum_i (f_{fi}-f_{0i})\epsilon_iV, \tag{1}$$

and we choose to call W_{G} the "Gardner free energy." Here, the plasma is rearranged incompressibly in the six-dimensional phase space such that the bins with the largest densities eventually occupy the lowest energy states, but each bin density is preserved. In practical situations, however, it would be highly unlikely ever to realize the maximum energy extractable under Gardner restacking.

Less energy, in general, might be extracted from a plasma if the wave were constrained to interact with particles only by diffusing them in phase space from regions of high phase-space density to regions of lower phase-space density. This is the so-called quasilinear diffusion[3] that occurs in resonant wave–particle interactions, and, in many circumstances, this would be the more likely type of plasma response to waves. The purpose of this paper is to examine the maximum extractable energy from the plasma slab under diffusive rearrangement. The restacking solution to the maximal extraction problem under merely the incompressibility constraint is easily and succinctly stated in Eq. (1). In contrast, under the diffusive constraint that interests us here, obtaining the maximum energy extractable is far more challenging. In fact, it turns out that a general solution to this problem is surprisingly difficult, so much so that, rather than a solution, what is offered here is mainly to identify, to motivate, and to pose precisely what is actually a quite fundamental problem of practical interest, and to indicate certain features that the solution must have.

The question of "free energy" in a plasma, different under different constraints, has been addressed in a number of other contexts. Rosenbluth and Rutherford[4] define three sources of free energy: (1) expansion free energy, arising from nonuniform density and temperature; (2) velocity space free energy, arising from non-Maxwellian velocity distributions; and (3) magnetic free energy, arising from the plasma diamagnetism. The free energy tapped in wave-driven plasmas, however, is not strictly considered in generality in any of the above three categories; in general, the free energy tapped by waves relies at once on the details of the distribution function in both velocity and configuration space, something of a combination of categories (1) and

(2). Hence, a consideration of the free energy in the expansion of a non-Maxwellian plasma could be more complicated than either the velocity space instabilities of a non-Maxwellian plasma or the spatial expansion of a Maxwellian plasma.

Even the spatial expansion of a Maxwellian plasma is more complicated when the velocity space features are considered. For example, more energy can be released, in principle, if specifically the energetic particles in the dense region diffuse with the less energetic particles in the underdense region. Waves can indeed diffuse particles in such a manner, and, even for Maxwellian velocity distributions, there could be a greater number of energetic particles in the denser region than slower particles in the underdense region. Just considering the expansion of a Maxwellian distribution[4] would indicate less free energy than might be extracted in the above manner.

The questions raised here must also be distinguished from the recent, interesting considerations of free energy in a plasma by Morrison and Pfirsch,[5–7] who derive expressions for the free energy in perturbations about plasma equilibria. What Morrison and Pfirsch[5] do is to impose precisely on the perturbation the constraint of "dynamic accessibility," taking into account the effects of resonant particles. With improvement on expressions for the energy in various plasma modes, more precise considerations of plasma stability may be made. Such calculations are relevant in initial value problems, where the growth of a plasma mode relies on the free energy available to the mode.

In the problem contemplated here, where the wave pulse enters and leaves the plasma from a boundary, such initial value problems do not arise. Since the plasma is driven externally, whether the waves employed would be stable or unstable were the plasma infinite and homogeneous does not turn out to be directly relevant. Also, what is of interest here is a global quantity, the maximum energy extractable, which is something that could not be approached from the viewpoint of local stability criteria.

This particular posing of the energy extraction problem, under the diffusive constraint, arose recently in the practical problem of extracting energy from the population of α particles that are produced in a fusion reactor.[8] Extracting energy from the α particles is thought to be valuable for two reasons: (i) less free energy in the α particles reduces the likelihood of unwanted deleterious instabilities of the plasma, and, (ii) the energy channeled into waves may perform certain useful tasks. One problem is to find the waves that might extract the largest amount of this free energy, and a statement of the maximum extractable energy would be useful in evaluating specific extraction schemes. As it turns out, none of the calculations in the literature of plasma free energy quite addresses the kind of problem that needs to be solved to find this maximum extractable energy.

The diffusion that we consider here is *formally* nonlocal; particles can diffuse from any high-density region of phase space to any low-density region of phase space, even if the two regions are not contiguous. As we show later, such a posing of the problem is entirely consistent with particles *physically* diffusing only locally: it turns out that when viewed on a microscopic scale the diffusion is local; when viewed on a more coarse scale, the diffusion can appear as nonlocal. Hence, the mathematical posing of the problem does not disallow nonlocal diffusion. Note that for the Gardner restacking solution it matters not whether the flow in phase space is local or not, since nonlocal rearrangements of density can always be constructed from local exchanges.

The paper is organized as follows: Before stating formally the exact problem to be solved, we introduce a number of examples: In Sec. II, we formulate this problem by taking as an example stimulated emission by a set of lasers. Here, the energy levels are discrete rather than continuous, as in a plasma. In Sec. III, certain insights into optimizing energy extraction are drawn through considering numerical examples of the case of discrete energy levels. In Sec. IV, we give a precise statement of the problem both for discrete and continuous distribution functions. In Sec. V, we discuss how the wave diffusion describes a subset of all possible phase-space conserving rearrangements, and so is consistent with incompressibility. In addition, we prove that the maximum energy released under diffusive rearrangements is bounded by the Gardner free energy, W_G. In Sec. VI, the bump-on-tail distribution is considered in one dimension. In Sec. VII, the conclusions and main results are discussed briefly.

II. STIMULATED EMISSION BY A SET OF LASERS

The problem posed in plasmas can similarly be posed with respect to stimulated emission by a set of lasers, and, for didactic reasons, a discussion with respect to the rearrangement by lasers of populations of discrete energy levels ought to precede discussion concerning the rearrangement of a plasma continuum.

Suppose, first, an atomic system with just three energy levels, the ground state at energy ϵ_0, the first excited state at ϵ_1, and the second excited state at ϵ_2, with initial population densities of, respectively, N_0, N_1, and N_2. Suppose further the availability of three lasers with frequencies ν_{10}, ν_{20}, and ν_{21}, that, respectively, can stimulate transitions between the first level and the ground state, the second level and the ground state, and the second level and the first level. What these lasers can do, essentially, is to exchange the population in any one level with the population in any other level. This happens if the resonant frequency is applied for just the right amount of time (the so-called "π phasing"). The question we pose is the following: What can be done with these lasers to extract the maximum energy from the atomic system?

The maximum energy extractable is, in this case, just the Gardner solution: Suppose for example that $N_1 > N_2 > N_0$, then the Gardner solution gives

$$\Delta W = \epsilon_0 N_0 + \epsilon_1 N_1 + \epsilon_2 N_2 - (\epsilon_0 N_1 + \epsilon_1 N_2 + \epsilon_2 N_0), \quad (2)$$

where the populations were restacked so that the largest population now occupies the ground state. The set of lasers available can accomplish this in two steps:

$$\text{Exchange case:}\quad \begin{array}{r} \\ \text{initial} \\ \text{step 1} \\ \text{step 2} \end{array} \begin{pmatrix} \epsilon_0 & \epsilon_1 & \epsilon_2 \\ N_0 & N_1 & N_2 \\ N_1 & N_0 & N_2 \\ N_1 & N_2 & N_0 \end{pmatrix},$$

where, in step 1, applying frequency ν_{10} for just the right amount of time exchanges the populations in energy levels 0 and 1, and then, in step 2, applying frequency ν_{21}, again for just the right amount of time, exchanges the populations in levels 1 and 2. This sequence for extracting the maximum energy can be expressed by the notation "(ν_{10},ν_{21})."

Note that these steps are not commutative, i.e., $(\nu_{10},\nu_{21})\neq(\nu_{21},\nu_{10})$. Thus, to extract the maximum energy, the lasers must be both carefully synchronized to achieve the π phasing and ordered in the correct sequence.

Now suppose that timing the lasers so precisely as to exchange populations is very difficult or impossible—perhaps the atoms cannot all be stimulated at once—then one can imagine that all each laser can do is to tend to equalize populations rather than to exchange them. In the previous example, what would be accomplished by (ν_{10},ν_{21}) is

Diffusive case:

$$\begin{array}{r} \\ \text{initial} \\ \text{step 1} \\ \text{step 2} \end{array} \begin{pmatrix} \epsilon_0 & \epsilon_1 & \epsilon_2 \\ N_0 & N_1 & N_2 \\ \dfrac{(N_1+N_0)}{2} & \dfrac{(N_1+N_0)}{2} & N_2 \\ \dfrac{(N_1+N_0)}{2} & \dfrac{(N_1+N_0)}{4}+\dfrac{N_2}{2} & \dfrac{(N_1+N_0)}{4}+\dfrac{N_2}{2} \end{pmatrix}.$$

Again, these steps are not commutative, i.e., $(\nu_{10},\nu_{21})\neq(\nu_{21},\nu_{10})$. Note, too, that if we denote the final population levels as N_0', N_1', and N_2', it is not guaranteed that $N_0'\geqslant N_1'\geqslant N_2'$; it may take further applications of these lasers to converge to such a solution.

In the case where the lasers exchanged populations, finding the optimal laser sequence was easy; not only was the minimum energy configuration apparent immediately—through the Gardner reordering—but, even if we did not guess immediately this state, we could simply have examined all accessible states, which are just permutations of the populations. With only three populations that is only $3!=6$ states. In the case here, where the lasers diffuse rather than exchange populations, to examine all accessible states would not be so easy, even with just three energy levels, since there are now an infinite number of accessible states!

III. NUMERICAL EXAMPLE FOR THREE-LEVEL SYSTEM

Suppose a three-level system, with a density vector $\mathbf{N}\equiv(N_0,N_1,N_2)$, and an energy vector $\epsilon\equiv(\epsilon_0,\epsilon_1,\epsilon_2)$. Thus, the energy is $W\equiv\mathbf{N}\cdot\epsilon$. For example, say that the energy vector is $\epsilon=(0,1,4)$. If the initial density vector is $\mathbf{N}^0=(0,2,5)$, then the initial energy is $W^0=\mathbf{N}^0\cdot\epsilon=22$. Let us try to solve, for this example, for the minimum energy under diffusive rearrangement.

One might try to apply frequency ν_{20} first, since that releases the maximum energy, ten units; if one does so, then it is noticed that the population in level two is still larger than in level one, so that frequency ν_{21} should be applied to release more energy. The steps are

$$\begin{array}{lll} & & \\ \text{sequence 1} & \text{initial} & W^0=22 \\ (\nu,_{20},\nu_{21}) & \text{step 1} & W^1=12 \\ & \text{step 2} & W^2=\frac{45}{4} \end{array} \begin{array}{c} \begin{array}{ccc} \epsilon_0=0 & \epsilon_1=1 & \epsilon_2=4 \end{array} \\ \begin{pmatrix} 0 & 2 & 5 \\ \frac{5}{2} & 2 & \frac{5}{2} \\ \frac{5}{2} & \frac{9}{4} & \frac{9}{4} \end{pmatrix} \end{array}$$

Note that since the populations are now monotonically decreasing in energy, no further energy can be extracted.

Alternatively, consider the sequence $(\nu_{10},\nu_{20},\nu_{21})$, which gives

$$\begin{array}{lll} & & \\ & \text{initial} & W^0=22 \\ \text{sequence 2} & \text{step 1} & W^1=21 \\ (\nu_{10},\nu_{20},\nu_{21}) & \text{step 2} & W^2=13 \\ & \text{step 3} & W^3=10 \end{array} \begin{array}{c} \begin{array}{ccc} \epsilon_0=0 & \epsilon_1=1 & \epsilon_2=4 \end{array} \\ \begin{pmatrix} 0 & 2 & 5 \\ 1 & 1 & 5 \\ 3 & 1 & 3 \\ 3 & 2 & 2 \end{pmatrix} \end{array}.$$

Again, note that since the populations are now monotonically decreasing in energy, no further energy can be extracted. But, this sequence of lasers has extracted $1\frac{1}{4}$ units more energy than the sequence above, (ν_{20},ν_{21}). A strategy apparently with merit is the following:

Strategy 1: Diffusion of particles first between similar population levels, all other things being equal, eventually releases more energy.

That being the case, another sequence that might be tried is $(\nu_{21},\nu_{20},\nu_{10})$, which gives

$$\begin{array}{lll} & & \\ & \text{initial} & W^0=22 \\ \text{sequence 3} & \text{step 1} & W^1=\frac{35}{2} \\ (\nu_{21},\nu_{20},\nu_{10}) & \text{step 2} & W^2=\frac{21}{2} \\ & \text{step 3} & W^3=\frac{77}{8} \end{array} \begin{array}{c} \begin{array}{ccc} \epsilon_0=0 & \epsilon_1=1 & \epsilon_2=4 \end{array} \\ \begin{pmatrix} 0 & 2 & 5 \\ 0 & \frac{7}{2} & \frac{7}{2} \\ \frac{7}{4} & \frac{7}{2} & \frac{7}{4} \\ \frac{21}{8} & \frac{21}{8} & \frac{7}{4} \end{pmatrix} \end{array},$$

which actually releases the most energy of all! In fact, it appears that this sequence releases the maximum energy.

Note that it is not possible to find a maximal sequence independent of the energy vector. For the energy vector here $\epsilon=(0,1,4)$, sequence 3 above appears to release the maximum energy, but for a more "flat" energy vector, say $\epsilon=(0,1,\frac{9}{8})$, sequence 2, which puts more atoms in the ground state, gives a lower final energy of $\frac{33}{8}$ whereas sequence 3, which removes the most atoms from the highest energy state, gives a final energy of $\frac{35}{8}$.

One might profitably examine one more sequence. Consider

$$
\begin{array}{lllccc}
 & & & \epsilon_0=0 & \epsilon_1=1 & \epsilon_2=4 \\
 & \text{initial} & W^0=22 & \left[\begin{array}{c} 0 \\ \\ 0 \\ \\ \frac{7}{4} \\ \\ \frac{21}{8} \\ \\ \frac{21}{8} \end{array}\right. & \begin{array}{c} 2 \\ \\ \frac{7}{2} \\ \\ \frac{7}{4} \\ \\ \frac{7}{4} \\ \\ \frac{35}{16} \end{array} & \left.\begin{array}{c} 5 \\ \\ \frac{7}{2} \\ \\ \frac{7}{2} \\ \\ \frac{21}{8} \\ \\ \frac{35}{16} \end{array}\right] \\
\end{array}
$$

$$
\begin{array}{lll}
 & \text{initial} & W^0=22 \\
 & \text{step 1} & W^1=\frac{35}{2} \\
\text{sequence 4} & \text{step 2} & W^2=\frac{63}{4} \\
(\nu_{21},\nu_{10},\nu_{20},\nu_{21}) & \text{step 3} & W^3=\frac{49}{4} \\
 & \text{step 4} & W^4=\frac{175}{16}
\end{array}
\qquad
\begin{bmatrix} 0 & 2 & 5 \\ 0 & \frac{7}{2} & \frac{7}{2} \\ \frac{7}{4} & \frac{7}{4} & \frac{7}{2} \\ \frac{21}{8} & \frac{7}{4} & \frac{21}{8} \\ \frac{21}{8} & \frac{35}{16} & \frac{35}{16} \end{bmatrix}.
$$

This sequence turns out relatively badly; here, with the opportunity to deplete the second level at step 2, the first level was depleted instead. Apparently, a second strategy with merit is the following:

Strategy 2: Depleting of particles the higher energy level first, all other things being equal, eventually releases more energy.

How might one prove that sequence 3 releases the most energy? One could probably offer, for this simple case, a rather cumbersome and inelegant proof, but we resist doing that without an algorithm for considering the N-level system, where N may be large. Evidently, the complexity of the problem increases rather substantially with increasing N.

IV. STATEMENT OF THE PROBLEM

Having motivated the importance of the problem, having identified where this problem might occur, and having indicated its complexity, finally, we are in a position to state the problem precisely, both for the case of discrete energy levels and for the case of a continuum in both energy and space. For the discrete case, in which diffusion equalizes the density in two energy states, the rearrangement problem may be stated as follows:

Statement of the problem, discrete diffusive rearrangement: For an energy level vector $\boldsymbol{\epsilon}\equiv\{\epsilon_i\}$ and a density vector, at the kth iteration $\mathbf{N}^k\equiv\{N_i^k\}$, find the iteration sequence $(a_1b_1,...,a_kb_k)$, meaning that, at the kth step, the density is iterated according to the rule

$$
N_i^k=\begin{cases} N_i^{k-1}, & \text{if} \quad i\neq a_k,b_k; \\ (N_{a_k}^{k-1}+N_{b_k}^{k-1})/2, & \text{if} \quad i=a_k,b_k; \end{cases} \tag{3}
$$

that minimizes $W^k=\boldsymbol{\epsilon}\cdot\mathbf{N}^k\equiv\Sigma_i\epsilon_iN_i^k$, for $k\to\infty$.

One can also state the problem in the continuum limit under general diffusive possibly nonlocal flow. Without loss of generality, as we show below, consider the one-dimensional density $f(v,t)$ that evolves according to

$$
\frac{\partial f}{\partial t}=\int K(v,v',t)[f(v',t)-f(v,t)]dv', \tag{4}
$$

where the kernel $K(v,v',t)$ has two important properties:

(1) $K(v,v',t)=K(v',v,t)$, which assures particle conservation;

(2) $K(v,v',t)\geqslant 0$, which assures that diffusion occurs in detailed balance, i.e., for any flow between v and v', $\partial f/\partial t\geqslant 0$ if $f(v,t)\leqslant f(v',t)$. The continuum rearrangement problem to be solved can then be stated as follows:

Statement of the problem, continuum diffusive rearrangement: For a density function $f(v,t)$ obeying Eq. (4), find the kernel $K(v,v',t)$ that minimizes

$$
W(t)=\int \epsilon(v)f(v,t)dv \tag{5}
$$

for $t\to\infty$. For a nonrelativistic plasma, one would take $\epsilon(v)=v^2/2$.

That the stating of the problem in one dimension is, in fact, without loss of generality can be demonstrated by proceeding to the continuum limit as follows: Let f evolve in the six-dimensional phase space, which is chopped into M boxes of volume $V=dx\,dy\,dz\,dv_x\,dv_y\,dv_z$. Associated with each box i is an energy ϵ_i, and a number $N_i(t)$, where the boxes may be ordered so that if $i>j$ then $\epsilon_i>\epsilon_i$. Define $f(s,t)\equiv N_i(t)/V$, for $iV<s<(i+1)V$, and obtain a one-dimensional density distribution function in the limit $M\to\infty$.

Note that the problem, even as put in one dimension, is quite formidable; in principle, to solve this exactly, all kernels $K(v,v',t)$ obeying properties (1) and (2) above must be searched, in order to find the kernel that evolves f to the minimum energy state. This is a search in three-dimensional function space. This might be posed variationally, but there does not occur to these authors any easy way of solving this problem.

Finally, note that f, as evolved through Eq. (4), obeys an H theorem.[9] Consider the quantity

$$
\begin{aligned}
\frac{d}{dt}\int f(v,t)^2\,dv
&=\int dv\int dv'\,K(v,v',t)[2f(v,t)f(v',t)-2f^2(v,t)] \\
&=\int dv\int dv'\,K(v,v',t)[2f(v,t)f(v',t)-f^2(v,t) \\
&\quad -f'^2(v,t)] \\
&=-\int dv\int dv'\,K(v,v',t)[f(v',t)-f(v,t)]^2\leqslant 0,
\end{aligned} \tag{6}
$$

where the first equality makes use of Eq. (4), the second equality may be written because $K(v,v',t)$ is symmetric in v and v', and the inequality follows from the positive definite nature of K. It follows that since a positive quantity is monotonically nonincreasing, it must reach a steady state.

V. RECONCILING DIFFUSIVE FLOW WITH PHASE SPACE INCOMPRESSIBILITY

In posing the problem of the extraction of plasma energy by waves traversing a plasma slab, it has been assumed that the waves can diffuse plasma from any region of phase space to any other region of phase space. Strictly speaking, diffusive flow cannot occur if the plasma is collisionless, and nonlocal diffusion is not physical. However, these considerations are on a microscopic scale; on a coarser scale, it will always appear that waves can, in principle, diffuse plasma from any region of phase space to any other region of phase space. Thus "fine-grained" entropy is conserved, even as what might be defined as "coarse-grained" entropy grows (see, e.g., Ref. 10).

Consider for a one-dimensional density distribution, a series of phase space boxes, each of size $\Delta x \Delta v$,

$$\boxed{8}\ \boxed{2}\ \boxed{4}\ ,$$

where we use horizontal placement to indicate the velocity axis, so that the different boxes represent different velocities. The numbers in each box give the average phase-space density within the box. One diffusive step could give

$$\boxed{8}\ \boxed{3}\ \boxed{3}\ .$$

How might such a diffusive rearrangement occur in a system conserving phase-space density such as a collisionless plasma? Suppose the plasma were more finely grained, e.g., divide each phase-space box into four smaller boxes, each of size $(\Delta x/2)(\Delta v/2)$, so that the density can be represented by

$$\boxed{\begin{matrix}8 & 8\\ 8 & 8\end{matrix}}\ \boxed{\begin{matrix}2 & 2\\ 2 & 2\end{matrix}}\ \boxed{\begin{matrix}4 & 4\\ 4 & 4\end{matrix}}\ .$$

Here, the second row of numbers represents the density at a slightly different spatial position, and the second column of numbers represents the density at a slightly different velocity. Now, a possible phase-space conserving rearrangement is

$$\boxed{\begin{matrix}8 & 8\\ 8 & 8\end{matrix}}\ \boxed{\begin{matrix}2 & 4\\ 4 & 2\end{matrix}}\ \boxed{\begin{matrix}4 & 2\\ 2 & 4\end{matrix}}\ ,$$

and, if one just looked at this phase-space conserving rearrangement on a slightly coarser scale, then one would simply see

$$\boxed{8}\ \boxed{3}\ \boxed{3}\ .$$

Thus, "diffusion" on one scale can be viewed as an incompressible flow on a more microscopic scale.

A second point related to "fine graining" is that diffusion need not be treated as if it could occur only between contiguous areas of phase space. For example, in the example above, consider the diffusive rearrangement

$$\boxed{8}\ \boxed{2}\ \boxed{4}\ \rightarrow\ \boxed{6}\ \boxed{2}\ \boxed{6}\ ,$$

in which the diffusion occurs between the noncontiguous regions of phase space. But exactly this rearrangement is possible by considering a finer graining over a small region abutting the larger region, as we show.

Suppose, in the original phase-space graining, each box is of size $\Delta x \Delta v$. Expand each box into N smaller boxes of size $(\Delta x/N)\Delta v$. In other words, with each row representing a spatial region of width $\Delta x/N$, the distribution can be represented by

$$\begin{matrix}8 & 2 & 4\\ 8 & 2 & 4\\ \vdots & \vdots & \vdots\\ 8 & 2 & 4\\ 8 & 2 & 4\,.\end{matrix}$$

Now, suppose that diffusion takes place in the horizontal direction across the bottom sliver, and in the vertical directions across the side regions only, then, the distribution evolves to

$$\begin{matrix}6- & 2 & 6-\\ 6- & 2 & 6-\\ \vdots & \vdots & \vdots\\ 6- & 2 & 6-\\ 6- & 6- & 6-\,,\end{matrix}$$

where by "6−" is a number slightly smaller than 6, approaching 6 for N large, since the distribution function tries to equilibrate in the bottom and side regions. But on a coarser scale, the resulting distribution vector just appears to be $\mathbf{N}=(6,2,6)$, as if the diffusion had occurred across noncontiguous regions.

What has been shown in this section is that diffusive, nonlocal rearrangements of phase space are obtainable through local phase-space conserving rearrangements on a smaller scale. The opposite, however, is clearly not true; the density vector $(2,5)$ cannot be rearranged diffusively on any scale to give $(5,2)$. Thus, the diffusive rearrangements form a subset of all possible phase-space conserving rearrangements of the plasma in phase space. Since the Gardner restacking solution gives the maximum energy extractable under the superset of phase-space conserving rearrangements, clearly the maximum energy extractable through diffusive rearrangements is necessarily bounded through Eq. (1) by W_G, the Gardner free energy.

VI. FREE ENERGY IN BUMP-ON-TAIL DISTRIBUTION

One of the classic examples of the release of energy through quasilinear diffusion by waves is the interaction of a spectrum of electrostatic waves with the so-called "bump-on-tail" distribution function. Here it is important to discriminate between an initial value problem, the case generally treated in the literature, and the boundary value problem, the situation of interest here.

In the initial value problem, a spectrum of unstable waves grows until there is no more energy to feed the instability. In addition to the change in energy in the group of particles resonant with the wave, there is also a change in energy of nonresonant particles, since these particles, nonetheless, oscillate in the wave. Of course, this bookkeeping of the energy could be accomplished equally well through lumping the oscillatory energy of the nonresonant particles into the wave energy.[11,12] In the boundary value problem treated here, the wave leaves the plasma slab entirely, so there is no remaining oscillatory motion within the wave; hence, the only change to the particle kinetic energy arises through the resonant particle rearrangement.

The "bump-on-tail" distribution function is a one-dimensional distribution in velocity, with a minimum at superthermal energies. The maximum extractable energy through diffusive rearrangement occurs when the minimum is filled in from higher energy. What about a distribution function with two or more minima at superthermal energies, a "several-bumps-on-tail" distribution? If the diffusion can be nonlocal, determining the maximum extractable energy is difficult; this is nothing simpler than the general statement of the problem in the continuum limit. However, if the diffusion were limited to local diffusion only, it turns out that this energy can be determined. If waves are employed only to diffuse particles from higher to lower energy, the temporal sequence of local diffusion is unimportant, since in any event, the lowest energy state is reached when there is a local flattening of abutting regions.

VII. SUMMARY AND DISCUSSION

What has been set forth here is the question of energy extraction from a plasma slab by means of waves traveling through it, such that the wave causes velocity and space diffusion within the plasma. This is the configuration that arises in certain problems of practical interest, for which knowledge of the maximum extractable energy would provide a standard by which to evaluate any particular means of extraction. In addition to relating to contemporary problems in plasma physics, this question relates to maximizing stimulated emission by a set of lasers.

Some consideration must be paid to a number of subtleties: on a microscopic scale, the collisionless rearrangement of a plasma must preserve phase-space density and occur locally, but viewed on a coarser scale, the plasma appears to be rearranged diffusively and nonlocally.

The problem has been posed here both for the discrete and the continuum cases. The latter posing is applicable to plasmas, while the former posing is applicable to stimulated emission in atomic systems as well as, indeed, any numerical and therefore necessarily discrete posing of the problem in plasmas. In both cases, the maximum extractable energy by diffusive means has been shown to be less than the Gardner energy.

What is interesting is that obtaining the precise maximum extractable energy under diffusion by waves, even in relatively simple situations, is a surprisingly formidable task. For local diffusion in several dimensions, or, equivalently, nonlocal diffusion in one dimension, the temporal sequence of the wave diffusion is critical. Different final states are reached depending on the wave history. While no solution has been obtained for finding the maximum extractable energy under such circumstances, certain insights have been drawn through the consideration of specific examples, and certain strategies for optimizing energy extraction have been discovered.

What has been identified in this work is a fundamental quantity of practical interest, the free energy of a plasma under the constraint of diffusion by waves. While it is difficult to calculate, it can be defined precisely. It is this quantity that is the free energy of interest in certain important problems.

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