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Approximate integrals of radiofrequency-driven particle motion in a magnetic field

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Abstract. For a particle moving in a non-uniform static magnetic field under the action of a radiofrequency wave, ponderomotive effects result from radiofrequency-driven oscillations nonlinearly coupled with Larmor rotation. Using the Lagrangian and Hamiltonian formalisms, we show how, despite this coupling, two independent integrals of the particle motion are approximately conserved. These are the magnetic moment of free Larmor rotation and the quasi-energy of the guiding center motion parallel to the d.c. magnetic field. Under the assumption of non-resonant interaction of the particle with the radiofrequency field, these integrals represent adiabatic invariants of the particle motion.

1. Introduction

Under the action of an intense radiofrequency (rf) drive, charged particles undergo fast oscillations superimposed on the average drift motion. If the drift is sufficiently slow and the amplitude of particle oscillations is small compared with the characteristic spatial scale of the applied external fields, then the particle average motion can be described in the framework of the guiding center approach. In this case, the average effect of the rf drive can approximately be replaced by the particle interaction with an effective potential (Gaponov and Miller 1958; Motz and Watson 1967).

One of the applications where the guiding center approximation finds use is the problem of the particle motion under the action of intense rf radiation superimposed on the particle interaction with a static non-uniform magnetic field† (Motz and Watson 1967). Among other reasons why such a field configuration is worth studying, it has been noted that the inhomogeneous magnetic field makes an asymmetric rf barrier possible, if the cyclotron resonance occurs within a localized rf field, with the possibility of a driving current (Suvorov and Tokman 1988; Litvak et al. 1993). Through the proper arrangement of these fields, an efficient one-way ponderomotive barrier could be produced (Fisch et al. 2003), with important practical consequences including the driving of non-inductive currents at efficiencies possibly greater than those obtained by other means (Fisch 1987). In these cases, the particle motion consists not only of the rf-driven oscillations, but also of the Larmor rotation in the plane perpendicular to the magnetic field lines, accompanied by the diamagnetic

† The term 'magnetic field' here is always applied to a static but not rf field.

acceleration of the particle guiding center parallel to the magnetic field. These types of motion can easily be studied *separately* when either the rf field or the magnetic field is negligible. In the absence of rapidly oscillating rf fields, it can be shown that the magnetic moment associated with particle Larmor rotation, $\mu = mv_\perp^2/2B_0$, represents an adiabatic invariant (Gardner 1959; Jackson 1975), where $\frac{1}{2}\,mv_\perp^2$ is the energy of particle motion transverse to the magnetic field $\mathbf{B}_0$. In the other case, when only the rf field is present, the adiabatic invariance of the quantity $\frac{1}{2}\,m\langle v\rangle^2 + \Phi$ can be proven (see, e.g., Lichtenberg and Lieberman 1992), where $\langle v\rangle$ is the guiding center velocity, and Φ is the so-called ponderomotive potential (Gaponov and Miller 1958; Motz and Watson 1967) given by

$$\Phi = \frac{e^2\big|\mathbf{E}_{\mathrm{rf}}^{(0)}\big|^2}{4m\omega^2}, \tag{1.1}$$

where e and m are the electric charge and the mass of the particle, and $\mathbf{E}_{\mathrm{rf}}^{(0)}$ and ω are the complex amplitude and the frequency of the rf field, respectively.

In the case when *both* magnetic and rf fields are present, the question of conservation of the adiabatic invariants becomes non-trivial because of possible coupling between the Larmor rotation and the rf-driven motion. As the Larmor frequency becomes comparable to the frequency of the rf field, a conventional hierarchy of adiabatic invariants (Lichtenberg and Lieberman 1992) cannot be developed and, in principle, chaotic motion may result from nonlinear interaction between the two types of oscillations.

Usually, when the motion of a rf-driven particle in a magnetic field is studied, two approximate integrals are derived (Motz and Watson 1967). These are the magnetic moment associated with the particle free Larmor rotation $\mu = m\mathbf{v}_{\mathrm{f},\perp}^2/2B_0$ (here $\mathbf{v}_{\mathrm{f},\perp} = \mathbf{v}_\perp - \mathbf{v}_{\mathrm{rf},\perp}$ is the velocity additional to the velocity of the rf-driven oscillations $\mathbf{v}_{\mathrm{rf}}$) and the particle 'quasi-energy'

$$\mathscr{E} = \frac{m\langle v_\parallel\rangle^2}{2} + \mu B_0 + \Phi, \tag{1.2}$$

with the effective potential Φ given by

$$\Phi = \sum_{\nu=0,\pm1} \frac{e^2\big|E_\nu^{(0)}\big|^2}{4m\omega(\omega+\nu\Omega)}. \tag{1.3}$$

Here $E_\nu^{(0)}$ is the amplitude of the electric rf field component with polarization $\boldsymbol{\tau}_\nu$,

$$\boldsymbol{\tau}_{\pm1} = (\mathbf{x}^0 \pm i\mathbf{y}^0)/\sqrt{2}, \quad \boldsymbol{\tau}_0 = \mathbf{z}^0, \tag{1.4}$$

where $\mathbf{x}^0$ and $\mathbf{y}^0$ are the unit vectors in the plane perpendicular to the magnetic field $\mathbf{B}_0 \approx \mathbf{z}^0 B_0(z)$, and are smooth on the scale of the oscillations amplitude; $\Omega = eB_0/mc$ is the Larmor frequency.

Although μ is often claimed to be an adiabatic invariant (Motz and Watson 1967; Watson and Kuo-Petravic 1968; Eubank 1969), this statement, rather than being proved rigorously, is usually made by analogy with the case of free Larmor rotation at zero rf field (see, though, the discussion in Grebogi et al. (1979)). Consequently, conservation of μ is never examined analytically (for numerical and experimental studies, see Eubank (1969) and Watson and Kuo-Petravic (1968)). Moreover, it remains unclear exactly what is the nature of the integral (1.2) and what are the approximations under which $\mathscr{E}$ can be considered as a conserved quantity.

These shortcomings of the conventional consideration result from the intrinsic limitations of the approach used for deriving the average ponderomotive force. Namely, the guiding center motion equations are often obtained by direct averaging of the true motion equations, Taylor-expanded with respect to the spatial coordinate (see, e.g., Motz and Watson (1967); Kildal (1999); Lamb et al. (1984); Dimonte et al. (1983); Litwin (1994)). The potential form of the ponderomotive force in this case is not deduced directly – it is rather guessed at (while the proof follows), which makes the complicated averaging procedure even more unclear. What we show, however, is that there exists an alternative, physically intuitive, formally simple and clear in derivation Lagrangian approach, leading to the same expression for the average force 'seen' by a slowly drifting particle. (The power of the Lagrangian approach in application to drift dynamics of particles moving in a non-uniform magnetic field (without rf radiation) has been demonstrated previously in a number of works – see, e.g., Littlejohn (1983) and Danilkin (1995).)

The purpose of the present paper is, first, to present a simple Lagrangian derivation of the known conservation laws for μ and $\mathscr{E}$; second, to give a systematic Lagrangian and Hamiltonian formulation of the particle average motion; and, third, to demonstrate how such a treatment gives the conditions under which μ and $\mathscr{E}$ can approximately be considered as adiabatic invariants. The paper is organized as follows. In Sec. 2 we show how the average potential (1.3) can be derived naturally and the conservation of μ and $\mathscr{E}$ can be proved in the framework of the Lagrangian approach. A more detailed calculation involving the Hamiltonian analysis is given in Sec. 3, which demonstrates the connection between the approximate integrals of the rf-driven particle motion and the theory of adiabatic invariants. We show that μ and $\mathscr{E}$ represent adiabatic invariants of the particle motion only under the assumption of negligible heating of a particle at high-order resonances, which always takes place when the particle travels along a non-uniform magnetic field under the action of a rf-drive. In Section 4, we summarize our main ideas. Some supplementary calculations are given in Appendices A and B.

2. Guiding center Lagrangian

To study the average motion of a charged particle under the action of rf radiation in a d.c. magnetic field, let us first concretely define the guiding center approximation. The key condition under which particle dynamics can be readily averaged over fast oscillations is that the particle displacement on an oscillation time scale is small compared with the spatial scale of the external fields, both d.c. and rf. Hence, the obvious conditions that are required can be put as

$$\frac{r_\sim}{\Delta} \ll 1, \quad \frac{|\langle \mathbf{v} \rangle|}{\omega\Delta} \ll 1, \quad \frac{|\langle \mathbf{v} \rangle|}{\Omega\Delta} \ll 1, \tag{2.1}$$

where $r_\sim$ is the amplitude of the particle oscillations and Δ is the least characteristic spatial scale of the electromagnetic field. In addition to these, however, one also needs the drift motion to remain slow in comparison with the beating period between Larmor rotation and rf-driven oscillations $\tau_\mathrm{b} = 2\pi/|\omega - \Omega|$, and τ_b itself to vary smoothly along the particle trajectory:

$$\frac{\langle v_z \rangle}{|\omega - \Omega|\Delta} \ll 1, \quad \frac{\langle v_z \rangle}{(\omega - \Omega)^2} \frac{d(\omega - \Omega)}{dz} \ll 1. \tag{2.2}$$

To develop the guiding center description under conditions (2.1) and (2.2), first consider the expression for the action

$$S = \int_{t_1}^{t_2} L \, dt, \tag{2.3}$$

where L is the Lagrangian of the particle motion. Consider the time scale $\Delta t = t_2 - t_1$ to be large compared with τ_b. Then, the major contribution to the action S (linear on Δt) is provided by the time-averaged part of the Lagrangian, $\langle L \rangle$, while the contribution of the oscillatory Lagrangian into the integral (2.3) remains small. Thus, the action S is approximately given by $S = \int_{t_1}^{t_2} \langle L \rangle \, dt$, from where it follows that $L_{\mathrm{d}} \equiv \langle L \rangle$ can be treated as the Lagrangian of the guiding center motion.

To calculate L_{d}, consider the full Lagrangian of a particle moving in a static magnetic field $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$ under the action of rf drive governed by the vector potential $\mathbf{A}_{\mathrm{rf}}$:

$$L = \frac{mv^2}{2} + \frac{e}{c}(\mathbf{v} \cdot (\mathbf{A}_0 + \mathbf{A}_{\mathrm{rf}})). \tag{2.4}$$

Take ϵ to be the largest of the small parameters defined in (2.1) and (2.2). In the limit $\epsilon \ll 1$, the vector potential $\mathbf{A}_0(\mathbf{r})$ can be approximated by a linear function of the particle transverse displacement:

$$\mathbf{A}_0(\mathbf{r}) = B_0(z)(\mathbf{z}^0 \times \mathbf{r})/2 + \mathcal{O}(\epsilon) \tag{2.5}$$

(see also Danilkin (1995)). Let us denote the rf-driven oscillatory displacement by $\mathbf{r}_{\mathrm{rf}}$, and introduce the new coordinate $\mathbf{R} = \mathbf{r} - \mathbf{r}_{\mathrm{rf}}$ together with the corresponding velocity $\mathbf{V} = d\mathbf{R}/dt$ and the quiver velocity $\mathbf{v}_{\mathrm{rf}} = d\mathbf{r}_{\mathrm{rf}}/dt$. Then the Lagrangian (2.4) takes the form

$$L = \frac{mV^2}{2} + \frac{e}{c}(\mathbf{V} \cdot \mathbf{A}_0(\mathbf{R})) + L_{\mathrm{rf}} + L_{\sim}, \tag{2.6}$$

$$L_{\mathrm{rf}} = \frac{mv_{\mathrm{rf}}^2}{2} + \frac{e}{c}(\mathbf{v}_{\mathrm{rf}} \cdot \mathbf{A}_{\sim}), \tag{2.7}$$

$$L_{\sim} = m\mathbf{V} \cdot \mathbf{p}_{\sim} + \frac{e}{c}(\mathbf{v}_{\mathrm{rf}} \cdot \mathbf{A}_0(\mathbf{R})) + \mathcal{O}(\epsilon), \tag{2.8}$$

where $\mathbf{p}_{\sim} = m\mathbf{v}_{\mathrm{rf}} + (e/c)\mathbf{A}_{\sim} = m\mathbf{v}_{\mathrm{rf},\perp} + (e/c)\mathbf{A}_{\sim,\perp} + \mathcal{O}(\epsilon)$ is the oscillatory momentum and $\mathbf{A}_{\sim} = \mathbf{A}_{\mathrm{rf}} + \mathbf{A}_0(\mathbf{r}_{\mathrm{rf}})$ is the oscillatory vector potential 'seen' by the particle.

In comparison with traditional averaging of motion equations (see, e.g., Motz and Watson (1967); Kildal (1999); Lamb et al. (1984); Dimonte et al. (1983); Litwin (1994)), the advantage of the Lagrangian approach consists of the fact that, in the guiding center Lagrangian, it is enough to keep only the zeroth-order terms with respect to ϵ. (The ponderomotive force, which is of the first order in ϵ, readily appears in the motion equation, as the guiding center Lagrangian is differentiated with respect to $Z \equiv R_z$.) In the limit of zero ϵ, the function $L_{\sim}$ represents a full time derivative, $L_{\sim} = d(\mathbf{R}_\perp \cdot \mathbf{p}_{\sim,\perp})/dt + \mathcal{O}(\epsilon)$, and thus can be taken out of the original Lagrangian. Therefore, the particle motion can be equivalently described in terms of the alternative Lagrangian function

$$\mathscr{L} = \frac{m\dot{Z}^2}{2} + L_\perp + L_{\mathrm{rf}} + \mathscr{O}(\epsilon), \tag{2.9}$$

$$L_\perp = \frac{mV_\perp^2}{2} + \frac{e}{c}(\mathbf{V}_\perp \cdot \mathbf{A}_0(\mathbf{R}_\perp, Z)). \tag{2.10}$$

From the form of the Lagrangian (2.9), it can be concluded that, in the limit $\epsilon \to 0$, a particle drifts along a magnetic field line with velocity $\langle v_z \rangle = \dot{Z}$, undergoes Larmor rotation in variables $(\mathbf{R}_\perp, \mathbf{V}_\perp)$ and experiences a ponderomotive force ∇L_{rf}. To derive the Lagrangian of the longitudinal drift motion, let us average the expression (2.9) over both Larmor and radiofrequency-driven oscillations, as well as over the beating between the two. Since $L_\perp$ has a form of the Lagrangian of Larmor motion in variables $(\mathbf{R}_\perp, \mathbf{V}_\perp)$, it can be shown (see Appendix A) that, after omitting the full time derivative,

$$\langle L_\perp \rangle = -\mu B_0, \quad \mu = \text{constant}. \tag{2.11}$$

Let us now derive the expression for the time-averaged function L_{rf}. Under the action of a rf field a particle undergoes oscillations, which can be represented in the complex form as $\mathbf{r}_{\mathrm{rf}} = (-e/m\omega^2)\mathsf{T}\mathbf{E}_{\mathrm{rf}}$, where the tensor T, for $\mathbf{E}_{\mathrm{rf}} \propto \exp(-i\omega t)$, is given by

$$\mathsf{T} = \begin{pmatrix} \dfrac{1}{1-b^2} & \dfrac{ib}{1-b^2} & 0 \\ \dfrac{-ib}{1-b^2} & \dfrac{1}{1-b^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \Omega/\omega. \tag{2.12}$$

Consider the most general expression for the radiofrequency field:

$$\mathbf{E}_{\mathrm{rf}} = \mathrm{Re}\big(E_+^{(0)}\boldsymbol{\tau}_+ + E_-^{(0)}\boldsymbol{\tau}_- + E_\parallel^{(0)}\boldsymbol{\tau}_0\big)e^{-i\omega t}, \tag{2.13}$$

where $E_\nu^{(0)}$ are some arbitrary complex amplitudes and $\boldsymbol{\tau}_\nu$ are the polarization vectors defined according to (1.4). In this case, one has

$$\langle L_{\mathrm{rf}} \rangle = \frac{e^2}{4m\omega^2}\{|\mathsf{T}\mathbf{E}_{\mathrm{rf}}|^2 - 2\,\mathrm{Re}[\mathbf{E}_{\mathrm{rf}}^* \cdot \mathsf{T}\mathbf{E}_{\mathrm{rf}}] + b\mathbf{z}^0 \cdot \mathrm{Im}[(\mathsf{T}^*\mathbf{E}_{\mathrm{rf}}^*) \times (\mathsf{T}\mathbf{E}_{\mathrm{rf}})]\}. \tag{2.14}$$

Since

$$\mathsf{T}\mathbf{E}_{\mathrm{rf}} = \frac{\mathbf{x}^0}{\sqrt{2}}\left(\frac{E_+^{(0)}}{1+b} + \frac{E_-^{(0)}}{1-b}\right) + \frac{i\mathbf{y}^0}{\sqrt{2}}\left(\frac{E_+^{(0)}}{1+b} - \frac{E_-^{(0)}}{1-b}\right) + \mathbf{z}^0 E_\parallel^{(0)}, \tag{2.15}$$

it can be shown that

$$|\mathsf{T}\mathbf{E}_{\mathrm{rf}}|^2 = \frac{|E_+^{(0)}|^2}{(1+b)^2} + \frac{|E_-^{(0)}|^2}{(1-b)^2} + |E_\parallel^{(0)}|^2, \tag{2.16}$$

$$\mathrm{Re}[\mathbf{E}_{\mathrm{rf}}^* \cdot \mathsf{T}\mathbf{E}_{\mathrm{rf}}] = \frac{|E_+^{(0)}|^2}{1+b} + \frac{|E_-^{(0)}|^2}{1-b} + |E_\parallel^{(0)}|^2, \tag{2.17}$$

$$\mathrm{Im}[(\mathsf{T}^*\mathbf{E}_{\mathrm{rf}}^*) \times (\mathsf{T}\mathbf{E}_{\mathrm{rf}})]_z = \frac{|E_+^{(0)}|^2}{(1+b)^2} - \frac{|E_-^{(0)}|^2}{(1-b)^2}. \tag{2.18}$$

Substituting these into (2.14), one obtains

$$\langle L_{\rm rf}\rangle = -\frac{e^2}{4m\omega^2}\left\{\frac{\left|E_+^{(0)}\right|^2}{1+b}+\frac{\left|E_-^{(0)}\right|^2}{1-b}+\left|E_\parallel^{(0)}\right|^2\right\} = -\Phi, \tag{2.19}$$

where Φ is the ponderomotive potential defined according to (1.3).

Hence, finally, the expression for the guiding center Lagrangian can be put in the form

$$L_{\rm d} = \frac{m\dot{Z}^2}{2} - \mu B_0(Z) - \Phi(Z) + \langle \mathcal{O}(\epsilon)\rangle, \tag{2.20}$$

yielding a motion equation in a potential form

$$m\frac{d^2Z}{dt^2} \approx -\frac{d}{dZ}(\mu B_0(Z) + \Phi(Z)), \tag{2.21}$$

which conserves the quasi-energy (1.2). Indeed, $\mathscr{E}$ coincides with the drift Hamiltonian of a particle, $H_{\rm d} = m\dot{Z}^2 - L_{\rm d}$, $\dot{Z} = \langle v_\parallel\rangle$ and, since $\partial H_{\rm d}/\partial t = 0$, the value of $\mathscr{E}$ represents an integral of the guiding center motion.

3. Action-angle variables

As shown above, the two approximate integrals of the guiding center motion, μ and $\mathscr{E}$, exist for a particle undergoing Larmor rotation under the action of a rf field. In this section, we show how these integrals appear naturally from the Hamiltonian description of the ponderomotive effects (see also Grebogi et al. (1979), where the rf field is treated as a perturbation). To be more precise, what is shown below is that, under certain conditions, μ and $\mathscr{E}$ can be considered as adiabatic invariants of the particle motion.

To proceed, let us develop the Hamiltonian formalism for particle dynamics starting from the Lagrangian (2.9). The canonical momentum of the motion, additional to the rf-driven oscillations, is given by $\mathbf{P} = m\mathbf{V} + (e/c)\mathbf{A}_0(\mathbf{R})$, and the Hamiltonian function can be put in the intuitively expected form

$$H = \frac{P_z^2}{2m} + \frac{1}{2m}\left(\mathbf{P}_\perp - \frac{e}{c}\mathbf{A}_0(\mathbf{R}_\perp, Z)\right)^2 - L_{\rm rf}(Z,t) + \mathcal{O}(\epsilon), \tag{3.1}$$

where Larmor rotation in the variables $(\mathbf{R}_\perp, \mathbf{P}_\perp,)$ is separated (at least, locally) from the rf-driven oscillations and the average motion parallel to the magnetic field. After the canonical transformation to the Larmor guiding center variables (see, e.g., Lichtenberg and Lieberman (1992); White et al. (2002)), the equivalent Hamiltonian takes the form

$$\overline{H} = \frac{P_z^2}{2m} + \Omega(Z)P_\phi - L_{\rm rf}(Z,t) + \epsilon\Delta\overline{H}, \tag{3.2}$$

where $P_\phi = (mc/e)\mu$ is the action variable corresponding to the canonical angle ϕ standing for the Larmor phase of the particle and $\Delta\overline{H}$ is a periodic function of ϕ and t (since the original particle Lagrangian is periodic with respect to these variables).

Our next step is to perform the so-called averaging transformation of the system determined by the Hamiltonian function $\overline{H}$ (for a relevant discussion, see Littlejohn

(1983)). Consider the action (2.3) written as

$$S = \int P_z\, dZ + P_\phi\, d\phi - \overline{H}\, dt. \tag{3.3}$$

From this representation (Landau and Lifshitz 1960), it follows that one can treat the quantity $-\overline{H}$ as a canonical momentum with the time t as the corresponding canonical coordinate, while the pair $(-P_z, Z)$ is treated as the new Hamiltonian $\widehat{H}$ and the new 'time': $\widehat{H} = -P_z(\phi, P_\phi; t, -\overline{H}; Z)$. Assuming, for clarity, that P_z is positive, one obtains

$$\widehat{H} = -\sqrt{2m(\overline{H} + L_{\mathrm{rf}}(Z,t) - \Omega P_\phi)} + \epsilon\Delta\widehat{H}, \tag{3.4}$$

with the small term $\epsilon\Delta\widehat{H}$ periodic in t and ϕ.

Let us perform another canonical transformation to represent the Hamiltonian function in terms of the action variable

$$P_\varphi = \frac{1}{2\pi}\oint \overline{H}\, dt \tag{3.5}$$

and the corresponding angle variable φ, yet to be defined. To do so, consider the generating function

$$F(P_\varphi, t) = -\int_0^t \overline{H}\, dt = -P_\varphi \omega t + F_\sim, \tag{3.6}$$

with $F_\sim$ having zero time average. The new Hamiltonian $\mathscr{H} = \widehat{H} + \partial F/\partial Z$ is given by $\mathscr{H} = \mathscr{H}_0 + \epsilon\mathscr{H}_\sim$, where

$$\mathscr{H}_0 = -\sqrt{2m(\omega P_\varphi - \Phi(Z) - \Omega P_\phi)}, \tag{3.7}$$

and $\mathscr{H}_\sim$ is periodic with respect to ϕ and t. Since

$$\varphi = \frac{\partial F}{\partial P_\varphi} = -\omega t - \mathscr{O}(\epsilon), \tag{3.8}$$

where the second term is periodic in t, $\mathscr{H}_\sim$ also appears to be periodic in φ. Finally, introducing the vector action $\mathbf{J} = (P_\phi, P_\varphi)$ and the corresponding angle variable $\boldsymbol{\theta} = (\phi, \varphi)$, one can put the Hamiltonian $\mathscr{H}$ in the form

$$\mathscr{H} = \mathscr{H}_0(\mathbf{J}; Z) + \epsilon\mathscr{H}_\sim(\mathbf{J}, \boldsymbol{\theta}; Z), \tag{3.9}$$

where the small term $\mathscr{H}_\sim$ is periodic in $\boldsymbol{\theta}$ and thus can be represented as a Fourier series

$$\mathscr{H}_\sim = \sum_{\mathbf{n}} \mathscr{H}_{\mathbf{n}}(\mathbf{J}; Z)\exp(i\mathbf{n}\cdot\boldsymbol{\theta}), \tag{3.10}$$

with summation taken over all possible pairs of integers $\mathbf{n} = (n_\phi, n_\varphi)$.

Consider now another canonical transformation to the new variables $(\overline{\boldsymbol{\theta}}, \overline{\mathbf{J}})$ with the generating function given by

$$\mathscr{S}(\overline{\mathbf{J}}, \boldsymbol{\theta}) = \overline{\mathbf{J}}\cdot\boldsymbol{\theta} + \sum_{\mathbf{n},k} \epsilon^k \mathscr{S}_{\mathbf{n},k}(\overline{\mathbf{J}}; Z)\exp(i\mathbf{n}\cdot\boldsymbol{\theta}). \tag{3.11}$$

For small ϵ, one can find such coefficients $\mathscr{S}_{\mathbf{n}}$, for which the new Hamiltonian $\overline{\mathscr{H}}$ represents a function of $\overline{\mathbf{J}}$ alone (see, e.g., Lichtenberg and Lieberman (1992) and the references therein). To the first order in ϵ one has

$$\overline{\mathbf{J}} = \mathbf{J} + \epsilon \sum_{\mathbf{n} \neq 0} \mathbf{n} \langle v_{\parallel} \rangle \frac{\mathscr{H}_{\mathbf{n}}(\overline{\mathbf{J}}; Z)}{\mathbf{n} \cdot \boldsymbol{\Xi}} \exp(i\mathbf{n} \cdot \overline{\boldsymbol{\theta}}) \tag{3.12}$$

(see also Grebogi et al. (1979)). Here $\overline{\boldsymbol{\theta}}$ is considered a function of $\boldsymbol{\theta}$, and $\boldsymbol{\Xi} = \langle v_{\parallel} \rangle \, \partial \mathscr{H}_0 / \partial \mathbf{J} = (\Omega, -\omega)$ is the frequency vector.

By construction, the new Hamiltonian $\overline{\mathscr{H}}(\overline{\mathbf{J}}, \overline{\boldsymbol{\theta}})$ does not depend on $\overline{\boldsymbol{\theta}}$ to any order in ϵ. Thus, from the canonical equation $d\overline{\mathbf{J}}/dZ = -\partial \overline{\mathscr{H}} / \partial \overline{\boldsymbol{\theta}}$, it follows that $\overline{\mathbf{J}}$ is conserved with exponential precision *if* the above procedure can be realized (see below). In this case, the value of $\overline{\mathbf{J}}$ represents a so-called adiabatic invariant of the particle motion. As one can see from the definition of the action variable $\overline{\mathbf{J}} = (\overline{P}_{\phi}, \overline{P}_{\varphi})$, the previously introduced quantities μ and $\mathscr{E}$ can be represented in the form

$$\mu = (e/mc) \overline{P}_{\phi} + \mathscr{O}(\epsilon), \qquad \mathscr{E} = \overline{P}_{\varphi} \omega + \mathscr{O}(\epsilon), \tag{3.13}$$

and thus also represent approximate integrals of the particle motion. If evaluated away from the region of non-zero Hamiltonian of interaction $\epsilon \mathscr{H}_{\sim}$, the $\mathscr{O}(\epsilon)$ terms vanish. Therefore, after the particle has experienced a *complete* transition between the two regions of non-zero ϵ, the overall changes of μ and $\mathscr{E}$ are exactly proportional to the changes of $\overline{P}_{\phi}$ and $\overline{P}_{\varphi}$ correspondingly:

$$\Delta \mu = (e/mc) \, \Delta \overline{P}_{\phi} \approx 0, \qquad \Delta \mathscr{E} = \omega \, \Delta \overline{P}_{\varphi} \approx 0. \tag{3.14}$$

Note, however, that under the condition of resonant interaction between the particle cyclotron motion and the rf field ($n_{\phi} \Omega = n_{\varphi} \omega$) the canonic transformation (3.11) cannot be accomplished because of its singularity. Hence, the conservation laws (3.14) are violated, and a particle becomes capable of exchanging energy with the rf field. Such non-adiabatic interaction always takes place when Ω varies along the particle trajectory, so that the particle consecutively passes resonant regions corresponding to different $\mathbf{n}$. Let us estimate the change of action $\mathbf{J}$ as a particle crosses a resonance region $n_{\phi} \Omega = n_{\varphi} \omega$. Suppose $n_{\phi}/n_{\varphi} \neq \pm 1$, so that one may take $\mathscr{H}_{\mathbf{n}} \approx$ constant. In this case, applying the steepest descent method when integrating the canonical equation for $\mathbf{J}$ and taking $B_0/B_0' = L_B \approx$ constant, one gets for $\mathbf{n} \neq 0$,

$$(\Delta \mathbf{J})_{\mathbf{n}} \approx 2 \epsilon \mathbf{n} |\mathscr{H}_{\mathbf{n}}| \sqrt{\pi \frac{|\langle v_{\parallel} \rangle| L_B}{n_{\varphi} \omega}} \cos \psi_{\mathbf{n}}, \tag{3.15}$$

where $\psi_{\mathbf{n}}$ is a constant determined by initial conditions. Note that, since $\epsilon \propto L_B^{-1}$, the change of the action variable scales like $L_B^{-1/2}$. If L_B is large enough, so that $(\Delta \mathbf{J})_{\mathbf{n}} \ll \mathbf{J}$, then the scattering on multiple resonances crossed at random moments of time can be considered a diffusive process in the $\mathbf{J}$ space. Indeed, in average over $\psi_{\mathbf{n}}$, the squared change of action grows linearly with trace Z:

$$\langle (\Delta \mathbf{J})^2_{\Sigma} \rangle \sim Z \frac{|\Delta \mathbf{J}|^2}{\Delta z}, \tag{3.16}$$

where $|\Delta \mathbf{J}|$ is the characteristic (over $\mathbf{n}$) change of action (3.15) and Δz is the characteristic distance between the resonances. To ensure that $\langle (\Delta \mathbf{J})^2_{\Sigma} \rangle$ is small, the amplitudes of harmonics $|\mathscr{H}_{\mathbf{n} \neq 0}|$ must be sufficiently small. It is only in this case that approximate conservation of adiabatic invariants (3.14) can be claimed.

In the end, it is important to emphasize that, although the above consideration was developed for strictly periodic rf fields, it can also be extended to include a more general situation of interest, that is the rf field consisting of non-commensurate multiple harmonics. In this case, the rf-driven motion no longer remains periodic, and the P_φ conservation theorem must be revised. (Note that the loss of periodicity for the rf-driven motion does not impact on the μ conservation theorem, since Larmor rotation stays periodic with the frequency Ω.) For this situation, one must redefine the action P_φ as the 'energy' $\overline{H}$ averaged over some arbitrary time interval $\Delta\tau$, that is large compared with the correlation time of the rf-driven oscillatory motion $\tau_\sim$ as

$$P_\varphi = \lim_{\Delta\tau/\tau_\sim \to\infty} \frac{1}{\Delta\tau}\int_{\Delta\tau} \overline{H}\, dt = \langle \mathscr{E} \rangle. \tag{3.17}$$

As shown in Appendix B, for aperiodic processes, the quantity (3.17) represents an approximate integral of the particle motion (under the same stipulations as discussed above). Hence, the quasi-energy of a particle $\mathscr{E}$ and the (modified) magnetic moment μ are approximately conserved throughout the particle motion in an arbitrary non-resonant rf field under the condition of small ϵ.

4. Summary

In this paper, we have shown how the Lagrangian formulation of ponderomotive effects can be used to derive the average potential. This formulation also clarifies, both physically and mathematically the origin of the well-known approximate integrals of the particle motion. These include the magnetic moment μ of free Larmor rotation (in addition to the externally driven motion), and the quasi-energy $\mathscr{E}$ of the guiding center motion parallel to the magnetic field. By developing the Hamiltonian formulation, we have shown that μ and $\mathscr{E}$ represent adiabatic invariants of the particle motion only under the assumption of negligible heating at high-order resonances, which otherwise results in diffusive variations of these quantities. With minor reservations, the conservation of μ and $\mathscr{E}$ is preserved for both periodic and aperiodic high-frequency fields, although, in the latter case, the periodicity of the particle motion may be lost completely.

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Appendix A. Expression for $\langle L_\perp \rangle$

The Larmor rotation, in addition to the rf-driven motion, can be described in terms of the Larmor radius $\rho = |\mathbf{R}_\perp|$, the gyrophase ϕ and the corresponding canonical momentum $P_\phi = m\rho^2(\dot{\phi} + \frac{1}{2}\,\Omega)$. Using these variables, one can rewrite (2.10) in the form

$$L_\perp = -\phi\frac{dP_\phi}{dt} - \mu B_0 + \frac{d}{dt}(\phi P_\phi), \tag{A 1}$$

where $\mu = m\mathbf{v}_{\mathrm{f},\perp}^2/2B_0$. Note that

$$\frac{dP_\phi}{dt} = \frac{d\mathscr{L}}{d\phi} = \sum_n in\mathscr{C}_n(Z,\dot{Z})\exp(in\phi), \tag{A 2}$$

where we used the Fourier transformation of $\mathscr{L}$ with respect to the angle variable ϕ. The ϕ-dependence of $\mathscr{L}$ can only originate from the inhomogeneity of the magnetic (and rf) field, which vanishes in the zeroth-order approximation in ϵ, and thus $\mathscr{C}_n = \mathscr{O}(\epsilon)$. Also note that $d\phi/dt = \Omega + \mathscr{O}(\epsilon)$, and therefore the time-averaged derivative of P_ϕ is small compared with ϵ:

$$\left\langle \frac{dP_\phi}{dt} \right\rangle = o(\epsilon). \tag{A 3}$$

From the obtained result, it follows that P_ϕ (or $\mu = (e/mc)P_\phi$) represents an approximate integral of the particle motion. (More careful discussion is given in Sec. 3; see also Grebogi et al. (1979) and Motz and Watson (1967).)

The contribution of the first term in (A 1) into the integral (2.3) taken over a large time $\Delta t = \mathscr{O}(\epsilon^{-1})$ scales like $\mathscr{O}(1)$. Since the contribution of the μB_0 term appears to be of the order of ϵ^{-1}, the first term in (A 1) can be neglected and, omitting the full-time derivative, one can approximate the average Lagrangian function (A 1) as

$$\langle L_\perp \rangle = -\mu B_0, \tag{A 4}$$

where $\mu =$ constant (see also Danilkin (1995)).

Appendix B. Approximate integral of aperiodic motion with a slowly varying parameter

Consider a dynamic system governed by the Hamiltonian function $\mathscr{H}(\mathscr{Q}, \mathscr{P}, \lambda(t))$ with a parameter $\lambda(t)$ slowly varying in time t. Assume that, for $d\lambda/dt = 0$, the system undergoes aperiodic oscillatory motion with a characteristic correlation time $\tau_\sim$. Assume also that these oscillations are statistically uniform on time scales that are large compared with $\tau_\sim$. Consider the action J, that is the canonical momentum $\mathscr{P}$ averaged over the fixed trajectory $\Delta\mathscr{Q} = \int_{\Delta T} d\mathscr{Q}$, along which the system travels during some large time $\Delta T \gg \tau_\sim$:

$$J = \lim_{\Delta T(\Delta\mathscr{Q})/\tau_\sim \to \infty} \frac{1}{\Delta\mathscr{Q}} \int_{\Delta\mathscr{Q}} \mathscr{P}_0 \, d\mathscr{Q}_0, \tag{B 1}$$

where the subindex 0 denotes quantities evaluated on the unperturbed trajectory with $d\lambda/dt = 0$. The above-imposed requirement of statistical uniformity of the oscillatory motion guarantees that the averaging procedure is well-defined, so that the limiting value of the integral (B 1) exists.

Let us prove that the action J represents an approximate integral of the system, i.e. remains constant if $\lambda(t)$ changes slowly compared with unperturbed oscillations, in the sense that

$$\epsilon = \Delta T \left| \frac{1}{\lambda} \frac{d\lambda}{dt} \right| \ll 1. \tag{B 2}$$

(The treatment of periodic oscillations, similar to that given below, can be found in Landau and Lifshitz (1960).) To do so, consider the time derivative

$$\frac{dJ}{dt} = \frac{1}{\Delta\mathscr{Q}} \int_{\Delta\mathscr{Q}} \frac{\partial \mathscr{P}_0}{\partial t} \, d\mathscr{Q}_0, \tag{B 3}$$

where we have omitted the limit sign for clarity and made use of the fact that the limits of integration do not depend on time. Since the integration is performed over the unperturbed trajectory, $\mathscr{P}_0$ must be considered as a function of $\mathscr{Q}_0$, parameter λ and energy E: $\mathscr{P}_0 = \mathscr{P}_0(\mathscr{Q}_0; \lambda, E)$. Then

$$\frac{dJ}{dt} = \frac{1}{\Delta\mathscr{Q}} \int_{\Delta\mathscr{Q}} \left(\frac{\partial\mathscr{P}_0}{\partial\lambda}\frac{d\lambda}{dt} + \frac{\partial\mathscr{P}_0}{\partial E}\frac{dE}{dt} \right) d\mathscr{Q}_0, \tag{B 4}$$

where the partial derivatives can be obtained by differentiating the definition of the energy $E = \mathscr{H}(\mathscr{Q}, \mathscr{P}; \lambda)$:

$$dE = \frac{\partial\mathscr{H}}{\partial\mathscr{Q}}\, d\mathscr{Q} + \frac{\partial\mathscr{H}}{\partial\mathscr{P}}\, d\mathscr{P} + \frac{\partial\mathscr{H}}{\partial\lambda}\, d\lambda. \tag{B 5}$$

Representing the above expression as the complete differential of $\mathscr{P}_0(\mathscr{Q}_0; \lambda, E)$, one gets

$$\frac{\partial\mathscr{P}_0}{\partial\lambda} = -\frac{\partial_\lambda\mathscr{H}_0}{\partial_{\mathscr{P}_0}\mathscr{H}_0}, \quad \frac{\partial\mathscr{P}_0}{\partial E} = \frac{1}{\partial_{\mathscr{P}_0}\mathscr{H}_0}, \tag{B 6}$$

where $\mathscr{H}_0 \equiv \mathscr{H}(\mathscr{Q}_0, \mathscr{P}_0, \lambda)$. Let us use the Hamiltonian equation $d\mathscr{Q}_0/dt = \partial\mathscr{H}_0/\partial\mathscr{P}_0$ to rewrite the above integral in the following form:

$$\frac{dJ}{dt} = \frac{1}{\Delta\mathscr{Q}} \int_{\Delta T} \left(-\frac{\partial\mathscr{H}_0}{\partial\lambda}\frac{d\lambda}{dt} + \frac{dE}{dt} \right) dt, \tag{B 7}$$

equivalent to

$$\frac{dJ}{dt} = \frac{\Delta T}{\Delta\mathscr{Q}} \left(\left\langle \frac{dE}{dt} \right\rangle - \frac{d\lambda}{dt} \left\langle \frac{\partial\mathscr{H}_0}{\partial\lambda} \right\rangle_\lambda \right), \tag{B 8}$$

where the subindex λ stands for averaging at a fixed value of the parameter λ, and, in the first-order approximation in ϵ, $d\lambda/dt$ is assumed to be constant on time ΔT. Since

$$\frac{dE}{dt} = \frac{d\mathscr{H}}{dt} = \frac{\partial\mathscr{H}}{\partial t} = \frac{\partial\mathscr{H}}{\partial\lambda}\frac{d\lambda}{dt} \tag{B 9}$$

and $\lambda(t)$ is a slow function, we again take $d\lambda/dt$ out of the averaging and consider the rest a function of fixed λ and energy. The latter allows $\mathscr{H}$ to be replaced by $\mathscr{H}_0$, so that one gets $\langle dE/dt\rangle = (d\lambda/dt)\langle\partial\mathscr{H}_0/\partial\lambda\rangle_\lambda$. Finally,

$$dJ/dt = o(\epsilon), \tag{B 10}$$

from where it follows that the action J represents an approximate integral of motion at small ϵ.

Note that, in the above derivation, we implicitly assumed that, if $\lambda(t)$ is changing slowly, the true phase-space trajectory can be adequately approximated by the trajectory with fixed λ on the whole time interval ΔT. For the situation discussed in the main text, this requirement is fulfilled due to the linearity of the particle local response to the rf fields. However, in a strongly nonlinear system, phase space trajectories may be unstable with respect to small variations of parameters, rendering the above analysis invalid.

References

Danilkin, I. S. 1995 *Plasma Phys. Rep.* **21**, 777.

Dimonte, G., Lamb, B. M. and Morales, G. J. 1983 *Plasma Phys.* **25**, 713.

Eubank, H. P. 1969 *Phys. Fluids* **12**, 234.

Fisch, N. J. 1987 *Rev. Mod. Phys.* **59**, 175.

Fisch, N. J., Rax, J. M. and Dodin, I. Y. 2003 *Phys. Rev. Lett.* **91**, 205004.

Gaponov, A. V. and Miller, M. A. 1958 *Sov. Phys.–JETP* **7**, 168.

Gardner, C. S. 1959 *Phys. Rev.* **115**, 791.

Grebogi, C., Kaufman, A. N. and Littlejohn, R. G. 1979 *Phys. Rev. Lett.* **43**, 1668.

Jackson, J. D. 1975 *Classical Electrodynamics*. New York: Wiley.

Kildal, A. 1999 *Contribut. Plasma Phys.* **39**, 349.

Lamb, B. M., Dimonte, G. and Morales, G. J. 1984 *Phys. Fluids* **27**, 1401.

Landau, L. D. and Lifshitz, E. M. 1960 *Mechanics*. New York: Pergamon Press.

Lichtenberg, A. J. and Lieberman, M. A. 1992 *Regular and Chaotic Dynamics*, 2nd edn. New York: Springer.

Littlejohn, R. G. 1983 *Phys. Fluids* **24**, 1730.

Littlejohn, R. G. 1983 *J. Plasma Phys.* **29**, 11.

Litvak, A. G., Sergeev, A. M., Suvorov, E. V., Tokman, M. D. and Khazanov, I. V. 1993 *Phys. Fluids* B **5**, 4347.

Litwin, C. 1994 *Phys. Plasmas* **1**, 515.

Motz, H. and Watson, C. J. H. 1967 *Advances in Electronics* **23**, 153.

Suvorov, E. V. and Tokman, M. D. 1988 *Fizika Plazmy* **14**, 950 (in Russian).

Watson, C. J. H. and Kuo-Petravic, L. G. 1968 *Phys. Rev. Lett.* **20**, 1231.

White, R., Chen, L. and Lin, Z. 2002 *Phys. Plasmas* **9**, 1890.