Quantumlike Dynamics of Classical Particles in Ponderomotive Potentials

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The average dynamics of a classical particle under the action of a high-frequency radiation resembles quantum particle motion in a conservative field with an effective de Broglie wavelength λ equal to the particle average displacement on the oscillation period. In a quasiclassical field, with a spatial scale large compared to λ, the guiding-center motion is adiabatic. Otherwise, a particle exhibits quantized eigenstates in ponderomotive potential wells, tunnels through "classically forbidden" regions, and experiences stochastic reflection from attractive potentials.

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A multiscale approach to describing particles driven by intense electromagnetic radiation relies on separating fast oscillatory motion of the particle in the ac field from its slow translational motion. Assume that the parameters of the oscillations vary along the particle trajectory $\mathbf{s}(t)$ on a sufficiently large scale L, that is,

$$\epsilon \equiv \max\{\lambda/L, d\lambda/ds\} \ll 1, \tag{1}$$

where λ is the particle average displacement on the oscillation period. The translational motion is then conveniently described in terms of the so-called "guiding-center" variables, for which the explicit time (or phase) dependence is removed from the motion equations to any power in ϵ [1]. This technique is widely used in theoretical and computational plasma physics to study particle dynamics in magnetic, rf, and laser fields; what is in practice often missed though is the intrinsically limited accuracy of the guiding-center approximation. The transformation to the new coordinates is an asymptotic procedure with an exponentially small error in ϵ. The true drift coordinates are then in principle definable only with limited accuracy, thus the dynamics of the guiding center as a quasiparticle may not follow the laws of classical mechanics. Strikingly, what this dynamics resembles instead is the motion of a *quantum* object.

For the particle motion in a nonuniform magnetic field, a similar analogy was previously drawn by Varma, as reviewed in Ref. [2]. However, the explanation of the "macroquantum" effects in terms of the actual wave functions remains controversial [3]. In contrast to Varma's quantum approach, we show that purely classical particles can exhibit *quantumlike* effects. In particular, we show that an ac-driven particle exhibits quantized eigenstates in a ponderomotive potential well, tunnels through "classically forbidden" regions, and experiences reflection from attractive potentials. To describe these effects quantitatively, we employ the results of our previous research [4]; yet the conceptual results offered here follow from very elementary points.

The traditional understanding of the particle interaction with a wave barrier can be explained as follows [5,6]. Under the condition (1), the particle dipole moment $\mathbf{p}$ induced by the ac field follows an "adiabatic" equation of state. It means that $\mathbf{p}$ can be approximately expressed as a local function of the particle location $\mathbf{r}$, which, in the simplest case, is proportional to the amplitude of the field: $\mathbf{p} \approx \boldsymbol{\alpha}(\omega; \mathbf{r})\mathbf{E}_0(\mathbf{r})$. (Here $\boldsymbol{\alpha}$ is the polarizability tensor, ω is the field frequency, and the conventional complex notation is implied.) The average force on the particle can be approximately described in terms of the ponderomotive, or Miller potential Φ, equal to the average energy of the dipole-field interaction:

$$\Phi \approx -\frac{1}{4}(\mathbf{E}_0^* \cdot \boldsymbol{\alpha} \cdot \mathbf{E}_0). \tag{2}$$

Unlike for a true potential, the conservative property of the Miller force is only approximate. Consider an elementary particle with charge e and mass m exhibiting one-dimensional (1-D) oscillations governed by $m\ddot{z} = eE(z)\sin\omega t$. (In this case $\Phi \approx e^2E^2/4m\omega^2$, assuming $eE/m\omega^2 L \ll 1$.) Suppose that at $t = 0$ the particle is placed at a local minimum of $\Phi(z)$ (say, $z = 0$) with velocity v. If $\mathcal{E} \equiv \frac{1}{2}mv^2 < \Phi_{\max} - \Phi(0)$, the particle will exhibit bounce oscillations, periodic in the limit $\epsilon \to 0$. At finite ϵ though, when the adiabatic approximation is violated, the particle, in fact, gains or loses energy each time it bounces off a ponderomotive wall. The bounce motion remains strictly periodic only for a countable set of resonant orbits, for which the amount of energy gained by the particle per bounce period equals the energy transferred back to the field [7]. Unlike other trajectories, the periodic orbits can be assigned definite energies [$\mathcal{E}_n \equiv \mathcal{E}(z = 0, t = q\tau_n)$; q is an integer, τ_n is the nth bounce period], and, therefore, can be viewed as the stationary energy levels (eigenstates) of the particle in the ponderomotive potential, similar to those exhibited by a quantum particle in a conservative field.

Remarkably, this analogy predicts observable physical effects if an ensemble of particles is contemplated. As ϵ

increases, stochastic oscillations may develop, resulting in the particles escaping from the interaction region. Regular dynamics is preserved in the first place *near* the periodic phase-space trajectories (corresponding to elliptic points of Poincaré mapping [1]), as shown in Fig. 1. This selective confinement provides that, like in a quantum well, only particles with $\mathcal{E} \approx \mathcal{E}_n$ remain inside the interaction region at $t \to \infty$ for sufficiently large ϵ.

Discrete energy levels in a Miller potential have a similar nature to those in a quantum well. In fact, the *average* ponderomotive force applies not to the particle itself, but to its guiding center, which can be assigned a phase, $\theta = \omega t$, like a quantum object. The quantization rule in a ponderomotive potential (i.e., the periodicity requirement for the bounce motion) can be written in terms of this phase increment over the bounce period: $\Delta\theta_n = 2\pi n$, where n is an integer. By definition, large n corresponds to the adiabatic ("quasiclassical") domain, for which $\tau \approx \oint dz/\bar{v}$ can be expressed in terms of the guiding-center velocity $\bar{v}$. Since $\lambda = 2\pi\bar{v}/\omega$, the quantization condition at $n \gg 1$ then coincides with the Bohr-Sommerfeld rule

$$\oint k dz \approx 2\pi n, \tag{3}$$

with the "uncertainty" of the guiding-center location $\lambda = 2\pi/k$ serving as the effective de Broglie wavelength.

The Bohr-Sommerfeld eigenspectra (and quantumlike effects in general) are also inherent to other types of ponderomotive barriers. Consider an ac field of the form $\mathbf{E}_{\rm ac} = \mathbf{x}^0 E(z) \sin\omega t$ applied in the presence of a uniform dc magnetic field $\mathbf{B}^0 = \mathbf{z}^0 B_0$, so that the Miller potential is given by $\Phi = e^2E^2/4m(\omega^2 - \Omega^2)$, where $\Omega = eB_0/mc$ is the corresponding gyrofrequency [5,8]. In this case, it is the beat period $\tau_{\rm b} = 2\pi/|\Delta\omega|$ (where $\Delta\omega = \omega - \Omega$) that determines the adiabaticity parameter (1). On average over the ac period, the Larmor period, and the beat period

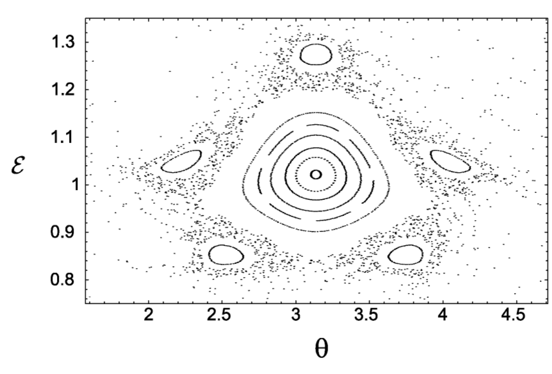

FIG. 1. Poincaré mapping (fragment) for a particle exhibiting 1D oscillations in the field $E(z, t) = \bar{E}_0 \sinh^2(z/L) \sin\omega t$ with $\hat{\epsilon} \equiv e\bar{E}_0/m\omega^2 L = 2$: $\mathcal{E}$ (a.u.) and $\theta = \omega t$ are taken at particle crossing $z = 0$ with $v > 0$. Stable oscillations (continuous curves) are observed near resonant orbits (one in the center and five on the sides); chaos is developed at the periphery. At larger $\hat{\epsilon}$, only particles near the central orbit are confined.

simultaneously, the particle exhibits conservative motion if $\lambda \equiv v_z\tau_{\rm b}$ is small compared to the spatial scale L of $E(z)$. Two adiabatic invariants are conserved in this case [8–10]. Those are the magnetic moment of free Larmor rotation $\mu = m(\mathbf{v}_\perp - \mathbf{v}_\sim)^2/2B_0$ and the quasienergy $\mathcal{H} = \frac{1}{2}mv_z^2 + \Phi(z) + \mu(B_0 - B_{\rm res})$, where $\mathbf{v}$ is the particle velocity; $\mathbf{v}_\sim$ is the velocity of the ac-induced particle oscillations transverse to $\mathbf{B}_0$; $B_{\rm res} = mc\omega/e$ is the magnetic field strength, at which the particle would be in exact cyclotron resonance with the ac field.

At adiabatic interaction, $\mu B_{\rm res}$ is constant (and so is μB_0 at $B_0 = \text{const}$) and hence could be omitted. Remarkably though, with this term, $\mathcal{H}$ is conserved also for $\epsilon \gtrsim 1$ at near-resonant drive ($\Lambda \equiv |\Delta\omega/\Omega|^{-1} \gg 1$), if $v/\omega L \ll 1$ [11]. An interesting type of a wave barrier is produced then, as explored both theoretically [9,12] and experimentally [10]. Averaging over the ac and Larmor periods, the particle motion equation takes the form $m\ddot{z} = -d\Phi_{\rm eff}/dz$ [4], where

$$\Phi_{\rm eff} = \Phi - \frac{\Delta\omega}{2\Omega} m|\psi - \psi_{\rm a}|^2 \tag{4}$$

is the effective potential, $\psi = (v_x + iv_y)e^{i\Omega t}$ is the slow amplitude of the particle transverse velocity in complex representation, $\psi_{\rm a} \approx -(eE/2m\Delta\omega)e^{-i\theta}$ is its leading-order adiabatic counterpart for the induced oscillations, $\theta = \Delta\omega t$ is the new phase assigned to the particle guiding center, and $t(z) = \int_0^z d\tilde{z}/v_z(\tilde{z})$. In the limit $\epsilon \to 0$, $\mu = m|\psi - \psi_{\rm a}|^2/2B_0$ is conserved, hence, the second, phase-dependent term in Eq. (4) produces no effect. The average dynamics then effectively decouples from the transverse quiver motion, and the particle "sees" the adiabatic potential Φ. If the condition (1) is broken though, the μ conservation is violated. The two types of motion then become strongly interconnected, and essentially phase-dependent dynamics takes place.

Assume for definiteness that $E(z = 0) = 0$, $E(z) = E(-z)$, $\omega < \Omega$, and consider a subclass of oscillations with $v_\perp(z = 0) = 0$. A particle starting with $v_z = v_0$ at $z = 0$ will be decelerated by the ac field, come to a stop at $z(t_{\rm s}) = A$, and be reflected. Since the particle motion may be phase dependent, the trajectory after the reflection generally will not be symmetric to that before the reflection. However, the symmetry does exist for some v_0. Having such a case requires that $\Phi_{\rm eff}[z(t)]$ is an even function of $t - t_{\rm s}$, which is equivalent to

$$\arg\psi(A) + \theta(A) = \pi n, \tag{5}$$

where n is an integer. If this condition is satisfied, the particle is returned to $z = 0$ with the longitudinal and transverse energies precisely matching their initial values $\mathcal{E}_{\|,0} = \frac{1}{2}mv_0^2$, $\mathcal{E}_{\perp,0} = 0$. If $\Phi(z)$ is even, the phase-space trajectory of the particle will form a closed loop on the plane (z, v_z) [Fig. 2(a)]. Like a quantum particle in a conservative field, a classical particle in a ponderomotive

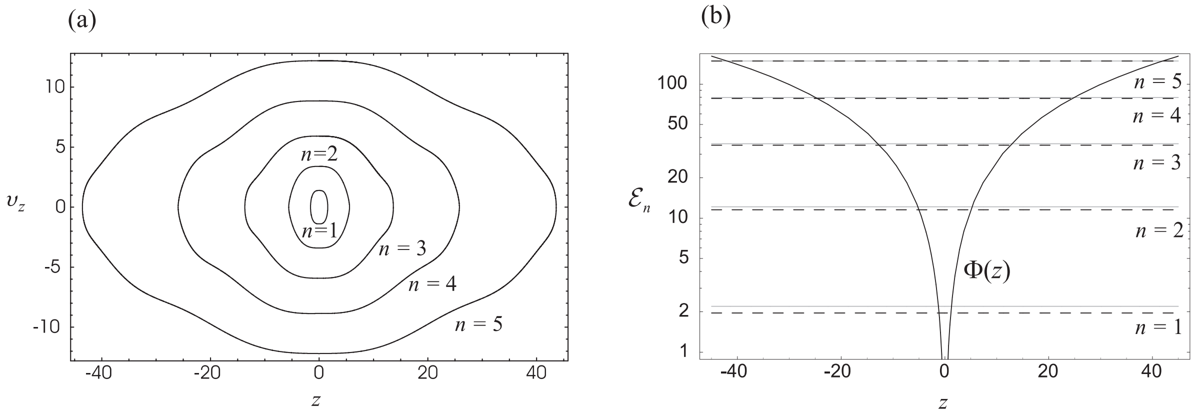

FIG. 2. First five stationary eigenstates of a guiding center trapped within a ponderomotive potential formed by an ac field with the amplitude $E(z) = q|z|^\alpha$, $\alpha = 0.6$, $q = 10^{-3}$: (a) phase plane (z, v_z), (b) energy diagram [dashed, numerical; solid gray, quasiclassical; solid black, Miller potential $\Phi(z)$] (E is measured in units $mc\Omega/e$; z is measured in units c/Ω; v_z is measured in units $\hat{v}$; $\mathcal{E}_\perp$ is measured in arbitrary units).

potential can exhibit periodic bounce oscillations only if its energy satisfies the quantization condition (5). In the adiabatic limit ($n \gg 1$), Eq. (5) yields $\Delta\omega t_s \approx \pi n$, or $\Delta\theta_n \approx 4\pi n$ for the change of θ over the whole bounce period. The quasiclassical approximation of Eq. (5) then reads $\oint k dz \approx 4\pi n$, where $k = 2\pi/\lambda$. This equation is again analogous to the Bohr-Sommerfeld quantization condition, yet now for even energy levels. As usual, to apply this rule for calculating the actual eigenspectrum $\mathcal{E}_n$ in a Miller potential, one can derive $v_z(z)$ from the adiabatic motion equations.

As an example, consider $E(z) \propto |z|^\alpha$, $\alpha > 0$. [For a nonanalytic $E(z)$ like this, some of the eigenstates are unstable. Minor modification of $E(z)$ near $z = 0$ restores the stability without changing most of $\mathcal{E}_n$.] By approximating $z(t)$ with a parabola on a half of the bounce period, one can write the quasiclassical quantization rule in a more precise form $\oint k dz = 4\pi(n + \alpha/2)$. The energy spectrum is then given by $\mathcal{E}_n = (n + \alpha/2)^{2\alpha/(1-\alpha)}\hat{\mathcal{E}}(\Lambda, \alpha)$, where $\hat{\mathcal{E}}$ is a constant depending solely on the parameters of the field. Hence, if $\alpha < 1$, $\mathcal{E}_n$ increases with n, and the quasiclassical limit is approached at $v_0 \to \infty$. On the other hand, if $\alpha > 1$, $\mathcal{E}_n$ decreases with n and becomes quasiclassical at $v_0 \to 0$. (At $\alpha = 1$, which would correspond to a linear pendulum in the limit $\epsilon \to 0$, a degeneracy is observed: in this case all trajectories are self-similar, and the bounce period is independent of v_0.) The difference between the two cases is due to the fact that for $E(z) \propto |z|^\alpha$ the scale L is effectively determined by the amplitude of bounce oscillations: $L = A \propto v_0^{1/\alpha}$. The condition (1) with $\lambda \equiv 2\pi v_z/\Delta\omega$ then can be put as $(v_0/\hat{v})^{\alpha-1} \ll 1$ [here $\hat{v} = (\hat{\mathcal{E}}/m)^{1/2}$], which is satisfied for $v_0 \gg \hat{v}$ if $\alpha < 1$ and $v_0 \ll \hat{v}$ if $\alpha > 1$ [Fig. 2(b)].

Remarkably, the quantum analogy extends further and applies also to free (nonconfined) particles: The average force on the guiding center is proportional to the induced transverse oscillations of a particle. If a particle incident on a localized wave barrier is fast enough ($\epsilon \gtrsim 1$), it will not have sufficient time to gain quiver energy from the field and hence will neither experience significant average acceleration. Such a particle will then be able to penetrate ("tunnel") through classically forbidden regions $\frac{1}{2}mv_0^2 < \Phi(z)$, just like a quantum particle having a de Broglie wavelength of the order of the field scale.

Quantumlike properties are also inherent to attractive barriers, which turn out to be capable of reflecting particles: The conservation of $\mathcal{H}$ yields $\Delta\mathcal{E}_\parallel = (\Delta\omega/\Omega)\Delta\mathcal{E}_\perp$, which connects the overall changes of $\mathcal{E}_\parallel$ and $\mathcal{E}_\perp$ at $t \to \infty$. In the adiabatic limit, the net energy change is exponentially small with respect to ϵ. Suppose though that $\epsilon \gtrsim 1$ and $\Phi < 0$ (i.e., $\omega < \Omega$). If $\Delta\mathcal{E}_\perp > 0$, a particle loses $\mathcal{E}_\parallel$ as a result of interaction. At some $v_0 = v_z(t \to -\infty)$, the deceleration can become sufficient to trap a particle in a potential well: a particle entering the ac field freely can be bounced back toward the stronger field at the exit (Fig. 3).

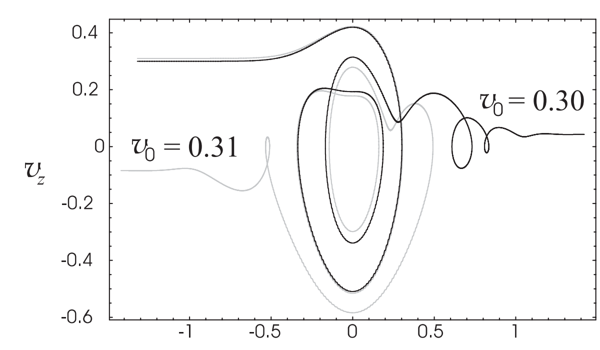

FIG. 3. Longitudinal velocity v_z vs z for a particle being trapped and released by a Gaussian attractive ponderomotive potential in a magnetic field with $\hat{\epsilon} = 6\pi$ ($E_{\max} = 0.001$, $\Lambda = 100$, $L = 0.33$; $\mathcal{E}_{\perp,0} = 0$; same notation as in Fig. 2): $v_z = 0.30$ (black) and $v_z = 0.31$ (gray).

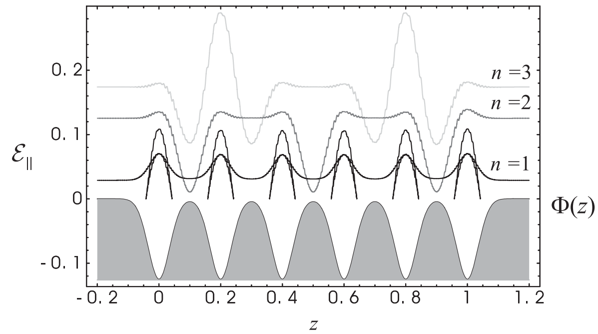

FIG. 4. First three stationary eigenstates of a free particle traveling through a crystal formed of multiple ponderomotive barriers (same notation and parameters as in Fig. 3): shaded, $\Phi(z)$; solid, longitudinal energy $\mathcal{E}_\parallel$ vs z.

The trapping condition can be derived as follows. Slow particles ($v_0 \ll \hat{v}$) are accelerated up to v_z of the order of $\hat{v} = (|\Phi|_{\max}/m)^{1/2}$. If $\hat{v}$ itself is large enough, that is, $\hat{\epsilon} \equiv \lambda(\hat{v})/L \gtrsim 1$, nonadiabatic effects have to reveal for all, even initially slow particles, some of which may then experience trapping. On the other hand, at $\hat{\epsilon} \lesssim 1$, slow particles remain adiabatic and hence cannot be trapped. As for fast particles ($v_0 \gg \hat{v}$), in both cases they have enough energy to overcome the deceleration and avoid trapping. Thus, if $\hat{\epsilon} \gtrsim 1$, at sufficiently small v_0 a particle can be trapped within a ponderomotive potential, but if $\hat{\epsilon} \lesssim 1$, trapping is impossible regardless of v_0.

What is the "destiny" of a "once trapped" particle? Because of the phase-space conservation requirement, particles may not stay trapped forever. However, if the number of bounce oscillations within a potential well is large, the post-trapping dynamics of a particle correlates little with its pretrapping dynamics. The direction in which the particle is released is almost uncorrelated with the initial velocity (Fig. 3). Hence, the particle can escape toward the direction opposite to v_0, which qualifies as reflection. The effect disappears under the condition (1), and again resembles a quantum phenomenon in that a particle can be reflected by an attractive potential for de Broglie wavelength of the order of L.

Because of clearly stochastic behavior inside a ponderomotive well, a particle traveling through a chain of such potentials would undergo a random walk, as each of the potentials can scatter a particle back and forth with roughly equal probability. Hence, a sufficiently long chain of barriers violating the condition (1) acts like a diffusive mirror. However, among v_0, for which stochastic dynamics is realized, there exists a countable set of regular trajectories, at which a particle can "collisionlessly" travel through a "crystal" formed by multiple barriers. These trajectories can then be attributed as stationary eigenstates of a free particle moving in a "ponderomotive crystal." Such eigenstates were found numerically for chains of both attractive and repulsive potentials (Fig. 4). Similarly to bounce oscillations within a potential well, the ground energy level ($n = 1$) of a transmitting particle is located at $v_0 \sim \hat{v}$. Higher levels ($n > 1$) are located at larger energies, and at $n \gg 1$ (corresponding to $v_0 \gg \hat{v}$) the particle motion becomes classical, that is, in this case, only slightly disturbed by the ponderomotive force.

The quantumlike effects described here run counter to what follows from the traditional adiabatic theory. Selective confinement, tunneling, and stochastic reflection cannot be easily captured by asymptotic methods, as those would not resolve the probabilistic nature of the guiding-center dynamics. Therefore, in addition to the academic interest in the demonstration that very general classical systems can exhibit quantum effects, capturing the effects we describe here will be a challenge to the existing computational and analytical techniques in plasma kinetic theory.

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