AS551 General Plasma Physics I
November 5, 2004

Homework #8, due Friday, Nov 12

1. Deriving fluid equations. Some research papers investigated the effects of the ionization of neutrals on a certain class of plasma turbulence. (Motivational background: interactions with neutrals are often justifiably ignored in the hot plasmas of fusion research, but they can become important near the interface of a plasma with a solid wall, such as near the divertor plates in a tokamak, or in plasma processing of semiconductor chips.) The essential features of the equations used in those papers can be seen in the 1-dimensional limit, which can be written as:

$$\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial z} = n n_0 s$$

$$mn\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial z} + enE_{\parallel}$$

where $s = \langle \sigma v \rangle$ is the reaction rate for electron-impact ionization of the neutrals so that $nn_0 s$ is the particle source rate due to ionization. (σ is the cross-section for the reaction, but you don't have to worry about the details and can just treat s as a constant here.) The ion particle density n, neutral density n_0, average ion velocity u, and pressure p are all functions of position z (along a magnetic field line) and time t.

Unfortunately, there is an error in the above equations which led to the claim of a spurious instability to drive turbulence. Your job is to to find this error by deriving the proper form of the above fluid equations from first principles by starting with the 1-dimensional Vlasov equation for the ion distribution function $f(z, v, t)$, modified to include an ionization source:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} + \frac{e}{m}E_{\parallel}\frac{\partial f}{\partial v} = nn_0 s\delta(v)$$

(Note that the reaction rate $s = \langle \sigma v \rangle$ has already been averaged over the electron velocities, and should be taken to be a fixed constant for your purposes.) The neutrals are very cold compared to the plasma, and so are treated as having zero velocity (hence the δ function) on the scale of the much hotter plasma ions.

You should discover a term which is missing from the above fluid equations. Give a brief description of the physics of this missing term.

2. A non-uniform equilibrium solution of the Vlasov Eq. Consider a slab of plasma that varies only in the x direction, in a simple uniform straight magnetic field in the $\hat{z}$ direction, with no electric field. A particle at position $\vec{x}$ with velocity $\vec{v}$ is gyrating around its guiding center located at position $\vec{R} = \vec{x} + \vec{v} \times \hat{z}/\Omega_c$, where Ω_c is the cyclotron frequency (as the particle moves, its position $\vec{x}$ and velocity $\vec{v}$ change, but its guiding center position $\vec{R}$ remains fixed—in this simple limit where there are no guiding center drifts).

(a) Explicitly show by substitution that a possible equilibrium solution of the Vlasov equation is of the form $f = g(v^2)h(\hat{x} \cdot \vec{R}) = g(v^2)h(x + v_y/\Omega_c)$. [This is a special case of a more general principle that any function of constants of the motion is an equilibrium solution. In this case the constants of the motion used are the energy v^2 and the guiding center position $\vec{R}$.]

(b) The xx component of the pressure tensor is defined as $p_{xx} = \int d^3v f m v_x^2$. Using the form of f in part a, differentiate this with respect to x to show that $\partial p_{xx}/\partial x = j_y B/c$, where j_y is the $\hat{y}$-directed current in the particles. (You may do some integration by parts to prove this. Manipulations similar to these arise in the "gyrokinetic" equation used to study turbulence in tokamaks and other plasmas.)