A Student Survives the Exam: A Counter-Intuitive Particle Drift Problem

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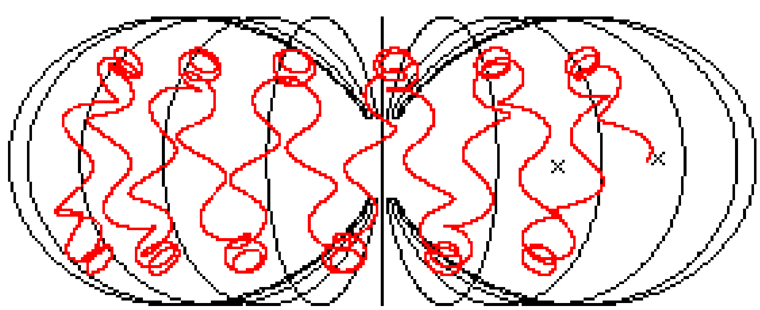
This is an intriguing little particle drift problem which is counter-intuitive at first to many people. To be a successful physicist, one needs to be able to think slowly—carefully and slowly.

The dialogue below is a composite (not meant to be any actual student) based on our experience in giving this problem to students in the introductory graduate course in plasma physics and in general oral exams (but we only gave it in the later stages of oral exams to students who were doing well, to try to avoid unncesessarily shaking their confidence).

The motion of particles in magnetic and electric fields is of course basic (but fundamental) physics. Understanding such motion is important for having good intuition, not just about various fusion plasma devices and astrophysical plasmas, but also about particle detectors and accelerators, microwave generators and plasma thrusters, and various electronic devices. In plasma physics (and in the dialogue below), we often make use of an implied guiding center approximation, in which the radius of gyration of a particle is small and one can consider the slower drifts of the "guiding center" about which the particle gyrates.

The problem:

> **Consider the magnetic field produced by a small ring of current enclosing the z axis. The magnetic field far from the ring has a dipole structure, so on the midplane $|B| \propto 1/R^3$, where R is the distance from the ring. Consider a small gyroradius particle deeply trapped (the component of the velocity parallel to $\vec{B}$ is small) in this dipole magnetic field (this is similar to a particle that is trapped near the equatorial plane of the earth's magnetosphere). Sketch the particle's orbit (which the student does as follows[1], to illustrate the gyromotion, the bounce-motion, and the precession due to grad-B/curvature drifts).**



> **The current in the ring is now slowly increased. Does the energy of the particle go up or down? (You can consider the limit $v_\parallel = 0$ to simplify the analysis.)**

[1]this figure is actually taken from the interactive "xspace" program used to teach space plasma physics, see http://www-ssc.igpp.ucla.edu/ssc/software/xspace.html and http://www-ssc.igpp.ucla.edu/pictures/edu_particle.gif

The eager young student assumes this is one of those simple "quicky" problems and says, "well, assuming that the current in the ring changes slowly compared to the cyclotron period of the particle, then the magnetic moment μ of the particle is an adiabatic invariant. Since $\mu = mv_\perp^2/2B$, the energy of the particle must go up because the magnetic field goes up."

Then the examiner asks, "Is that your final answer?", and the fun begins.

Trying to be helpful, the examiner asks where B is evaluated in this formula for μ. But the student just says "at the guiding center position", and doesn't see at first why this would change the answer. The examiner next asks, 'What are the invariants of the motion?" The student replies, "E, μ, and P_ϕ,—no wait, that would mean that the energy E is fixed. Sorry, those are the constants of the motion only in static fields. With time-varying fields, there are the first, second, and third adiabatic invariants, corresponding to μ, to the bounce motion along the field line, and to the particle's precessional drift around the z axis. But we are taking the limit of $v_\parallel = 0$ so the bounce motion is probably irrelevant." Then the student asks, "can we assume that the magnetic field is changing slowly compared to the precession frequency of the particle so that the third adiabatic invariant is conserved?". She is assured that yes, the problem states that the magnetic field is changing slowly, so she can assume it is slow even compared to the precession frequency.

Then the student explains the physics of the third adiabatic invariant (using this to buy time to think about the problem and see if she missed something). "The third adiabatic invariant is similar in a way to the first adiabatic invariant. The first adiabatic invariant corresponds to conservation of magnetic flux inside the small gyroradius of the particle. Because of the particle's rapid gyromotion, it can essentially be treated as a small, ideally-conducting ring of current, which must conserve the magnetic flux going through the ring (assuming the magnetic field is changing slowly compared to the cyclotron period). It accomplishes this conservation of flux by adjusting its gyroradius. (The general interpretation of adiabatic invariants is what Einstein called conservation of "action", but in our case we can think of them as related to conservation of magnetic flux inside an ideally conducting loop). So too the third adiabatic invariant corresponds to conservation of magnetic flux inside the region enclosed by the particle as it undergoes grad-B and curvature drifts around the z-axis. [These drifts are what cause the precession around the earth of energetic particles trapped in the earth's magnetosphere.] "

"In this problem, since the magnetic field is everywhere going up, in order for the precessional drift to enclose the same amount of magnetic flux, the particle's orbit around the z-axis must shift in to smaller R. So the energy ($= \mu B$) of this particle goes up even faster than I first thought."

But the examiner counters, "shouldn't the particle also conserve the amount of flux outside of its orbit?" This is a bit disconcerting at first. People don't usually talk about the flux outside of a loop. But since $\nabla \cdot \vec{B} = 0$, the flux outside of a loop must exactly balance the flux inside of a loop, so the two ways of looking at it must be equivalent. "Yes,", the fast-thinking student agrees, "the flux outside of its orbit should be conserved also." But if this is true, then as the magnetic field increases, the particle's precession orbit would have to expand to larger R to conserve this flux. The student knows there must be a solution to this conundrum, and, after pondering for a moment, realizes that she had forgotten about the magnetic flux coming up through the center of the current ring, which has the opposite sign of the magnetic flux coming down outside of the ring. The student now realizes that it must be that as the current in the ring is increased and the magnetic field everywhere is increased, then the particle's orbit must actually expand to conserve the amount of flux outside (or inside) of its orbit.

Doing a quick blackboard calculation of the amount of magnetic flux outside of the particle's orbit, assuming it is at large enough R that a dipole approximation $|B| \propto 1/R^3$ is accurate, the student finds that, although B everywhere is going up, the particle moves far enough out at the same time that the B it sees actually drops.

Thus the student arrives at the unsettling conclusion that even though the magnetic field everywhere went up, the particle's energy actually goes down.

Being a little unsure of this result, the student looks for a simpler explanation to make sure it's true, saying "I suppose another way of looking at this is to consider the fact that the time-changing magnetic field induces an electric field in the θ direction which would then lead to an $E \times B$ drift of the particle". Doing a quick calculation, the student verifies that indeed the sign of this $E \times B$ drift is outwards. "I probably should have thought of this from the start (meaning the fact that the particle doesn't stay at fixed R but moves outwards), because the $E \times B$ velocity can be thought of as the velocity at which magnetic field lines move (related to the fact that particles and magnetic fields move together in the "frozen-in field lines" condition of ideal MHD). Increasing the current in the central ring means that additional magnetic field lines are being generated and the other magnetic field lines must move outwards to maintain magnetic force balance."

[In our experience, about half of the students have their intuition backwards about which direction the magnetic field lines move as the current in the dipole ring increases. They seem to think that since magnetic field lines must move closer together in order for the magnetic field strength to go up, the field lines must all squeeze in closer to the dipole ring at the origin. But this would require a source of magnetic field lines at infinity, and neglects the fact that every field line that comes down through the midplane at large R also comes back up through the midplane inside of the ring. The right way to think about this is to realize that magnetic fields are being generated at the surface of the current ring, and must then move outwards. This of course must be true to satisfy causality: if you suddenly change the current in the ring, the magnetic fields at large R can't change until at least the time it would take for light to propagate that distance. Because our intuition can sometimes be wrong, it is good to be able to check the answer in multiple ways, as the student did here by calculating the $E \times B$ drift and the flux outside of the particle's orbit.]

Thus the student has arrived at the startling conclusion that, even though the magnetic field everywhere went up and $\mu = E/B$ is conserved, the particle's energy actually goes down, because it moves outwards to a region of lower B faster than the B at a fixed position is increasing. The examiner, being satisfied with the depth and carefulness of the student's answers to this problem, now moves on to the next problem.