Lecture Notes
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What is a Plasma?

The Fourth State of Matter

A plasma is a gas of charged particles. Often it is referred to as the fourth state of matter, the first three states being solid, liquid and gas. When solid matter is heated it may lose its given shape and become liquid. Further heating can separate the molecules even farther apart so that the material no longer occupies a given volume – it becomes gaseous. Further heating yet causes dissociation into ions and electrons, producing the plasma.

The idea of a state of matter requires some clarification: it is not a property of a material so much as it is an approximation of a process. The precise, but impossibly complicated, description of a material would evolve the coupled equations of motion of each of the smallest units. What is taken as the smallest unit, perhaps a molecule, itself represents an approximation. Further approximations are made through assumptions about the correlations between these units. For example, if we expect the material to maintain its shape, then an enormous simplification is possible in requiring that the relative positions of the smallest units are unalterable - in other words, the material is characterized as a solid.

Characterizing a wax candle as a solid makes tractable a mathematical description of a spinning and rotating wax candle that is thrown into the air and caught again. But at roughly room temperature over months or years the candle will flow and assume the shape of its container. In other words, on a long time scale it behaves as a liquid.

A second example might be gravel. Certainly, gravel feels solid, but in describing the flow of gravel in the unloading of a gravel truck, it should be called a liquid. The designation of a state of matter is helpful nomenclature that defines not the material, but the best approximation in simplifying the exact equations of motion in a given problem of interest. Accordingly, for some applications, a copper bar might be a solid, while for others (namely electrical conduction), it might be an electron plasma.

Gas Law

A plasma is to first approximation an ideal gas, albeit a very special gas that exhibits far more varied behavior than can an ordinary gas. Still, the most important approximation is that the plasma is very nearly an ideal gas (that is, the correlation between particles is small), and the field of plasma physics largely involves reducing the N-body equations of motion to a tractable set of equations based on this approximation.

The assumption of small correlations in an ordinary gas leads to the ideal gas equations and, specifically, to Boyle's Law. Boyle's Law states that the gas pressure is proportional to the product of the density n and the temperature T. Before pointing out the similarity between the plasma and the ideal gas, we offer a simple derivation of this law.

Consider a particle in a cube with sides of length L. There are two faces perpendicular to the x-axis and a particle with velocity v_x will collide with each of these faces every bounce time $\tau = 2L/v_x$. The momentum transferred to the cube wall in an elastic collision will be $2mv_x$, where m is the particle mass. The total force on a side of the cube will be the total momentum transferred per unit time by all of the N particles that might be found in the cube. In other words, the force

$$F_x = N\frac{2mv_x}{\tau} = \frac{mN}{L}v_x^2, \tag{1}$$

is exerted on each of the planes perpendicular to the x axis.

Similar forces are exerted on each of the faces perpendicular to the y and z axes. If there is a distribtion of particle velocities, then we can write the total force total outward force exerted on any of the six faces if the system is isotropic as

$$F_{\mathrm{TOT}} = \frac{mN}{L}\langle v_x^2\rangle, \tag{2}$$

where the angle brackets indicate an average over all particle energies.

The pressure exerted by the gas is the force per unit surface area. The total surface area on any side is L^2, so the pressure is

$$P = F_{\mathrm{TOT}}/L^2 = nkT, \tag{3}$$

where $n \equiv N/L^3$ is the particle density, and we used the definition of the temperature T of the system, $kT/2 \equiv m\langle v_x^2\rangle/2$, where k is Boltzmann's constant and $m\langle v_x^2\rangle$ is the average energy per degree of freedom. This is Boyle's law for an ideal gas ($PV = NkT$).

The ideal gas law applies in the limit of very dilute gases where the particle may be taken as noninteracting. Corrections to the ideal equations, using the so-called virial coefficients, take into account multi-body encounters. Consider, for example, that a 180° collision between finite size particles of diameter d effectively decreases by d the distance that the particles need traverse to reach the walls of the cube. Reaching the wall sooner means that the pressure will be higher.

Attractive forces between particles can effectively increase the distance traversed (also on the order of a molecular diameter) before the particle reaches the wall. The bounce frequency in this case is decreased, which implies that the pressure will be lower.

The percentage correction to the pressure will be given by the percentage change in the particle trajectory due to a collision. In each collision the trajectory is altered a distance d, whereas the total distance covered between collisions is the mean free path, λ_{mfp}. Thus, if

$$d/\lambda_{\mathrm{mfp}} \ll 1 \tag{4}$$

then the corrections to the ideal gas law should be small.

Note that the mean free path is related to the interparticle spacing $n^{-1/3}$. A particle impinging upon any thin slice in the gas has a probability of about $d^2n^{2/3}$ of colliding in that slice. Succeeding slices, spaced by $n^{-1/3}$, stop the particles that survive the first slice. The mean free path, or characteristic penetration length, is given by the number of layers likely to be survived, i.e.

$$\lambda_{\text{mfp}} = n^{-1/3}\left(\frac{n^{-1/3}}{d}\right)^2 . \tag{5}$$

Using Eq.(6), the condition in Eq.(5) may be rewritten as

$$\frac{d}{\lambda_{\text{mfp}}} = \left(\frac{d}{n^{-1/3}}\right)^3 \ll 1$$

or equivalently

$$nd^3 \ll 1, \tag{6}$$

which indicates that the gas must be dilute.

The Plasma Approximation

In the ideal gas, the forces between neutral molecules, whether repulsive or attractive, tend to operate over at most atomic distances, so that the trajectory alteration is always about d. For charged constituents, a new length arises in the problem – the closest distance to which, in view of their coulombic repulsion, two charged particles can approach each other.

The distance of closest approach, b, can be found by requiring that the potential energy, $e^2/\epsilon_0 b$, when the particles are close but stationary, is equal to the kinetic energy, mv^2, of two completely separated but moving particles. This exercise gives a typical distance of closest approach for typical particles with mean energy $m\langle v^2\rangle$ as

$$b = e^2/\epsilon_0 m\langle v^2\rangle. \tag{7}$$

Completely in analogy with the derivation of Eq.(4) we can write a condition

$$1 \gg \frac{b}{\lambda_{\text{mfp}}} = \left(\frac{b}{n^{-1/3}}\right)^3 . \tag{8}$$

Here, we use for λ_{mfp} the mean free path that results only from 90° scattering collisions. In fact, in a plasma, a more detailed analysis including all collisions, results in a substantially shorter mean free path – but for the arguments here the present simplified analysis suffices.

Let us introduce a very important plasma parameter, the Debye length λ_D, which can be defined such that $b \equiv 1/n\lambda_D^2$. Thus, we have

$$\left(\frac{b}{n^{-1/3}}\right)^3 \equiv \left(\frac{1}{n\lambda_D^3}\right)^2 , \tag{9}$$

so that Eq.(8) implies that $n\lambda_D^3 \gg 1$, or that there are many particles in a so-called Debye sphere. That there are many particles in a Debye shere is often referred to as the "plasma approximation" which allows us to describe the "ideal plasma." If this limit is not satisfied, we have a so-called "strongly-coupled plasma" – something that is far more difficult to describe mathematically.

The Debye length is the characteristic distance over which the plasma enforces charge neutrality. To see this, assume that the ions constitute a homogeneous background and are too sluggish to participate in the relaxation of a charge imbalance, while electrons obey the Gibb's distribution

$$n_e = n_0 \mathrm{e}^{\mathrm{e}\phi/\mathrm{kT}} = \mathrm{n}_0(1 + \mathrm{e}\phi/\mathrm{kT}). \tag{10}$$

The linearization is allowed because the plasma kinetic energy is large compared to its potential energy (this is a consequence of $n\lambda_D^3 \gg 1$). Using Eq.(10) together with Poisson's equation,

$$\nabla \cdot \epsilon_0 \mathbf{E} = e(n_i - n_e), \tag{11}$$

we can write an equation for the potential as

$$\nabla^2 \phi = \frac{e^2 n_0}{kT\epsilon_0}\phi \equiv \lambda_D^{-2}\phi, \tag{12}$$

which indicates that the potential decays in the characteristic distance λ_D, as oppositely charged particles form a cloud of that radius to neutralize any charge excess.

The Debye length is a characteristic feature of charged liquids as well as plasmas. Any charged fluid near equilibrium will obey both Eq.(10) and Poisson's equation. Even if the linearization is not valid, λ_D remains the characteristic decay length for the potential, although the decay may not be exponential. The linearization, however, tends to be valid in plasmas because of their high temperature.

The characteristic rate at which the charge imbalance is restored is given by the electron plasma frequency, ω_{pe}. Assume again that only electrons participate in restoring a charge imbalance. In addition to Poisson's equation, we write the particle conservation equation,

$$\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{v} = 0, \tag{13}$$

where the electron velocity obeys

$$\frac{d\mathbf{v}}{dt} = \frac{e\mathbf{E}}{m_e}. \tag{14}$$

These equations are easily solved by a linearization $n = n_0 + n_1 \exp(i\omega t), v = v_0 + v_1 \exp(i\omega t)$, etc. which removes the nonlinearity in Eqs.(13) and (14). The small signal quantities are then found to oscillate at ω_{pe}, where

$$\omega_{pe}^2 = e^2 n/m_e\epsilon_0. \tag{15}$$

The characteristic length λ_D and time $1/\omega_{pe}$ are related, not surprisingly, by the average electron velocity $v_T \equiv \langle v^2 \rangle^{1/2}$, *i.e.*, $\lambda_D = v_T/\omega_{pe}$

In summary, the long range coulomb force allows all particles within a Debye sphere to interact at once. Since $n\lambda_D^3 >> 1$, there are many such particles. This appears to be completely different from the ideal gas, wherein only two particles interact at a time as

$nd^3 \ll 1$. However, the difference between these two media appears less stark in view of the equivalent plasma condition $nb^3 \ll 1$, which closely resembles the requirement for the ideal gas to be dilute. In other words, while many particles interact at once in a plasma but not in an ordinary gas, it is the weakness of the particle correlations that, in fact, forms the basis for describing accurately both media.

Plasma Confinement

All magnetic confinement schemes exploit the fact that a plasma is a collection of charged particles. A charged particle cannot move in a straight line perpendicular to a magnetic field. The magnetic field forces the particle into a circular orbit and the larger the magnetic field, the tighter the orbit. Motion parallel to the magnetic field is unaffected, so that charged particles in general execute helical motion along the magnetic field.

A uniform magnetic field evidently reduces the problem of confinement in three dimensions to a problem of confinement in merely one dimension. Particles are guided along the field, but their orbits are limited in extent in the perpendicular direction. This is the fundamental one dimensional confinement problem and there are many approaches to solving it. Mirror machines try to reflect the particles between two points. Toroidal confinement schemes, such as the tokamaks at PPL, employ magnetic fields that are themselves bent into circles.

Particles escape the magnetic trap because they collide. In practical schemes of economical size, collisions are so infrequent that particles traverse many system lengths (10–100) before colliding. On the other hand, scarcely do collisions result in nuclei fusing. These inequalities may be written as

$$L_{\text{fusion}} >> \lambda_{mfp} >> R.$$

The first inequality implies that particles tend to assume a Maxwell-Boltzmann equilibrium distribution before fusing. Schemes that rely upon other distributions of energy must overcome this serious difficulty. The second inequality (R is a measure of system size) implies that extreme care must be taken to assure that particles are confined but for collisional effects and implies that the single particle orbits must themselves be closed. Single particle orbits will be taken up in the next lecture.

Of course, collisions eventually destroy any magnetic confinement, but luckily indefinite confinement is not necessary. More energy must be gained from fusion than expended in creating the conditions for fusion – and the plasma must be confined only long enough for this breakeven criteria to be true.

This confinement requirement is known as the Lawson condition and it may be derived as follows: The total particle energy density is given by $3nT$. If this energy is lost from the system in a characteristic time τ_e, then the power density required to sustain the plasma is

$$P_d = 3nT/\tau_e.$$

At the same time, fusion power is being produced with density

$$P_f = n^2 g(T),$$

where g is some function of temperature related to the probability of colliding ions fusing. The density enters squared because a single ion's likelihood of fusing is clearly proportional to the number of background ions. Since each of n ions fuse with probability proportional to n, the probability of any reaction occuring is proportional to n^2. Evidently, since $P_f > P_d$ for breakeven, we must have

$$n\tau_e > h(T),$$

where h is some function of temperature. This is the Lawson condition. At typical fusion temperature (10—30 keV) and density ($10^{14}\mathrm{cm}^{-3}$, the plasma need be confined on the order of seconds.

Plasma Oscillations

We return to examples of one-dimensional plasma problems. An interesting case is that of plasma oscillations in a cold plasma which we touched on last lecture. Perturb an initially homogeneous plasma by displacing an infinite slab of electrons from some point $z = z_0$ to $z = z_1$ The initial disturbance is is a decrease in density near z_0 and an increase near z_1. From the small signal cold plasma equations derived last lecture we know that all perturbed quantities merely oscillate in time at the plasma frequency. For example,

$$n = n_0 + \tilde{n}(z, t = 0) \cos \omega_{pe} t,$$

where $\tilde{n}$ is the original density perturbation. Where the perturbation was initially zero, e.g. not near $z = z_O$ or $z = z_1$, it remains zero subsequently. The disturbance is nonpropagating and both density and field perturbations remain localized.

A z-directed magnetic field in this problem changes nothing. All motion is along the field so it can play no role. The problem remains entirely one-dimensional. However, were the field to point in some other direction, the response of the plasma would be far more complicated – in fact, the initial perturbation would propagate in the form of waves that will be considered only much later in this course.

We have examined two fundamental quantities, ω_{pe} and λ_D. Cold plasma oscillations occur at the plasma frequency, ω_{pe}. Shielding in a warm plasma occurs over a Debye length, λ_D. A magnetic field can change simple one-dimensional plasma oscillations into complex two-dimensional behavior. The shielding problem, however, is unaffected by a magnetic field. The derivation above of the shielding relied only upon two equations: Poisson's equation and the Gibb's equilibrium distribution of charged particles. A magnetic field does not contribute to the particle potential energy, so it cannot affect the equilibrium distribution. It also does not enter Poisson's equation. Hence, the magnetic field plays no role in Debye shielding, which remains, essentially, a one-dimensional problem. A consequence of this is that in a cylindrical plasma, even with a large axial magnetic field, electrons can stray a Debye length out of the cylinder.

Collisions

In a truly one dimensional world there are, in effect, no elastic hard-sphere collisions between like particles – if they do collide, their resulting trajectories are merely interchanged. There are no other trajectories that conserve both energy and momentum.

Thus, in a one dimensional world a system of like particles maintains its initial velocity distribution indefinitely and there is no relaxation towards a Maxwellian distribution.

In more dimensions than one there is a tendency to relax towards an equilibrium distribution. The added dimension allows the fastest particles to become even faster in like particle collisions. The trick is to hit these particles with a glancing blow, an opportunity made available by the extra dimension. Such events, however, are relatively unlikely so that the number of very fast particles must be exceedingly small, in accord with the Maxwellian distribution.

An interesting question, which we leave as an exercise, is whether in a truly one-dimensional world hard-sphere collisions between unlike particles are indeed sufficient to relax the system to a Maxwellian equilibrium distribution of speeds.

Of course, with respect to the considerations here, a plasma is not a one-dimensional system even when immersed in a very strong magnetic field. Particles are contained in their excursions perpendicular to the large field, but not in their velcoities. Glancing blows are possible. Therefore, in a strong field, with only like particle collisions, a plasma can relax to a Maxwellian. In one-dimensional problems, collisions are treated not as if they really occured in one-dimensional space, but in terms of their effect on the dimension of interest.

Particles streaming through a plasma encounter what is called dynamical friction, or a tendency to be slowed down by collsions. The frequency of collsions between a particle with velocity v and a stationary background of density n is given by

$$\nu = v/\lambda_{mfp} = nb^2 v \sim n/v^3.$$

The slowing down equation could be written as

$$\frac{d\mathbf{v}}{dt} = -\upsilon\mathbf{v} = -\alpha\frac{\mathbf{v}}{v^3}, \tag{16}$$

where α is some constant. This would be the slowing down due to stationary scatterers – to find the total slowing down a summation must be made over all scatterers. The dynamical friction F due to a Maxwellian background of scatterers rises and then falls as a function of energy. For $v >> v_T$, where v_T is the typical background velocity, it is clear that $F \sim 1/v^2$. For $v << v_T$ it turns out that F is proportional to v as contributions from nearby scatterers tend to cancel. One way to see this involves noting an analogy between Eq. (16) and the coulomb attraction force. But in any event, the result in this limit is not surprising since, for small v, restoring forces are usually proportional to v. In fact, many systems exhibit such behavior, *i.e.*, where the chance of something happening first increases and then decreases with speed. The fusion cross-section is another function that has a maximum with respect to speed.