Notes for Astrophysical Sciences 554: Irreversible Processes Plasmas

G. W. Hammett, et.al.
Princeton University Plasma Physics Laboratory
P.O. Box 451, Princeton, NJ 08543 USA

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1 Answers to a question about the Chapman-Enskog procedure

One of the students in Irreversible Processes asked the following question, and I thought the whole class might benefit from thinking about this. Below is my answer (with some improvements since the email answer I sent out earlier).

A student wrote:

> I have a question about using the Krook operator in equations derived using the Chapman-Enskog ordering. Originally, in my notes, I have that the Krook operator is $C[f] = -\nu(f - f_M \int dv f)$ but it seems that when we use it in the correction equation we only use $-\nu f_1$. Why do we assume that the integral of f_1 over v is 0 and should we assume that this is true for all higher order f's?

This is a very important and subtle question. I tried to explain it some on p. 2 of http://w3.pppl.gov/~hammett/courses/irrev01/notes.pdf but perhaps it should be explained further. The basic idea is as follows.

Start with the 1-D kinetic equation:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = -\nu(f - f_M \int dv f)$$

where $f_M(v)$ is an unshifted Maxwellian with unit density, so this Krook model has been designed to conserve particles, but does not conserve momentum.

Taking $\nu \sim 1/\epsilon$, and expanding $f = f_0 + f_1 + f_2 + \ldots$ in powers of ϵ, then to lowest order in the large ν limit, we have the equation

$$0 = -\nu(f_0 - f_M \int dv f_0)$$

so that the general solution is of the form $f_0 = n_0 f_M = n_0(x,t) f_M(v)$. Note that the density for f_0 in this equation is mathematically undetermined. (You can replace f_0

with cf_0 for arbitrary constant c and still have a solution.) We will use this freedom to choose the density of f_0 so that it is equal to the full density of f:

$$\int dv f_0 = n_0 = \int dv f = n(x,t)$$

Since $f = f_0 + f_1$, this means that f_1 won't contain any density, $\int dv f_1 = 0$. Neither will any other higher-order f_j's, since $\int dv f_j = 0$ must be satisfied independently at every order.

Perhaps the best simple argument for this is just that we certainly have the freedom to choose the density of f_0 to be the same as the density of the total f, and that experience has found that this choice simplifies the next order correction equations. We will show in the next section that it is in fact possible to allow n_0 to differ from the total density n, but this complicates the higher order equations some.

The discussion of these issues in Krommes Ch. 31 ("Transport and the one-component plasma") around Eq. 15 is useful to review. Krommes refers to this property (that $\int dv f_j = 0$ for all higher order perturbations $j \geq 1$) as meaning that all of the perturbations are "orthogonal to the hydrodynamic subspace", i.e. the space of functions for which the collision operator C vanishes and therefore the space of functions for which C can't be inverted. Regarding this choice (of $\int dv f_0$ being chosen to be the same as the density in the full f), Krommes says: "It is not completely obvious that this procedure is systematic and justifiable, and this is one of the annoying aspects of the Chapman-Enskog procedure. It can however, be justified. See the later discussion of the projection-operator method in Chap. 33."

There are several other contexts when similar issues arise to what we are considering here, such as gyro-averaging or bounce-averaging. For example, when we did the Chapman-Enskog-Braginskii calculation for a strong magnetic field, we had equations of the form

$$-\frac{q}{mc}\vec{v} \times \vec{B} \cdot \frac{\partial f}{\partial \vec{v}} + \hat{C} f = -\Omega_c \frac{\partial f}{\partial \phi} + \hat{C} f = S,$$

(where ϕ is the gyroangle and S is some unspecified source term). In class we showed one way to expand this equation in the limit of large Ω_c (large cyclotron frequency) and work out the transport coefficients perpendicular to the magnetic field. For large Ω_c, the first term on the LHS dominates. However, this can only be used to determine components of f which vary with ϕ, since $\partial/\partial\phi$ vanishes on any function which is constant in ϕ. (One thus has to go to higher order to determine the ϕ independent part of f.) In other words, the operator $\partial/\partial\phi$ has a "null space" and thus can't be the dominant operator in all of phase space. People often use projection operator techniques such as Krommes discusses to break up the problem into the subspace for which the main operator vanishes and the subspace in which it dominates (or some other convenient splitting).

1.1 Additional rationales

In addition to the main reason given above, there are several other ways to think of why it is useful to impose this constraint. One is to consider starting with an arbitrary non-Maxwellian f. Then on a very short time scale, one can neglect the $v\partial f/\partial x$ term relative to the collision term, and the kinetic equation is just:

$$\frac{\partial f}{\partial t} = -\nu(f - f_M \int dv f)$$

because ν is so large, f will quickly relax to a Maxwellian f_0, but this f_0 must have the same density as the initial f.

Another way to think of it is that since we are expanding $f = f_0 + f_1$, and assuming that f_1 is "small", we want to choose f_0 so that it is as "close" as possible to f. Since the only free parameter in f_0 is the density, we get a best fit (in some sense) by choosing the density of f_0 to be the same as the density in the total f.

Perhaps the most precise way to think about this is to write the next order "correction equation":

$$df_0/dt + vdf_0/dx = S = Cf_1$$

where C is the linearized collision operator, and S the "source term", i.e. the left-hand side of the equation. Formally, we have to invert C to find f_1:

$$f_1 = C^{-1}S$$

However, since C vanishes if it operates on a Maxwellian of arbitrary density, its inverse is not well defined and f_1 could be a specific solution to $C^{-1}S$ plus an arbitrary constant n_1 times a Maxwellian f_M. Determining that constant requires going to higher order (we will show how this can be done in the next section). But by requiring that $n_0 = n$ (which we are certainly free to do), where n is the full density, we automatically constrain $n_1 = \int dv f_1 = 0$ and we are saved the trouble of having to go to the even higher order equation to determine n_1. We can then use the full density conservation law to determine the time evolution of $n_0 = n$.

This is all related to the issue of how many fluid moment equations are kept before a closure is introduced. If the collision operator conserves only density, then only a density fluid equation is kept and a closure approximation for the flux $nu = -D\partial n/\partial x$ will be calculated. In this case only the density of f_0 is constrained, and f_1 can be constrained to have no density. If the collision operator conserves density, momentum, and energy, then 3 fluid conservation laws for all 3 quantities are kept and closures are calculated only for higher order moments (like the heat flux or anisotropic parts of the pressure tensor). In this case f_0 will be a shifted Maxwellian specified by 3 parameters (the density, average momentum, and temperature), and f_1 can be constrained to have no density, no average momentum, and no average energy, i.e. f_1 is orthogonal to the null space of C. In the notes at http://w3.pppl.gov/~hammett/courses/irrev01/notes.pdf I also did a case where the 1-D collision operator conserved density and momentum, but not energy. In that case we kept a density and a momentum fluid conservation law, and collisions led to a viscosity term in the momentum equation.

1.2 Allowing non-zero density in f_1

In this section, we will show how the calculation proceeds if one does not impose the constraint that $\int dv f_0 = \int dv f$ (and thereby show why things are simpler if one does use this constraint).

Start again with the 1-D kinetic equation. We will slightly modify the assumed ordering for the case at hand, where the collision operator conserves only density, to:

$$\underset{\sim \epsilon^1}{\frac{\partial f}{\partial t}} + \underset{\sim \epsilon^0}{v\frac{\partial f}{\partial x}} = \underset{\sim \frac{1}{\epsilon}}{-\nu(f - f_M \int dv f)}$$

If the collision operator conserves momentum and energy as well, then there can be hydrodynamic waves with $\partial/\partial t \sim \omega \sim v_t \partial/\partial x$. But in this case where the collision operator doesn't conserve momentum, we will find $\partial/\partial t \sim D\partial^2/\partial x^2 \sim 1/\nu \sim \epsilon$, so this is a consistent ordering. (One could work with a $\partial/\partial t \sim \epsilon^0$ ordering, but it complicates things a bit in going to higher order.)

To lowest order we find $f_0 = n_0 f_M = n_0(x,t) f_M(v)$. We won't require $n_0 = n$ exactly, but it should be close to n so that n_1 can be small by comparison (as f_1 should be small compared to f_0 in the ϵ expansion). To next order we have

$$v\frac{\partial f_0}{\partial x} = -\nu(f_1 - f_m \int dv f_1)$$

The general solution to this is

$$f_1 = -\frac{v}{\nu}\frac{\partial n_0}{\partial x} f_M + n_1 f_M$$

where $n_1(x,t)$ is an arbitrary coefficient (you should be able to convince yourself that this is the completely general solution). We actually don't need to know n_1 if we only want to calculate the first order flux $(nu)_1 = \int dv f_1 v = -D\partial n_0/\partial x$, where $D = v_t^2/\nu$, and $v_t^2 = \int dv f_M v^2$. The fluid density equation $\partial n/\partial t + \partial(nu)/\partial x = 0$ (an exact expression) in our ordering expands to

$$\frac{\partial n_0}{\partial t} = -\frac{\partial}{\partial x}\int dv f_1 v = D\frac{\partial^2 n_0}{\partial x^2}$$

and at higher orders is

$$\frac{\partial n_j}{\partial t} = -\frac{\partial}{\partial x}\int dv f_{j+1} v$$

To determine n_1, we have to go to the next order equation

$$\frac{\partial f_0}{\partial t} + v\frac{\partial f_1}{\partial x} = -\nu(f_2 - f_M \int dv f_2)$$

Substituting for f_0 and f_1 leads to

$$D\frac{\partial^2 n_0}{\partial x^2} f_M + v\left[-\frac{v}{\nu}\frac{\partial^2 n_0}{\partial x^2} + \frac{\partial n_1}{\partial x}\right] f_M = -\nu(f_2 - f_M \int dv f_2)$$

Denoting the left-hand side of this equation as S, the solution is $f_2 = -S/\nu + n_2 f_M$, i.e., f_2 is only determined by this equation up to an arbitrary multiple of f_M. It can be determined at next order, but if we are only interested in the flux, we can easily calculate it to be $(nu)_2 = \int dv v f_2 = -D\partial n_1/\partial x$, so that

$$\frac{\partial n_1}{\partial t} = D\frac{\partial^2 n_1}{\partial x^2}$$

This can be combined with the similar equation for $\partial n_0/\partial t$, so that only a single diffusion equation for the combined quantity $n_0 + n_1$ needs to be kept. In fact, there will be similar terms at all higher orders, which can all be combined together to provide a single diffusion equation for $n = n_0 + n_1 + n_2 + \ldots$. [However, there will also be additional terms at higher orders, starting with the f_3 term, that lead to things like the Burnett correction terms, as discussed in Sec. 1.3 of http://w3.pppl.gov/~hammett/courses/irrev01/notes.pdf.]

So while it appears that it is possible to relax the constraint $\int dv f_0 = \int dv f$, this section at least shows why it is more convenient to use such a constraint.