AS551 General Plasma Physics 1

January 18, 2001

1:30 - 4:30 p.m.

Final Exam

(closed book, except for NRL formulary given to you)

180 points total: 1 point $\approx$ 1 minute. Do not spend too much time on the short (5-15 point) problems that need only brief answers. Note that you have a choice on the final question, do ONLY problem 10A or 10B.

1. [**20 points**] Write down a complete set of ideal MHD equations. Briefly describe each equation (a sentence or phrase or two for each Eq. is sufficient).

Briefly describe 2 of the important assumptions made in the derivation of these equations, and their impact on the properties of the MHD equations.

2. [**10 points**] Write down a phase-space conservation law for the distribution function of particles responding to an arbitrary force $\vec{A}(\vec{x}, \vec{v}, t)$. What condition must the force satisfy in order for the usual phase-space volume conservation property of the Vlasov equation to hold?

3. [**5 points**] Express the ratio of the ion thermal velocity to the Alfvén velocity in terms of another common plasma parameter (assume equal ion and electron temperatures).

4. [**10 points**] Provide an order-of-magnitude estimate of the ratio of the distance of closest approach for a 90 degree scattering event to the average distance between particles. Express this ratio in terms of a common plasma parameter.

5. [**10 points**] Provide an order-of-magnitude estimate of the ratio of the ion mean-free-path to the electron mean-free-path (assuming equal electron and ion temperatures and ion charge $Z = 1$ for simplicity).

6. [**10 points**] In MHD equilibrium, $\nabla p = \vec{j} \times \vec{B}/c$ implies the current is proportional to a pressure gradient. But the single particle guiding center drifts depend only on the magnetic fields and not on the pressure gradient of other particles. Briefly explain this apparent paradox and illustrate it with a sketch.

7. [**10 points**] Consider an initially uniform magnetic field pointing in the x direction in an ideally conducting plasma. Due to some external force, the plasma develops a sheared flow with $\vec{u} = \hat{y} u_0 \times (1 - |x|/L)$ for $|x| < L$, and $\vec{u} = 0$ otherwise. (Assume this velocity remains fixed and neglect the back reaction force of the magnetic field.) Sketch the magnetic field at a later time. Has the magnetic pressure in the $|x| < L$ region gone up or down or stayed the same?

8. [**10 points**] Consider the magnetic field produced by a small ring of current enclosing the z axis. The magnetic field far from the ring has a dipole structure, so on the midplane $|B| \propto 1/R^3$, where R is the distance from the ring. Sketch the particle orbits in the (R, z) and (x, y) planes in the guiding center approximation, $\rho \ll R$, for a particle with $v_\parallel \ll v_\perp$.

The current in the ring is now increased very slowly (so all adiabatic invariants are conserved). Does this particle move in or out?

9. [45 points] Shear Alfvén Waves. Start with the equations of ideal MHD, but add a viscous drag term to the momentum equation, of the form:

$$\rho \frac{d\vec{u}}{dt} = \ldots + \chi \rho \nabla^2 \vec{u} \tag{1}$$

where ... are the usual terms in the MHD momentum equation. (All of the other MHD equations remain unchanged.) Show how to derive the ω vs. k dispersion relation for incompressible shear-Alfven waves. In the limit of small but non-zero viscosity coefficient χ, calculate what the damping rate is for the wave. With a sketch and a sentence or two, briefly describe the role of the magnetic field in the dynamics of this wave.

10. [50 points] Instabilities. Do EITHER 10A or 10B, not both.

10A. Two-Stream Instability

Consider a uniform plasma of cold electrons and a beam of ions, where the equilibrium distribution functions for electrons and ions vs. velocity in the z direction are $f_e(v_z) = n_e \delta(v_z)$ and $f_i(v_z) = n_e \delta(v_z - u_0)$. Starting with the Vlasov equation, linearize it for electrostatic perturbations $\propto \exp(ikz - i\omega t)$ and show how to derive the dispersion relation

$$1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{(\omega - ku_0)^2} = 0 \tag{2}$$

In the limit of small but finite $|\omega_{pi}/\omega_{pe}|$ and $|ku_0/\omega_{pe}|$, calculate the growth rate of the instability.

10B. Rayleigh-Taylor Instability

Consider a plasma suspended against gravity by a magnetic field. Denote g as the acceleration due to gravity (in the $-\hat{y}$ direction). There is a stationary equilibrium where quantities vary only in the $\hat{y}$ direction and the equilibrium magnetic field is in the $\hat{z}$ direction. Consider small amplitude perturbations with the perturbed flow in the (x, y) plane so the magnetic field remains in the $\hat{z}$ direction. Write down the linearized momentum equation. Taking the curl of this equation to eliminate the plasma and magnetic pressure, and looking at its $\hat{z}$ component, show how to derive the equation,

$$\rho_0 \frac{\partial}{\partial x} \frac{\partial u_y}{\partial t} - \frac{\partial}{\partial y} \left[\rho_0 \frac{\partial u_x}{\partial t} \right] = -\frac{\partial \tilde{\rho}}{\partial x} g \tag{3}$$

Assuming incompressibility and Fourier transforming in time and x, show how to derive a second order differential equation for u_y which determines the eigenfrequency. For an exponential density profile, $\partial \rho_0 / \partial y = \rho_0 / s$, calculate an instability growth rate in the limit of very large wave number k_x.

Answers:

Overall this exam might be a bit on the easy side, but I think it did test a broad range of plasma conceptual ideas and problem-solving skills. Here are answers to some of the conceptual issues that some people had.

1. Standard.

2. Starting from a general conservation law for a distribution function $f(\vec{x}_6, t)$ in a 6 dimensional phase space:

$$\frac{\partial f}{\partial t} + \nabla_6 \cdot (\dot{\vec{x}}_6 f) = 0 \tag{4}$$

$$\frac{\partial f}{\partial t} + \vec{x} + \cdot \nabla f + \vec{A} \cdot \nabla_v f + f \nabla_v \cdot A = 0 \tag{5}$$

and we get the usual form that conserves phase-space volume ($Df/Dt = 0$) only if $\nabla_v \cdot A = 0$. This will be true for a Hamiltonian system.

3. $v_{ti}/v_A \approx (\beta/2)^{1/2}$, where β (ratio of plasma to magnetic pressure) is *a* common plasma parameter.

4. Closest approach distance b determined by $T \sim e^2/b$. Average interparticle spacing $n^{-1/3}$. Their ratio scales as $1/\Lambda^{2/3}$, where $\Lambda \sim n\lambda_d^3$ is *the* plasma parameter and is usually a very big number.

5. $\nu_e/\nu_i \sim \nu_{ei}/\nu_{ii} \sim \sqrt{m_i/m_e}$, so the ion and electron mean free path's turn out to be comparable.

6. Spitzer explained this apparent paradox, emphasizing that "fluid flows $\neq$ guiding center drifts". Picture is on p. 99 of Goldston and Rutherford.

7. Standard frozen-in field line result. B_x is constant (to conserve flux through an $x =$constant plane) while B_y goes up, so the magnetic pressure does go up.

8. Wanted you to sketch grad B and curvature drift around the ring (and some bouncing motion along field line of the trapped particles).

The question of the direction of the particle motion is a fascinating subtle problem that requires careful thinking. Many people have the wrong intuition about it, but about 25% of you got the direction of the particle motion right. [Because its somewhat non-intuitive, I didn't take off much for those of you who got the sign wrong but invoked some of the right physics.]

It turns out that the particle moves outward. In fact, it moves outwards to regions of lower $|\vec{B}|$ faster than the magnetic field at a fixed position is increasing, and the net result is that the energy of the particle actually drops. A harder variant of this problem is described at http://w3.pppl.gov/~hammett/gpp1/counter-intuititive.

9. This is a straightforward calculational problem. One should be careful to note that the shear Alfven-wave is not restricted to have $k_\perp = 0$. The dispersion relation $\omega^2 = k_\parallel^2 v_A^2$ is valid for arbitrary $\vec{k}$, it just turns out to depend only on the parallel component of $\vec{k}$.

10 A. This is a variant of the two-stream instability as done in class and described in

Goldston and Rutherford. The derivation of the dispersion relation is straightforward. Writing it in terms of normalized quantities, $\Omega = \omega/\omega_{pe}$, $\epsilon = \omega_{pi}^2/\omega_{pe}^2 = m_e/m_i$, and $\alpha = ku_0/\omega_{pe}$, it can be written as

$$(\Omega^2 - 1)(\Omega - \alpha)^2 = \epsilon^2\Omega^2 \tag{6}$$

Taking the $\epsilon = 0$ limit, to lowest order the roots are $\Omega = \pm 1$ and $\Omega = \alpha$ (a double root). Substituting $\Omega = \Omega_0 + \Delta\Omega$, where $\Omega_0 = \alpha$ (one can show that the $\Omega_0 = \pm 1$ roots are stable), one finds that the disperion relation to next order is

$$(\alpha^2 + 2\alpha\delta\Omega - 1)(\delta\Omega)^2 = \epsilon^2\alpha^2 \tag{7}$$

As described in the lecture notes and Goldston and Rutherford, this will give a growth rate as a function of k (or α) with the most unstable mode k such that $\alpha = 1$, in which case this becomes a cubic equation for $\delta\Omega$ to lowest order. I was trying to simplify this problem for you some by saying you should look at the small α limit, in which case this simplifies to $\delta\Omega = \pm i|\epsilon\alpha|$, or in the original notation, the growth rate is $|\omega_{pi}ku_0/\omega_{pe}|$.

Knowing how to find approximate solutions to complicated equations (like a quartic or complicated integrals) is a very powerful skill that you can learn a lot more about from Prof. White's asymptotic math class (descended from a course that Kruskal taught).

10 B. This problem is done in Goldston and Rutherford and in the lecture notes.