AS551 General Plasma Physics I
November 10, 2002

Homework #7, due Wednesday, Nov 20

1. Kinetic derivation of finite-temperature corrections to plasma oscillations. Assume the ions are stationary, while the electrons obey the Vlasov equation, which in the 1-D electrostatic limit is

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} + \frac{e}{m}E_{\|}\frac{\partial f}{\partial v} = 0$$

In class I showed how to to derive the standard dispersion relation for electron plasma oscillations, $\omega^2 = \omega_{pe}^2$, from this Vlasov equation by linearizing ($f = f_0 + f_1$, etc.), Fourier-transforming, and expanding a resonant denominator that appeared in an integral

$$\frac{1}{\omega - k_z v} = \frac{1}{\omega}\left(1 + \frac{k_z v}{\omega} + \left(\frac{k_z v}{\omega}\right)^2 + \left(\frac{k_z v}{\omega}\right)^3 + \cdots\right).$$

In class I only kept terms through first order in $k_z v$ in the expansion of the resonant denominator, while you should keep terms through order $(k_z v)^3$ to find higher order corrections. At this order, the dispersion relation can be written as a 4th order polynomial in ω, but it is quadratic in ω^2 and so can be easily solved. You should find 2 pairs of roots for ω. One pair is the finite temperature correction to electron plasma oscillations. The other pair of solutions are unphysical. Why? For the physical solutions, sketch the resulting ω vs. k_z. What is the maximum group velocity from this simple dispersion relation? What is the maximum phase velocity?

2. Fluid derivation of finite-temperature corrections to plasma oscillations. In class I showed how to derive the following 1-D fluid conservation laws from the 1-D Vlasov equation:

$$\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial z} = 0$$

$$\frac{\partial (mnu)}{\partial t} + \frac{\partial (mnu^2)}{\partial z} = -\frac{\partial p}{\partial z} + enE_{\|}$$

You know how to set $p = 0$ and linearize these two equations for the electron density and electron momentum to derive electron plasma oscillations (for high frequency electron plasma oscillations, the ions are usually assumed to be a stationary background that just neutralizes the equilibrium electron density $n_{i0} = n_{e0}$). The energy-conservation/pressure equation (which I will discuss more in the future and is discussed in the notes) can be written as

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial z} + \Gamma p\frac{\partial (u)}{\partial z} + \frac{\partial q_{\|}}{\partial z} = 0$$

where $q_{\|}$ is the heat flux, though we will set $q_{\|} = 0$ for simplicity here. In the notes we show that $\Gamma = 3$ appears to be the result for 1-dimension, though we will discuss later why one might want to choose other values of Γ in some cases.

Keeping finite electron pressure, linearize these three equation (using $n = n_0 + \tilde{n}$, $u = 0 + \tilde{u}$, $p = p_0 + \tilde{p}$, $E_{\|} = 0 + \tilde{E}_{\|}$), and Fourier transform them and solve the resulting equations to find a dispersion relation $\omega(k_{\|})$ for electron plasma oscillations. Compare with the kinetic result you found in problem 1.

3. Deriving fluid equations. Some recent papers investigated the effects of the ionization of neutrals on a certain class of plasma turbulence. (Motivational background: interactions with neutrals are often justifiably ignored in the hot plasmas of fusion research, but they can become important near the interface of a plasma with a solid wall, such as near the divertor plates in a tokamak, or in plasma processing of semiconductor chips.) The essential features of the equations used in those papers can be seen in the 1-dimensional limit, which can be written as:

$$\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial z} = n n_0 s$$

$$mn\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial z} + enE_\parallel$$

where $s = \langle \sigma v \rangle$ is the reaction rate for electron-impact ionization of the neutrals so that $nn_0 s$ is the particle source rate due to ionization. The ion particle density n, neutral density n_0, average ion velocity u, and pressure p are all functions of position z (along a magnetic field line) and time t.

Unfortunately, there is an error in the above equations which led to the claim of a spurious instability to drive turbulence. Your job is to to find this error by deriving the proper form of the above fluid equations from first principles by starting with the 1-dimensional Vlasov equation for the ion distribution function $f(z, v, t)$, modified to include an ionization source:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} + \frac{e}{m}E_\parallel \frac{\partial f}{\partial v} = nn_0 s\delta(v)$$

(Note that the reaction rate $s = \langle \sigma v \rangle$ has already been averaged over the electron velocities, and should be taken to be a fixed constant for your purposes.) The neutrals are very cold compared to the plasma, and so are treated as having zero velocity (hence the δ function) on the scale of the much hotter plasma ions.

You should discover a term which is missing from the above fluid equations. Give a brief description of the physics of this missing term.

4. Phase-mixing. For a non-interacting neutral gas in 1-dimension, the kinetic equation for the distribution function f(x,v,t) is

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$$

Consider just a single Fourier component with the initial condition $f(x, v, t = 0) = f_0(v)e^{ikx}$.

(a) Assume the initial velocity distribution is a Maxwellian, $f_0(v) \propto exp(-v^2/(2v_t^2))$. Calculate the time history of the particle density $n(x, t) = \int dv f$.

(b) Now assume that $f_0(v) = H(v+v_0)H(v_0-v)/(2v_0)$, where H is the Heaviside step function, and calculate $n(x = 0, t)$.

(c) Now assume that $f_0(v)$ is given as a sum of delta functions: $f_0(v) = \Sigma_{j=-N}^{N} \delta(v-j\Delta v)/(2N+1)$, and calculate $n(x = 0, t)$ again.

(d) Plot the results for $n(x = 0)$ vs. time which you got from part (c) for $N = 1, 3, 10$, and compare with the plot of the results from part (b). The solutions in part (c) should be periodic in time. What is the recurrence period τ_{repeat}, and how does it scale with N? Discuss whether or not this difference between the continuum approximation of (b) and the discrete result for part (c) (which would seem more realistic for a real plasma with a discrete number of particles) is important.

(The recurrence problem is an issue which numerical codes must face if they use a fixed grid in velocity space. It is often treated by adding a small amount of velocity-space diffusion, either explicitly or implicitly through the numerical method in use. Particle codes try to reduce this problem by assigning random initial velocities to all particles, and not initializing the particles to lie on a fixed velocity grid whose pattern repeats throughout real space.)