Different types of convective derivatives

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A student asked me today, "What is the difference between D/Dt and d/dt?"

A common practice in fluid dynamics is to denote "convective time derivatives" by d/dt, where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \tag{1}$$

and $\vec{u}$ is the fluid velocity. Thus the density conservation equation can be written as

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{u}) = 0 \tag{2}$$

or as

$$\frac{dn}{dt} + n\nabla \cdot \vec{u} = 0 \tag{3}$$

which indicates that the density moving along with a fluid element is constant unless there is compression or rarefaction going on, $\nabla \cdot \vec{u} \neq 0$.

The combination $\partial/\partial t + \vec{u} \cdot \nabla$ occurs frequently. It is called a convective derivative because it is the rate of change of some quantity moving along with a particular element of the fluid. For example, consider a certain point A moving along with the fluid. The position $\vec{X}_A(t)$ of this point satisfies

$$\frac{d\vec{X}_A}{dt} = \vec{u}(\vec{X}_A, t) \tag{4}$$

If there is some quantity $\psi(\vec{x}, t)$, and we want to know how this quantity varies in time if evaluated at the moving position A, then we have (using the chain rule)

$$\frac{d}{dt}\psi(\vec{X}_a(t), t) = \frac{\partial \psi}{\partial t} + \nabla\psi \cdot \frac{d}{dt}\vec{X}_a(t) = \frac{\partial \psi}{\partial t} + \vec{u} \cdot \nabla\psi \tag{5}$$

One can also consider the Vlasov/Liouville equation as being a generalized convective derivative, following a point in 6-D fluid phase space instead of 3-D fluid regular space. To distinguish this phase-space convective derivative from the regular fluid convective derivative, I denote it by D/Dt, where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} + \vec{a} \cdot \frac{\partial}{\partial \vec{v}} \tag{6}$$

and $\vec{a} = (q/m)(\vec{E} + \vec{v} \times \vec{B}/c)$ is the Lorentz force acceleration, so that the Vlasov equation is $Df/Dt = 0$. Denoting $\vec{v} = \dot{\vec{x}}$ and $\vec{a} = \dot{\vec{v}}$, we see this can be written as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{\vec{x}} \cdot \frac{\partial}{\partial \vec{x}} + \dot{\vec{v}} \cdot \frac{\partial}{\partial \vec{v}} \tag{7}$$

which perhaps makes a little clearer how we can use the chain rule to interpret this as a derivative of a quantity at some point that moves in phase space satisfying $d/dt(\vec{x}, \vec{v}) = (\vec{v}, \vec{a})$.