AS551 General Plasma Physics 1
Dec. 6., 2004 (err. corr. 12/17)

Problem Set # 10 (due Tuesday Jan 4, 2005)

1. Properties of solutions to a diffusion equation. Consider the 1-D convection diffusion equation with fixed coefficients u and D:

$$\frac{\partial n}{\partial t} = -u\frac{\partial n}{\partial x} + D\frac{\partial^2 n}{\partial x^2} \tag{1}$$

Assume that the density profile is sufficiently localized that it satisfies $n(x,t) \to 0$ as $x \to \pm\infty$, and is normalized so that $\int dx\, n = 1$. Define the average position of the particles by $\bar{x}(t) = \int dx\, n(x,t)x$, and the average variation around this position by $\sigma^2(t) = \int dx\, n(x,t)(x - \bar{x}(t))^2$. Calculate $\partial\bar{x}/\partial t$ and $\partial\sigma^2/\partial t$, to show how they are related to u and D. (You should find that these result do not depend on the details of the distribution of the particles, $n(x,t)$.)

2. A particular solution to a diffusion equation. By direct solution show that

$$n(x,t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}}\exp(-x^2/(2\sigma^2(t)))$$

is a solution of the diffusion equation, Eq.1 in the $u = 0$ limit. Show how you would modify this solution to make it correct for a nonzero velocity u. (You've just determined the Green's function for Eq.1, i.e., the solution for the initial condition $n(x,0) = \delta(x)$.)

3. Collisional transport in air. Work out a rough estimate of the collisional diffusion coefficient for air (this is often called "molecular diffusion"), using the collision frequency, mean free path, and thermal speed of molecules under typical atmospheric conditions. Compare this to an estimate of the diffusion coefficient due to turbulent air motions observed outside on a typical afternoon (assuming the turbulence corresponds to a random walk process with $\Delta x \sim 10$ m and $\Delta t \sim \Delta x/v_{turb}$ with $v_{turb} \sim 10$ km/hour. You are in charge of homeland security in a certain city. Using this estimate of the turbulent diffusion coefficient, roughly how long would it take for an aerosol released in the air to be spread over an area of 1 square kilometer (after being spread over such a large volume, the toxicity is assumed to be reduced to acceptable levels...).

4. Fusion random walk estimates. Particle diffusion coefficients roughly of order $D = 1m^2/s$ are typical of tokamaks. Fluctuations are observed in the plasma with a time scale of order $\Delta t \approx 10^{-5}s$. Assuming this is the step time of some random walk process, what is the step size Δx needed to explain the observed D? Assuming this is due to randomly fluctuating $E \times B$ drifts, so that $v_{E\times B} \approx \Delta x/\Delta t$, use this to estimate the ratio $v_{E\times B}/v_{ti}$ (where v_{ti} is the thermal ion speed) for a typical 10 keV fusion plasma.

5. Steady-state diffusion with a source. Particle transport in a cylinder of plasma of radius a is modeled with the equation

$$\frac{\partial n}{\partial t} = D\nabla^2 n + S$$

where the particle diffusion coefficient D and the particle fueling rate S are constants independent of time or space. (a) Find the steady-state solution $n(r)$ to this equation in cylindrical geometry with the boundary condition that $n(r = a) = 0$ (where plasma particles are assumed to be lost to the wall). (b) Define the integrated particle density by $N = \int dr 2\pi r n$, and define the average particle confinement time by $\tau_p = N/(S\pi a^2)$ (or in other words, the total fueling rate must be $S\pi a^2 = N/\tau_p$ to maintain the plasma against diffusive losses). Show how τ_p can be expressed in terms of D and a. (c) Although we've assumed a boundary condition where the density vanishes at the wall, the flux of particles to the wall, $\Gamma = -D\partial n/\partial r$, is non zero. Calculate the flux Γ.

5. Goldston & Rutherford problem 12.6. (Effect of ions with charge Z on transport coefficients) (Approximate answers using scalings from random-walk arguments are sufficient here.)

6. Temperature and power in a conduction-limited regime using Braginskii's heat conduction. Consider a plasma bounded by walls at $z = \pm L/2$. The plasma is confined by a magnetic field in the z direction, and has a cross-sectional area A. Different effects dominate in various parameter regimes, here we will consider a conduction-limited regime (there are also sheath-limited regimes, etc.), where the dominant terms in the energy balance equation are

$$\frac{3}{2} n \frac{\partial kT}{\partial t} = \frac{\partial}{\partial z} \kappa_{\parallel}^{e} \frac{\partial kT}{\partial z} + H.$$

Here, $\kappa_{\parallel}^{e}$ is Braginskii's electron parallel thermal conductivity, and $H = P\delta(z)/A$ is an external heating source with total power P concentrated near $z = 0$. Approximating $\lambda = \log(\Lambda)$ as constant in Braginskii's formula for the collision time τ_e, (a) find the functional form of $T(z)$ in steady state, and sketch your solution. (b) Taking $T(z) = 0$ at the walls, calculate how much heat flux (in Watts/cm^2) would be required to maintain a plasma with $T(0) = 10$ eV, $n(0) = 10^{12}/\text{cm}^3$, and $L = 100$ cm. (Note: this power ultimately ends up on the walls, which may then require water cooling tubes to prevent overheating... Power loads on the wall can also be reduced by placing the wall at a shallow angle to the magnetic field.)