# Outline of Landau Damping

12/10/2004
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Ref: Goldston + Rutherford Ch. 24
(pretty good..., see also Stix, Plasma Waves, A. Bers, et. al.)

Start: Linearized Vlasov Eq. incorporating init. cond.

$$\frac{\partial f_1}{\partial t} + i k v_{\parallel} f_1 = -\frac{q}{m} E \frac{\partial f_0}{\partial v_{\parallel}} + \delta(t)\, f_1(\underset{\sim}{v}, t=0)$$

Fourier transform, with $\omega$ in upper half complex plane to insure convergence as $t \to +\infty$ (equiv. to Laplace transforms).

$$\hat{f}_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{+i\omega t} f_1(t)$$

Plug into Poisson Eq. to get:

related to initial conditions

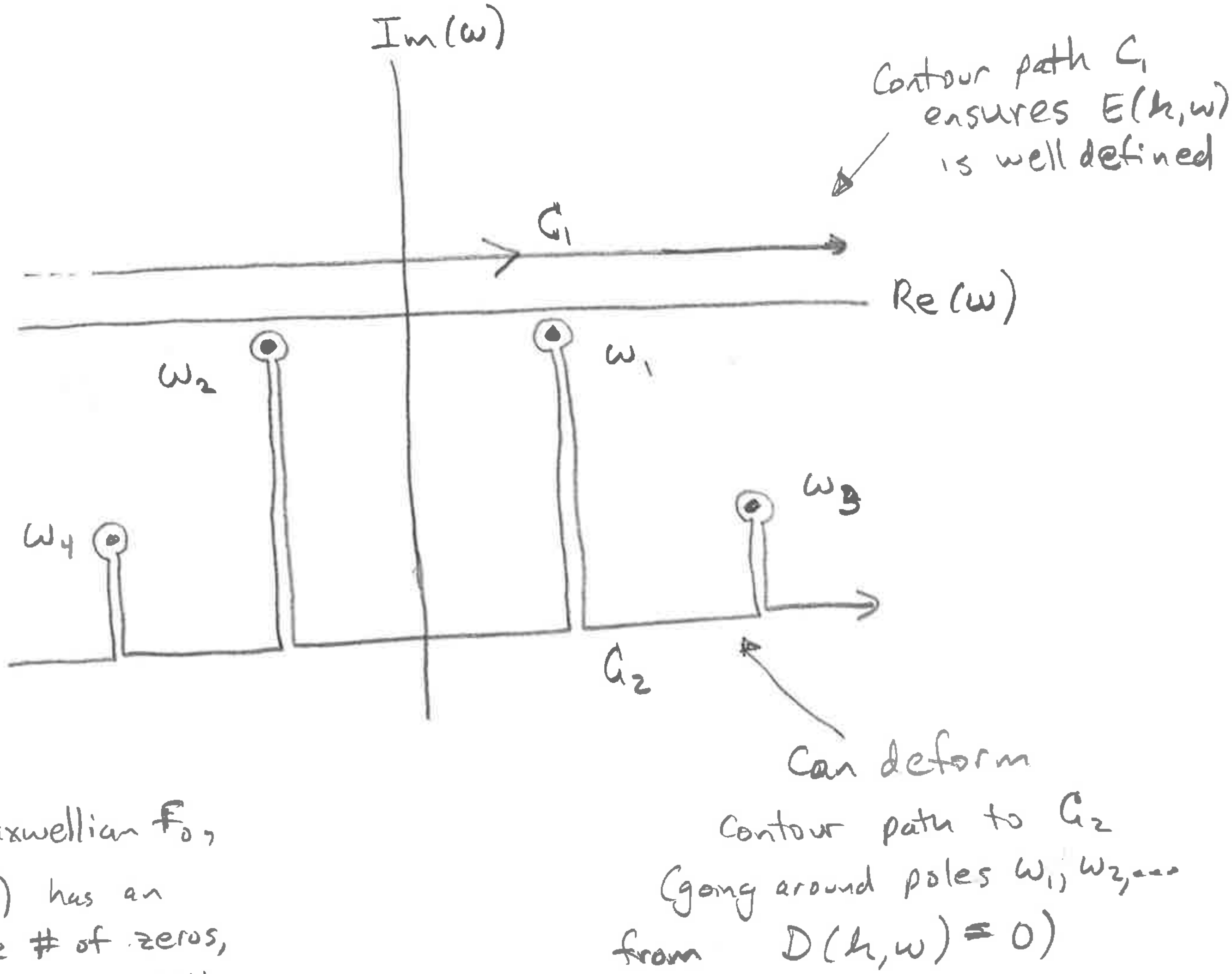
$$\hat{E}(k,\omega) = \frac{N(k,\omega)}{D(k,\omega)} = \frac{\dfrac{2\sqrt{2\pi}\, q}{k} \displaystyle\int d^3v \frac{f_1(\underset{\sim}{v}, t=0)}{\omega - k v_{\parallel}}}{1 + \dfrac{4\pi q^2}{m k} \displaystyle\int d^3v \frac{\frac{\partial f_0}{\partial v_{\parallel}}}{\omega - k v_{\parallel}}}$$

Plasma dielectric

$(D \to D_V$ = Vlasov version, defined only for $\mathrm{Im}(\omega) > 0)$.

# Inverse Fourier Transform

$$E(k,t) = \frac{1}{\sqrt{2\pi}} \int_{C_1} d\omega \, e^{-i\omega t} \frac{N(k,\omega)}{D(k,\omega)}$$




(For Maxwellian $F_0$, $D(k,\omega)$ has an infinite # of zeros, but most are highly damped.)

Deformed path shows in long time limit

$$E(k,t) \approx A_1 e^{-i\omega_1 t} + A_2 e^{-i\omega_2 t} + \ldots$$

where $\omega_1$ (& $\omega_2$) are the least-damped roots of $D(k,\omega)$.

Caveat: Integral form of $D_1(k,\omega)$ is discontinuous across $\mathrm{Im}(\omega)=0$. Have to use analytic continuation.

③

Look for existence of a weakly damped mode

$$\omega = \omega_r + i\varepsilon$$

$$D_v(k,\omega) = 1 + \frac{4\pi q^2}{mk}\int d^3v \, \frac{\frac{\partial f_0}{\partial v_{\parallel}}}{\omega_r + i\varepsilon - k v_{\parallel}}$$

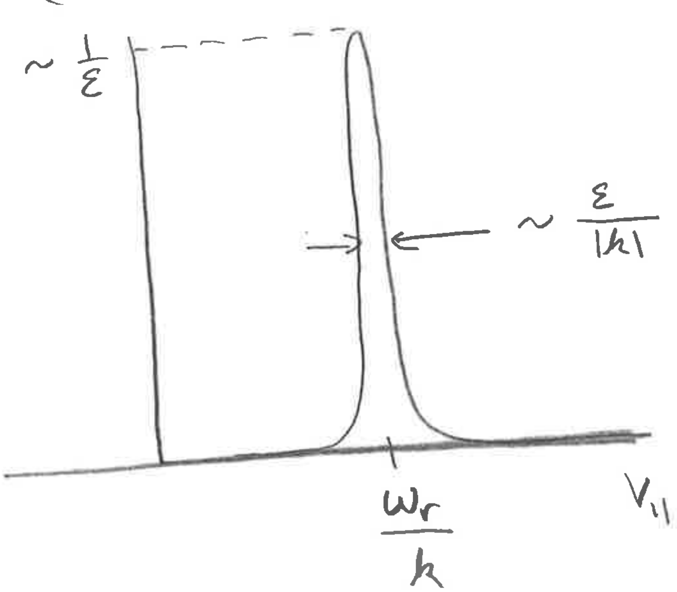
$$= 1 + \frac{4\pi q^2}{mk}\int dv_{\parallel} \, \frac{\frac{\partial f_0}{\partial v_{\parallel}}(\omega_r - i\varepsilon - k v_{\parallel})}{(\omega_r - k v_{\parallel})^2 + \varepsilon^2}$$

Imag part:

Useful I.D. $\lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2+\varepsilon^2} = \pi\,\delta(x)\,\mathrm{Sign}(\varepsilon)$

Cause of discontinuity in integral form of $D_v(k,\omega)$

$$\frac{\varepsilon}{(\omega_r - k v_{\parallel})^2 + \varepsilon^2}$$




$$= \pi\,\delta(\omega_r - k v_{\parallel})\,\mathrm{Sgn}(\varepsilon)$$

$$= \frac{\pi}{|k|}\,\delta\!\left(\frac{\omega_r}{k} - v_{\parallel}\right)\mathrm{Sgn}(\varepsilon)$$

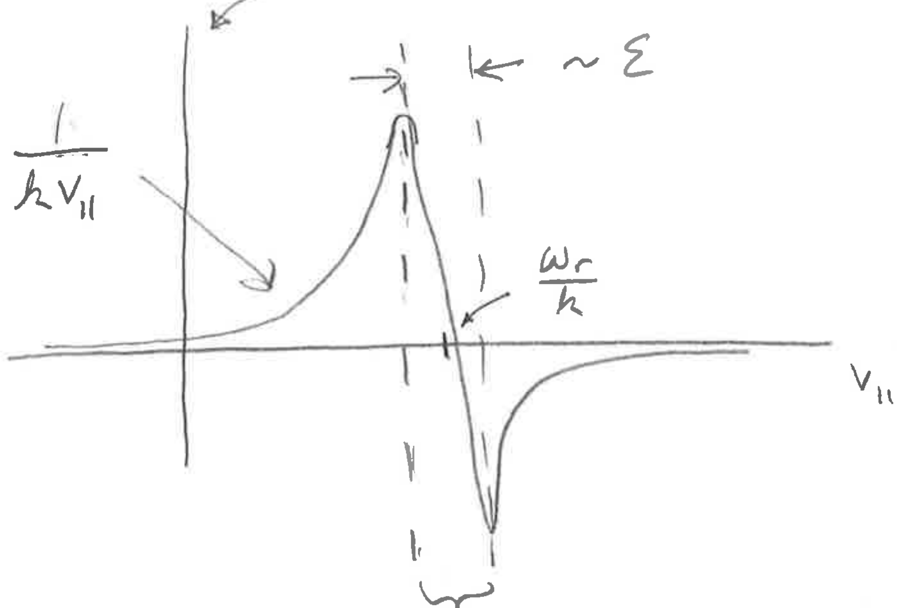
Fourier transforms defined for $\varepsilon > 0$

So

$$\mathrm{Im}(D_v) = -\frac{4\pi q^2}{mk}\,\frac{\pi}{|k|}\,\left.\frac{\partial f_0}{\partial v_{\parallel}}\right|_{v_{\parallel} = \frac{\omega_r}{k}}$$

Real part of integral:

$$\int dv_\parallel \, g(v_\parallel) \frac{\omega_r - k v_\parallel}{(\omega_r - k v_\parallel)^2 + \varepsilon^2} \Rightarrow \text{P.V.} \int dv_\parallel \frac{g(v_\parallel)}{\omega_r - k v_\parallel}$$




Contribution to integral vanishes as $\varepsilon \to 0$
(for smooth $g(v_\parallel)$)

$$\text{P.V.} \int_{-\infty}^{\infty} dx \frac{g(x)}{x - x_0} \equiv \lim_{\varepsilon \to 0} \left[ \int_{-\infty}^{x_0 - \varepsilon} dx \frac{g(x)}{x} + \int_{x_0 + \varepsilon}^{\infty} dx \frac{g(x)}{x} \right]$$

Cauchy
"Principal Value"

useful Plemelj I.D.:

$$\lim_{\varepsilon \to 0} \frac{1}{\omega + i\varepsilon - \omega_0} = \text{P.V.} \left( \frac{1}{\omega - \omega_0} \right) - i\pi \, \delta(\omega - \omega_0) \, \text{Sgn}(\varepsilon)$$

Look for high freq. waves $\frac{\omega_r}{k} >> V_t$

$$\frac{1}{\omega_r - kv_{\parallel}} \approx \frac{1}{\omega_r}\left(1 + \frac{kv_{\parallel}}{\omega_r} + \frac{k^2 v_{\parallel}^2}{\omega_r^2} + \frac{k^3 v_{\parallel}^3}{\omega_r^3} + \cdots\right)$$

& real part of $D_v(k,\omega)$ is (from 12/11/03 notes):

$$\simeq 1 - \frac{\omega_{pe}^2}{\omega_r^2}\left(1 + \frac{3k^2 v_t^2}{\omega_r^2}\right)$$

Combine w/ imag. part:

$$\lim_{Im(\omega)\to 0} D_v(k,\omega) \simeq 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{3k^2 v_t^2}{\omega^2}\right) - i\frac{4\pi q^2}{mk}\frac{\pi}{|k|}\frac{\partial f_0}{\partial v}\bigg|_{v=\frac{\omega}{k}}$$

(on real axis, $\omega_r \to \omega$), so

$$D(k,\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{3k^2 v_t^2}{\omega^2}\right) + i\sqrt{\frac{\pi}{2}}\frac{\omega_{pe}^2\,\omega}{|k|^3 V_t^3}\, e^{-\frac{\omega^2}{k^2 2V_t^2}}$$

This expression is a continuous function of $\omega$ & so is the proper analytic continuation of $D_v(k,\omega)$ from $Im(\omega) > 0$ to $Im(\omega)$ arbitrary

(for small $Im(\omega)$ & large $\frac{\omega}{kv_t}$ where this approx. was derived)

$$D(k,\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{3k^2 V_{te}^2}{\omega^2}\right) + \underbrace{i\sqrt{\frac{\pi}{2}}\frac{\omega_{pe}^2\,\omega}{|k|^3 V_{te}^3}\, e^{-\overbrace{\frac{\omega^2}{k^2 2V_{te}^2}}}}$$

$$0 =$$

exponentially
Small for
$\frac{\omega}{k V_t} \gg 1$

& so can neglect
while solving
real part to
desired order

This is big
for $\frac{k V_t}{\omega} \ll 1$

& so have to
keep to 2<sup></sup>nd order

Exact:

$$\omega^2 = \omega_{pe}^2\left(1 + \frac{3k^2 V_{te}^2}{\omega^2}\right) - i\,\omega^2 g(\omega)$$

$$\omega_r^2 \cong \omega_{pe}^2 + 3k^2 V_{te}^2$$

$$\omega \approx \omega_r + i\underbrace{\omega_i}_{\text{Small}} \qquad \omega^2 \simeq \omega_r^2 + 2\omega_r i\,\omega_i$$

$$\boxed{\omega_i \simeq -\frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{\omega_{pe}^4}{|k|^3 V_{te}^3}\, e^{-\left(\frac{\omega_{pe}^2}{k^2 2V_{te}^2} + \frac{3}{2}\right)}}$$

$$e^{-3/2} \approx 0.22$$

missed in original Landau 46
& G&R.

(pointed out by J. D. Jackson 1960, J. Nucl. Energy, part C, 1, 171)

# A Simple Phase-Mixing Paradigm

(Carl Oberman reminded me of this view of Landau damping)

Consider a 1-D kinetic Eq. for $f(z, v, t)$, with no $\vec{E}$ field:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} = 0$$

Exact solution is just $f(z, v, t) = f_0(z - vt, v)$.
Consider single Fourier mode in $z$ with Maxwellian distr. in v:

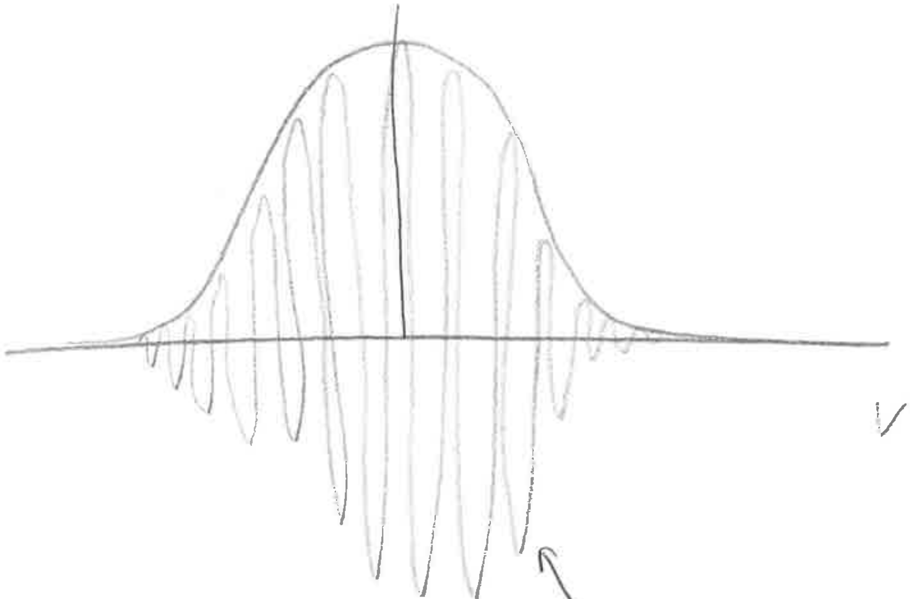
$$\begin{aligned} f_0 &= f_M(v)(1 + \alpha\cos(kz)) \\ &= f_M(v)(1 + \alpha\,\mathrm{Real}\,e^{ikz}) \\ f &= f_M(v)(1 + \alpha\,\mathrm{Real}\,e^{ik(z-vt)}) \end{aligned}$$

At any fixed $v$, $f$ oscillates in time with $\omega = kv$ & no damping.
However, any $v$-moment of $f$ will exponentially decay in time:

$$n(z,t) = \int dv\, f = n_0 + \alpha n_0\,\mathrm{Real}\,\frac{e^{ikz}}{\sqrt{2\pi v_t^2}} \underbrace{\int dv}_{\text{mixing}}\, \underbrace{e^{-ikvt}}_{\text{Phases}} e^{-v^2/(2v_t^2)}$$

$$n(z,t) = n_0 + \alpha n_0 \cos(kz) e^{-k^2 v_t^2 t^2/2}$$

$$f_0(v) \approx e^{-v^2/2v_t^2}$$

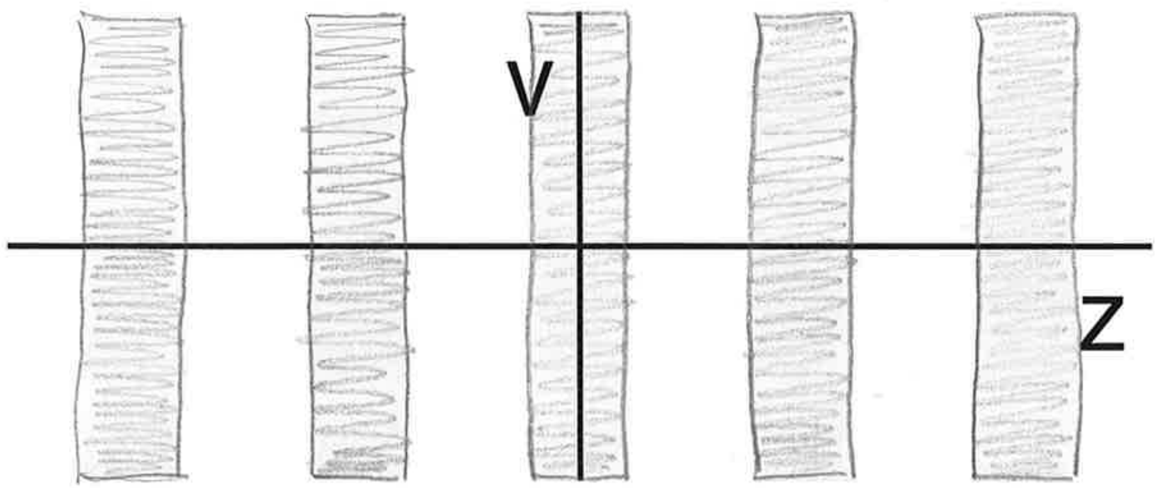



$$f(v) \approx e^{-ikvt}\, e^{-v^2/2v_t^2}$$

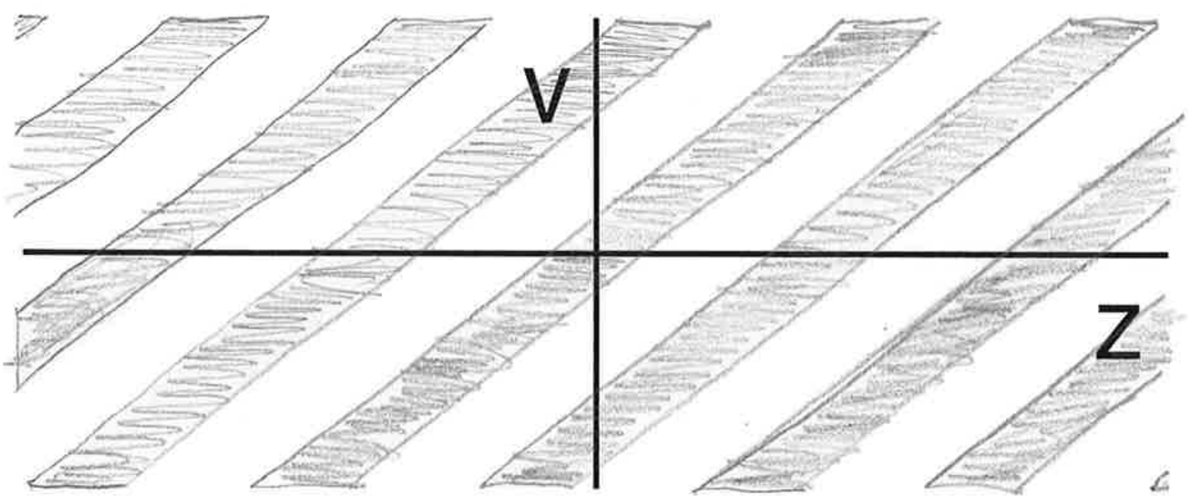
Highly oscillatory at large t.

Weak amount of collisional scattering will wipe out!

$$\left(\frac{\partial f}{\partial t}\right)_c \approx \nu\, v_t^2 \frac{\partial^2 f}{\partial v^2}$$




Initial perturbation with $k = 3$.




Advected perturbation at time $t = 1$.

Figure 4.1: Illustration of ~~the Plasma Echo~~ Phase mixing. An initial density perturbation is shown in the first picture with spatial structure having wave number $k = 3$. After some time, the perturbation has tilted in phase space, so the perturbation averaged over velocities has decayed.

Summary: Reversible, Hamiltonian, entropy conserving systems can display behavior that looks like its irreversible (entropy increasing) if there is some kind of "coarse-graining" or averaging

Landau-damping & phase-mixing + echoes are interesting!