Investigate the effects of a finite-correlation-time noise on a Langevin equation.
Using Krommes' symmetrization trick to get Galilean invariance.

Equations and notation used here follow the paper

"Non-white noise and a multiple rate Markovian closure theories for turbulence", by Gregory W. Hammett and John C. Bowman. Submitted to Physics of Fluids, Feb. 2002, and available at www.arxiv.org.

In particular, this maple script was used to check and produce some of the results in Section 3 on the effects of finite-correlation-time noise on a Langevin equation, and to make the plots found there.

Note that some of the notation used here is a little bit different than in the paper. In particular, the various rates are denoted by eta in the paper but by nu here.

rednoisek12 like rednoisek11, but with some additional plots for the paper.

rednoisek11 like rednoisek10, but using the direct solution instead of the iterative solution for nu_eff(nu, nu_f)

```
> restart; kernelopts(version); interface(version); ssystem("date");
```

Maple 7.00, IBM INTEL LINUX, May 28 2001 Build ID 96223

Maple Worksheet Interface, Maple 7.00, IBM INTEL LINUX, May 28 2001 Build ID 96223

[0, "Tue Feb 12 12:29:25 GMT 2002\n"]

```
> # We are working with the Langevin equation of the following form.

  diff(psi(t),t) = - nu * psi + conjugate(f);
```

$$\frac{\partial}{\partial t}\psi(t) = -\nu\,\psi + \bar{f}$$

```
> #
  # where nu is the decay rate and conjugate(f) is the noise
  (forcing) term.
  # Note that the conjugate(f) convention is being used, to make it
  similar to
  # the practice in the DIA/EDQNM/RMC.
  #
  # Note that the damping rate is denoted by eta in the paper, but
  by nu here.

>   # Next is the 2-time correlation function for complex arguments
```

```
  that I found both by
    # Fourier transforming and integrating around the poles, and by
  a longer calculation.
    # Also note that I am using the DIA convention that conjugate(f)
  appears in the
    # Langevin equation, so the resonance condition is
  omega+omega_f=0.
```

```
> c2 := proc(tau,nu,nu_f) ( (nu_f+conjugate(nu_f))
     /(conjugate(nu_f)-nu) *( exp(-nu*tau)
  /(nu+conjugate(nu))/(nu+nu_f)

  -exp(-conjugate(nu_f)*tau)/(nu_f+conjugate(nu_f))/(conjugate(nu)+c
  onjugate(nu_f)) ) )  end ;
```

$$c2 := \mathbf{proc}(\tau, \nu, nu\_f)$$
$$(nu\_f + \mathrm{conjugate}(nu\_f))*(\exp(-\nu*\tau) / ((\nu + \mathrm{conjugate}(\nu))*(\nu + nu\_f))$$
$$- \exp(-\mathrm{conjugate}(nu\_f)*\tau) / ((nu\_f + \mathrm{conjugate}(nu\_f))*(\mathrm{conjugate}(\nu) + \mathrm{conjugate}(nu\_f)))) / (\mathrm{conjugate}(nu\_f) - \nu)$$
$$\mathbf{end\ proc}$$

```
> c0 := proc(nu,nu_f) ( (nu + conjugate(nu)+nu_f+conjugate(nu_f))
            /(nu+conjugate(nu)) / (nu + nu_f) / conjugate(nu+nu_f)
  ) end ;
```

$$c0 := \mathbf{proc}(\nu, nu\_f)$$
$$(\nu + \mathrm{conjugate}(\nu) + nu\_f + \mathrm{conjugate}(nu\_f)) / ((\nu + \mathrm{conjugate}(\nu))*(\nu + nu\_f)*$$
$$\mathrm{conjugate}(\nu + nu\_f))$$
$$\mathbf{end\ proc}$$

```
> c2(tau,nu,nu_f);
```

$$\frac{(nu\_f + \overline{nu\_f})\left(\dfrac{\mathbf{e}^{(-\nu\,\tau)}}{(\nu + \bar{\nu})(\nu + nu\_f)} - \dfrac{\mathbf{e}^{(-\overline{nu\_f}\,\tau)}}{(nu\_f + \overline{nu\_f})(\nu + \overline{nu\_f})}\right)}{\overline{nu\_f} - \nu}$$

```
> normal(c2(0,nu,nu_f)); c0(nu,nu_f);
```

$$\frac{nu\_f + nu\_f + \nu + \bar{\nu}}{(\nu + \overline{nu\_f})(\nu + \bar{\nu})(\bar{\nu} + nu\_f)}$$
$$\frac{nu\_f + nu\_f + \nu + \bar{\nu}}{(\nu + \bar{\nu})(\nu + nu\_f)(\nu + \overline{nu\_f})}$$

```
> numer(normal(c2(0,nu,nu_f))) - numer(c0(nu,nu_f));
```

$$0$$

```
> denom(normal(c2(0,nu,nu_f))) - expand(denom(c0(nu,nu_f)));
```

$$(\nu + \overline{nu\_f})(\nu + \bar{\nu})(\nu + nu\_f) - \nu^2\,\bar{\nu} - \nu^2\,\overline{nu\_f} - \nu\,nu\_f\,\bar{\nu} - \nu\,nu\_f\,\overline{nu\_f} - \nu^2\,\bar{\nu} - \nu\,\bar{\nu}\,\overline{nu\_f} - \nu^2\,nu\_f$$
$$- \nu\,nu\_f\,\overline{nu\_f}$$

```
> simplify(%);
```

$$0$$

```
> c2(0,g+I*w,g_f+I*w_f);
```

$$(g\_f+I\,w\_f+\overline{(g\_f+I\,w\_f)})\left(\frac{1}{(g+I\,w+\overline{(g+I\,w)})\,(g+I\,w+g\_f+I\,w\_f)}-\frac{1}{(g\_f+I\,w\_f+\overline{(g\_f+I\,w\_f)})\,((g+I\,w)+\overline{(g\_f+I\,w\_f)})}\right)\Big/\left((\overline{(g\_f+I\,w\_f)}-g-I\,w)\right)$$

```
> evalc(%);
```

$$2\,\frac{g\_f\,(g\_f-g)\left(\frac{1}{2}\,\frac{g+g\_f}{g\,((g+g\_f)^2+(w+w\_f)^2)}-\frac{1}{2}\,\frac{g+g\_f}{g\_f\,((g+g\_f)^2+(-w\_f-w)^2)}\right)}{(g\_f-g)^2+(-w\_f-w)^2}+\frac{2\,g\_f\,(-w\_f-w)\left(-\frac{1}{2}\,\frac{w+w\_f}{g\,((g+g\_f)^2+(w+w\_f)^2)}+\frac{\frac{1}{2}\,(-w\_f-w)}{g\_f\,((g+g\_f)^2+(-w\_f-w)^2)}\right)}{(g\_f-g)^2+(-w\_f-w)^2}+I\left(-2\,\frac{g\_f\,(-w\_f-w)\left(\frac{1}{2}\,\frac{g+g\_f}{g\,((g+g\_f)^2+(w+w\_f)^2)}-\frac{1}{2}\,\frac{g+g\_f}{g\_f\,((g+g\_f)^2+(-w\_f-w)^2)}\right)}{(g\_f-g)^2+(-w\_f-w)^2}+\frac{2\,g\_f\,(g\_f-g)\left(-\frac{1}{2}\,\frac{w+w\_f}{g\,((g+g\_f)^2+(w+w\_f)^2)}+\frac{\frac{1}{2}\,(-w\_f-w)}{g\_f\,((g+g\_f)^2+(-w\_f-w)^2)}\right)}{(g\_f-g)^2+(-w\_f-w)^2}\right)$$

```
> simplify(%);
```

$$\frac{g+g\_f}{g\,(g^2+2\,g\_f\,g+g\_f^2+w\_f^2+2\,w\_f\,w+w^2)}$$

```
> # This has a nice simple form, and shows c2(t=0) is purely real,
  as it should be.
  # Use this to redefine c2 so c2(tau=0)=1:

> c2 := proc(tau,nu,nu_f)
        local g, g_f, c0;
        g:=Re(nu) ;
        g_f:=Re(nu_f) ;
        c0:=(1/g + 1/g_f)/2/((g+g_f)^2 + (Im(nu)+Im(nu_f))^2);
  1/c0*1/(conjugate(nu_f)-nu) *( exp(-nu*tau)
  /(nu+conjugate(nu))/(nu+nu_f)

  -exp(-conjugate(nu_f)*tau)/(nu_f+conjugate(nu_f))/(conjugate(nu)+c
```

```
  onjugate(nu_f) ) )    end ;
```

$$c2 := \mathbf{proc}(\tau, \nu, nu\_f)$$

$$\mathbf{local}\ g, g\_f, c0;$$

$$g := \Re(\nu);$$

$$g\_f := \Re(nu\_f);$$

$$c0 := 1 / 2*(1 / g + 1 / g\_f) / ((g + g\_f)\hat{}2 + (\Im(\nu) + \Im(nu\_f))\hat{}2);$$

$$(\exp(-\nu*\tau) / ((\nu + \mathrm{conjugate}(\nu))*(\nu + nu\_f))$$

$$- \exp(-\mathrm{conjugate}(nu\_f)*\tau) / ((nu\_f + \mathrm{conjugate}(nu\_f))*(\mathrm{conjugate}(\nu) + \mathrm{conjugate}(nu\_f)))$$

$$) / (c0*(\mathrm{conjugate}(nu\_f) - \nu))$$

$$\mathbf{end\ proc}$$

```
> c2(0,nu,nu_f);
```

$$2\frac{((\Re(\nu)+\Re(nu\_f))^2+(\Im(\nu)+\Im(nu\_f))^2)\left(\frac{1}{\overline{(\nu+\nu)}\,(\nu+nu\_f)}-\frac{1}{\overline{(nu\_f+nu\_f)}\,\overline{(\nu+nu\_f)}}\right)}{\left(\frac{1}{\Re(\nu)}+\frac{1}{\Re(nu\_f)}\right)\overline{(nu\_f-\nu)}}$$

```
> c2(0,1,2);  # verify that this rescaled c2 is 1 at tau=0
```

$$1$$

```
> c2(0,2.0+3.0*I, 4.0+7.0*I);
```

$$1.000000000 - .9615384615\ 10^{-10}\ I$$

```
> c2(1,1-I,2-I);
```

$$\left(\frac{52}{15}-\frac{104}{15}I\right)\left(\left(\frac{3}{26}+\frac{1}{13}I\right)\mathbf{e}^{(-1+I)}-\left(\frac{3}{52}-\frac{1}{26}I\right)\mathbf{e}^{(-2-I)}\right)$$

```
> # recursive definition of nu_eff with symmetrized weight
  nu_eff_rec := proc(nu,nu_f, nu_eff)
       ( nu*conjugate(nu_f)*(nu+nu_f +
  conjugate(nu)+conjugate(nu_f))
         + conjugate(nu_eff) * (nu * conjugate(nu_f) -
  nu_f*conjugate(nu)) )
       / ( (nu +conjugate(nu_eff)) * (nu + nu_f + conjugate(nu) +
  conjugate(nu_f))
      + (conjugate(nu_f)+conjugate(nu))*(nu_f+conjugate(nu_f)) )
  end;
```

$$nu\_eff\_rec := \mathbf{proc}(\nu, nu\_f, nu\_eff)$$

$$(\nu*\mathrm{conjugate}(nu\_f)*(\nu + \mathrm{conjugate}(\nu) + nu\_f + \mathrm{conjugate}(nu\_f))$$

$$+ \mathrm{conjugate}(nu\_eff)*(\nu*\mathrm{conjugate}(nu\_f) - nu\_f*\mathrm{conjugate}(\nu))) / ($$

$$(\nu + \mathrm{conjugate}(nu\_eff))*(\nu + \mathrm{conjugate}(\nu) + nu\_f + \mathrm{conjugate}(nu\_f))$$

$$+ (\mathrm{conjugate}(nu\_f) + \mathrm{conjugate}(\nu))*(nu\_f + \mathrm{conjugate}(nu\_f)))$$

$$\mathbf{end\ proc}$$

```
> nu_eff_rec(nu,nu_f,nu_eff);
```

$$\frac{\bar{\nu}\,\overline{nu\_f}\,(nu\_f + nu\_f + \bar{\nu} + \nu) + \overline{nu\_eff}\,(\nu\,\overline{nu\_f} - \bar{\nu}\,nu\_f)}{(\nu + nu\_eff)\,(\overline{nu\_f} + \overline{nu\_f} + \bar{\nu} + \nu) + (\nu + \overline{nu\_f})\,(nu\_f + nu\_f)}$$

```
> nu_eff_rec(1.0, 0.2+5*I, 0.0);
```

$$2.064549714 - 2.732951588\,I$$

```
> # A complete definition of nu_eff (which iterates internally till
  it converges):
  nu_effc := proc(nu, nu_f)
       local nu_eff, nu_eff0, error_rms, i;
       if not (type(nu,complexcons) and type(nu_f,complexcons)) then
           RETURN('procname(nu, nu_f)'); # if arguments aren't
  (complex) numbers, return unevaluated.
       fi;
       nu_eff := 0.0;  # Initial guess
       for i from 1 to 100 do
          nu_eff0 := nu_eff;  nu_eff := evalf(nu_eff_rec(nu, nu_f,
  nu_eff));
          error_rms := abs(nu_eff0-nu_eff);
          if ( i > 5 and error_rms < 1.0e-8 ) then break fi;
       od;
       if(error_rms>1.0e-8) then print("Warning in nu_eff:  error, #
  of iterations =",error_rms, i); fi;
       nu_eff
  end;
```

nu_effc := **proc**(ν, *nu_f*)
local *nu_eff*, *nu_eff0*, *error_rms*, *i*;
  if not (type(ν, *complexcons*) **and** type(*nu_f*, *complexcons*)) **then**
    RETURN('procname(ν, *nu_f*)')
  end if;
  nu_eff := 0.;
  for *i* **to** 100 **do**
    nu_eff0 := *nu_eff*;
    nu_eff := evalf(nu_eff_rec(ν, *nu_f*, *nu_eff*));
    error_rms := abs(*nu_eff0* – *nu_eff*);
    if 5 < *i* **and** *error_rms* < .10*10^(-7) **then break end if**
  end do;
  if .10*10^(-7) < *error_rms* **then**
    print("Warning in nu_eff: error, # of iterations =", *error_rms*, *i*)
  end if;
  nu_eff
end proc

```
> # Direct evalution of nu_eff from nu and nu_f:
```

```
nu_effd := proc(nu,nu_f)
      local g, w, g_f, w_f , g_eff, w_eff;
      g   := Re(nu)   ; w   := Im(nu);
      g_f := Re(nu_f) ; w_f := Im(nu_f);
g_eff :=
-1/2*(g^4+g_f^2*w^2+w^2*g^2+6*g^2*g_f^2+4*g^3*g_f+4*g*g_f^3-2*g*g_
f*w^2+g_f^4-4*g*w*w_f*g_f+w_f^2*g_f^2+g^2*w_f^2+2*g^2*w*w_f-2*w_f^
2*g_f*g+2*g_f^2*w*w_f-sqrt((g+g_f)^2*(g^6-2*w_f^4*g_f*g+w^4*g^2+w^
4*g_f^2+2*w_f^2*g_f^4+4*g_f^4*w*w_f+4*g_f^2*w^3*w_f+6*w^2*w_f^2*g_
f^2+4*w*w_f^3*g_f^2-12*g*g_f*w_f^2*w^2-8*g*g_f*w_f*w^3-8*g*g_f*w*w
_f^3+2*g^4*w_f^2+4*g^4*w*w_f+6*g^2*w_f^2*w^2+4*g^2*w_f*w^3+4*g^2*w
*w_f^3+2*g^4*w^2+2*g_f^4*w^2+g^2*w_f^4+w_f^4*g_f^2+g_f^6-2*w^4*g*g
_f-4*g_f^2*g^2*w^2+8*g*g_f^3*w^2+8*g^3*g_f*w^2+31*g^2*g_f^4+44*g^3
*g_f^3+8*w_f^2*g_f*g^3-4*w_f^2*g_f^2*g^2+8*w_f^2*g_f^3*g+31*g^4*g_
f^2-8*w_f*g_f^2*g^2*w+16*w_f*g_f^3*g*w+16*g^3*g_f*w*w_f+10*g^5*g_f
+10*g*g_f^5)))*(g+g_f)/(g^4+g_f^2*w^2+w^2*g^2+6*g^2*g_f^2+4*g^3*g_
f+4*g*g_f^3-2*g*g_f*w^2+g_f^4-4*g*w*w_f*g_f+w_f^2*g_f^2+g^2*w_f^2+
2*g^2*w*w_f-2*w_f^2*g_f*g+2*g_f^2*w*w_f)          ;
w_eff :=
-(g*w*g_eff-g_eff*w_f*g_f+g^2*w_f+g*w_f*g_eff+w_f*g_f*g-g*g_f*w-g_
f^2*w-g_f*w*g_eff)/(g^2+g_f^2+2*g*g_f) ;
evalf(evalc(g_eff+I*w_eff));
end ;
```

nu_effd := **proc**(ν, *nu_f*)

local *g*, *w*, *g_f*, *w_f*, *g_eff*, *w_eff*;

g := ℜ(ν);

w := ℑ(ν);

g_f := ℜ(*nu_f*);

w_f := ℑ(*nu_f*);

g_eff := –1 / 2*(*g*^4 + *g_f*^2*w^2 + *w*^2*g^2 + 6*g^2*g_f^2 + 4*g^3*g_f + 4*g*g_f^3
– 2*g*g_f*w^2 + *g_f*^4 – 4*g*w*w_f*g_f + *w_f*^2*g_f^2 + g^2*w_f^2 + 2*g^2*w*w_f
– 2*w_f^2*g_f*g + 2*g_f^2*w*w_f – sqrt((*g* + *g_f*)^2*(4*g^4*w*w_f – 2*w^4*g*g_f
+ 8*g*g_f^3*w^2 + 4*w*w_f^3*g_f^2 + 8*w_f^2*g_f*g^3 – 4*w_f^2*g_f^2*g^2
– 12*g*g_f*w_f^2*w^2 – 8*g*g_f*w_f*w^3 – 8*g*g_f*w*w_f^3 – 8*w_f*g_f^2*g^2*w
+ 16*w_f*g_f^3*g*w + 16*g^3*g_f*w*w_f + g^6 + g_f^6 – 2*w_f^4*g_f*g
+ 6*g^2*w_f^2*w^2 + 4*g^2*w_f*w^3 + 4*g^2*w*w_f^3 + 8*g^3*g_f*w^2
+ 8*w_f^2*g_f^3*g + w^4*g^2 + w^4*g_f^2 + g^2*w_f^4 + w_f^4*g_f^2
+ 2*w_f^2*g_f^4 + 2*g^4*w_f^2 + 2*g^4*w^2 + 2*g_f^4*w^2 + 31*g^2*g_f^4
+ 44*g^3*g_f^3 + 31*g^4*g_f^2 + 10*g^5*g_f + 10*g*g_f^5 + 6*w^2*w_f^2*g_f^2
+ 4*g_f^2*w^3*w_f + 4*g_f^4*w*w_f – 4*g_f^2*g^2*w^2)))*(*g* + *g_f*) / (*g*^4
+ *g_f*^2*w^2 + *w*^2*g^2 + 6*g^2*g_f^2 + 4*g^3*g_f + 4*g*g_f^3 – 2*g*g_f*w^2

```
        + g_f^4 - 4*g*w*w_f*g_f + w_f^2*g_f^2 + g^2*w_f^2 + 2*g^2*w*w_f - 2*w_f^2*g_f*g
        + 2*g_f^2*w*w_f);
    w_eff := -(g*w*g_eff - g_eff*w_f*g_f + g^2*w_f + g*w_f*g_eff + w_f*g_f*g - g*g_f*w
        - g_f^2*w - g_f*w*g_eff) / (g^2 + g_f^2 + 2*g*g_f);
    evalf(evalc(g_eff + I*w_eff))
end proc
> nu_effd(1, 4+10*I); nu_effc(1, 4+10*I);
                          1.300884281 - .438938864 I
                          1.300884279 - .4389388620 I
> nu_effd(1,1); nu_effc(1,1);
                                .414213562
                                .4142135621
> nu_effd(1,1+16*I); nu_effc(1,1+16*I);
                          7.124038405 - 8. I
                          7.124038406 - 7.999999971 I
> nu_effd(1,1+50*I); nu_effc(1,1+50*I);
                          24.03996805 - 25. I
        "Warning in nu_eff:  error, # of iterations =", .001354631653, 101
                          24.03994978 - 24.98372051 I
> tau_eff := proc(nu,nu_f) 1/nu_effd(nu,nu_f) end ;
              tau_eff := proc(ν, nu_f) 1 / nu_effd(ν, nu_f) end proc
> tau_eff(1.0,1.0+I);
                          1.000000000 + 1.000000000 I
> # Older, more complex definition of tau_eff.
  tau_effa := proc(nu,nu_f)
        local g, g_f, c0;
        g:=Re(nu) ;
        g_f:=Re(nu_f) ;
        c0:=(1/g + 1/g_f)/2/((g+g_f)^2 + (Im(nu)-Im(nu_f))^2);
  1/c0*1/(nu_f-nu) *( 1/nu/(nu+conjugate(nu))/(nu+conjugate(nu_f))

  -1/nu_f/(nu_f+conjugate(nu_f))/(conjugate(nu)+nu_f) )    end ;
tau_effa := proc(ν, nu_f)
local g, g_f, c0;
    g := ℜ(ν);
    g_f := ℜ(nu_f);
    c0 := 1 / 2*(1 / g + 1 / g_f) / ((g + g_f)^2 + (ℑ(ν) - ℑ(nu_f))^2);
    (1 / (ν*(ν + conjugate(ν))*(ν + conjugate(nu_f)))
        - 1 / (nu_f*(nu_f + conjugate(nu_f))*(conjugate(ν) + nu_f))) / (c0*(nu_f - ν))
end proc
> # Older, more complex, definition of tau_eff, now including
```

```
  Galilean invariance.
  # Perhaps the best fit?
  tau_effg := proc(nu,nu_f)
        local g, g_f, c0, w, dw, tau_eff;
        w := Im(nu);
        dw := Im(nu_f)-Im(nu);
        g:=Re(nu) ;
        g_f:=Re(nu_f) ;
        c0:=(1/g + 1/g_f)/2/((g+g_f)^2 + (dw)^2);
        tau_eff := 1/c0*1/(g_f-g+I*dw)/2*( 1/g/g/(g+g_f-I*dw)

  -1/g_f/(g_f+I*dw)/(g+g_f+I*dw) );
        # That is tau_eff in the w=0 frame.  Now correct it:
        tau_eff := 1/(1/tau_eff + I*w);
        # print("c0, tau_eff = ", c0, tau_eff);
        tau_eff
  end ;
```

tau_effg := **proc**(ν, *nu_f*)

local *g*, *g_f*, *c0*, *w*, *dw*, *tau_eff*;

w := ℑ(ν);

dw := ℑ(*nu_f*) − ℑ(ν);

g := ℜ(ν);

g_f := ℜ(*nu_f*);

c0 := 1 / 2∗(1 / *g* + 1 / *g_f*) / ((*g* + *g_f*)^2 + *dw*^2);

tau_eff := 1 / 2∗(1 / (*g*^2∗(*g* + *g_f* − *I*∗*dw*)) − 1 / (*g_f*∗(*g_f* + *I*∗*dw*)∗(*g* + *g_f* + *I*∗*dw*))) / (*c0*∗ (*g_f* − *g* + *I*∗*dw*));

tau_eff := 1 / (1 / *tau_eff* + *I*∗*w*);

tau_eff

end proc

```
> # Try a redefinition of the simple version of tau_eff to handle
  the weak-turbulence problem
  # but further modified to handle a kind of Galilean invariance:
  tau_eff_wt := proc(nu,nu_f)
     local w, nu_eff;
     w := Im(nu);
     nu_eff := w*I+1/(1/Re(nu) + 1/(nu_f-w*I)) ;
     # print("w, nu_eff =", w, nu_eff);
     1/nu_eff
   end;
```

tau_eff_wt := **proc**(ν, *nu_f*)

local *w*, *nu_eff*;

w := ℑ(ν); *nu_eff* := *I*∗*w* + 1 / (1 / ℜ(ν) + 1 / (*nu_f* − *I*∗*w*)); 1 / *nu_eff*

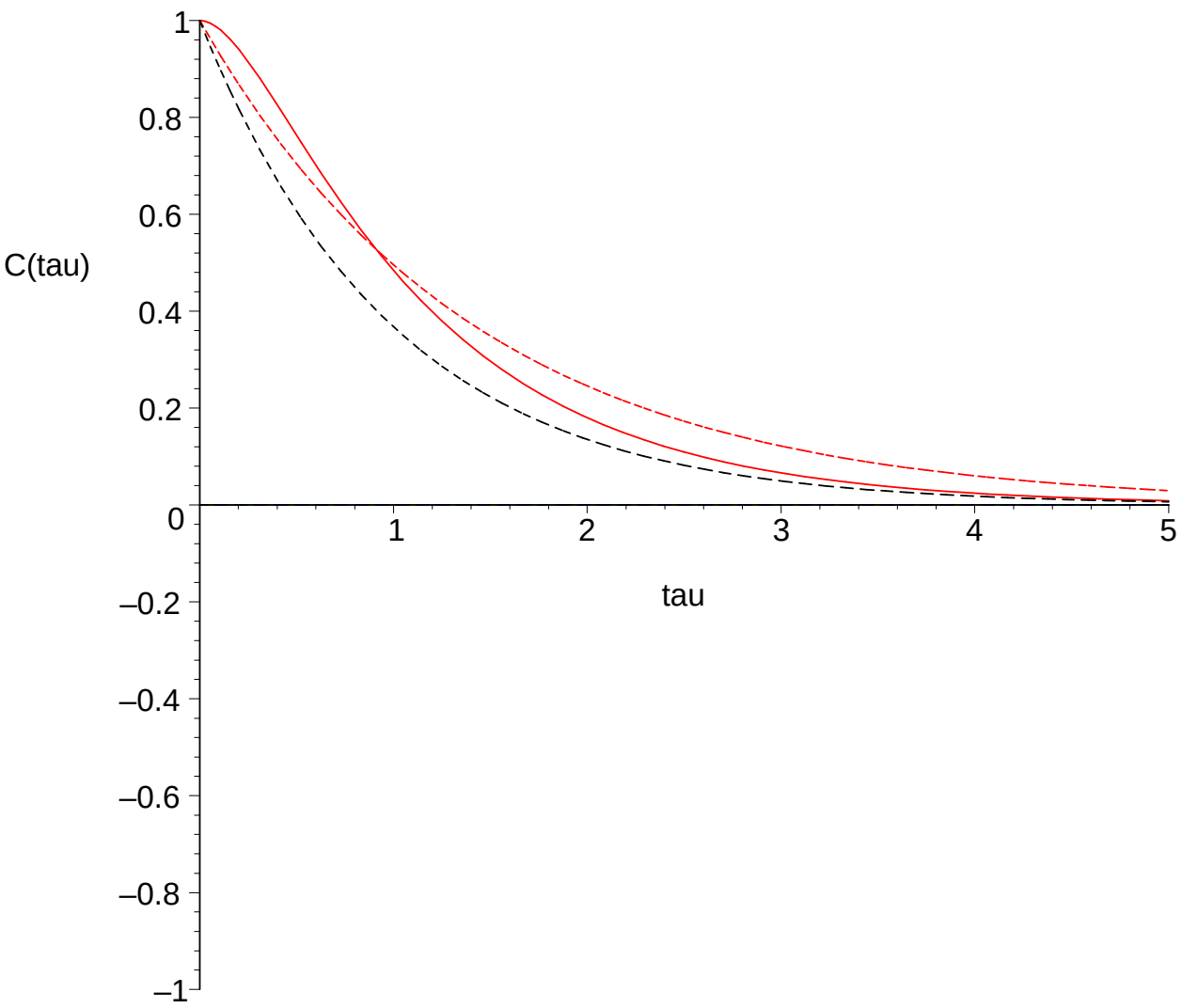
end proc

```
> # 1/tau_eff(g+I*w,g_f+I*w_f);  # In this case, it is 1/tau_eff
  which has a simpler form:
> # evalc(%);
> # simplify(%); # Thus 1/tau_eff is fairly simple in this case,
  though tau_eff looks more complicated,
  # if evalc is used in its evaluation:
> # simplify(1/%);
> # simplify(expand(evalc(%)));
> nu:=1 ; nu_f := 4;
> nu_eff := 1/tau_eff(nu,nu_f);
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=black, linestyle=2):
  p2c := plot(Im(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p2,p1c,p2c,p1b,p2b});
```

$$\nu := 1$$

$$nu\_f := 4$$

$$nu\_eff := .7015621181$$



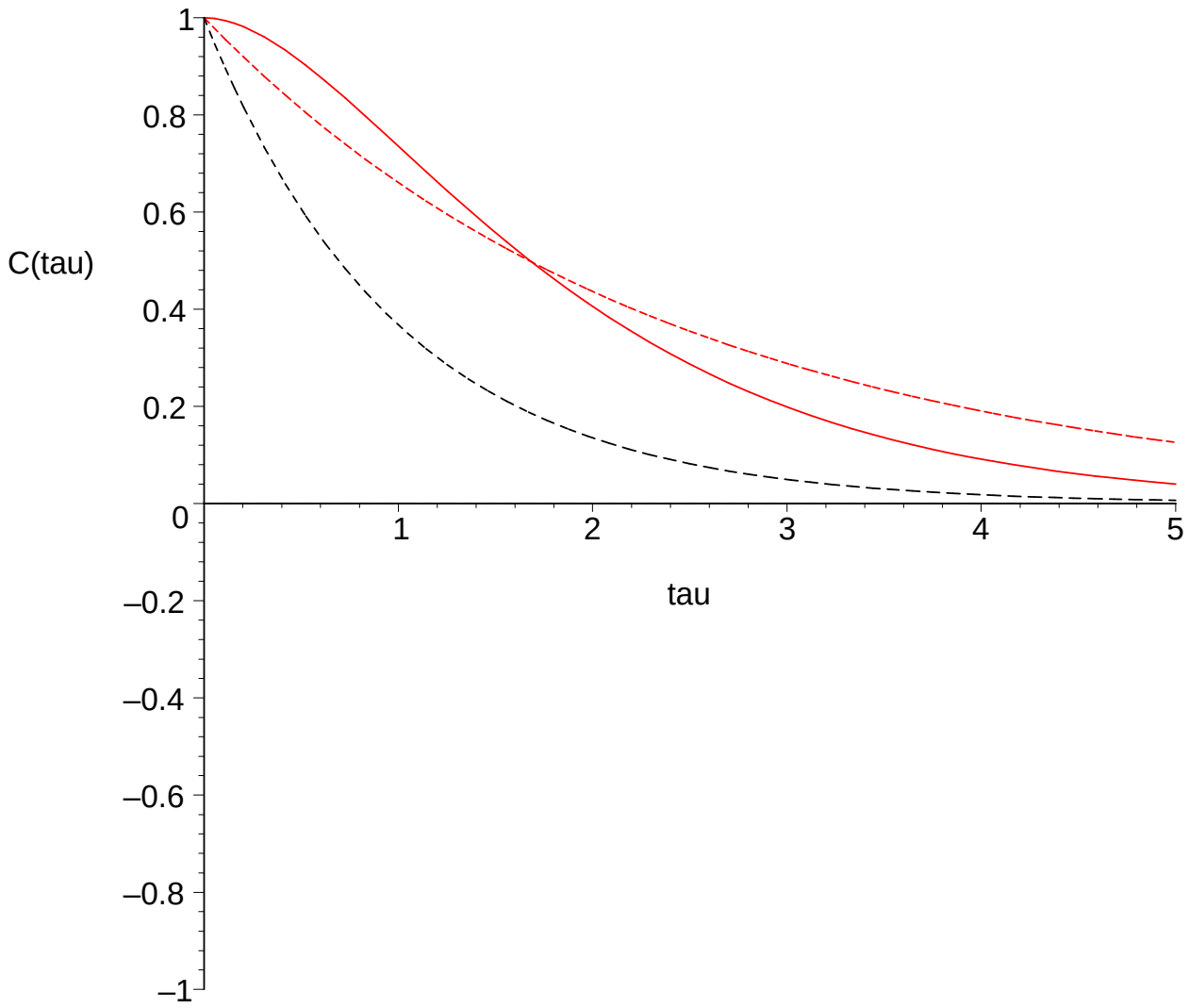

```
> nu:=1 ; nu_f := 1.000001;
> nu_eff := evalf(1/tau_eff(nu,nu_f));
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=black, linestyle=2):
```

```
  p2c := plot(Im(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p2,p1c,p2c,p1b,p2b});
```

$$\nu := 1$$

$$nu\_f := 1.000001$$

$$nu\_eff := .4142137694 - 0.\,I$$



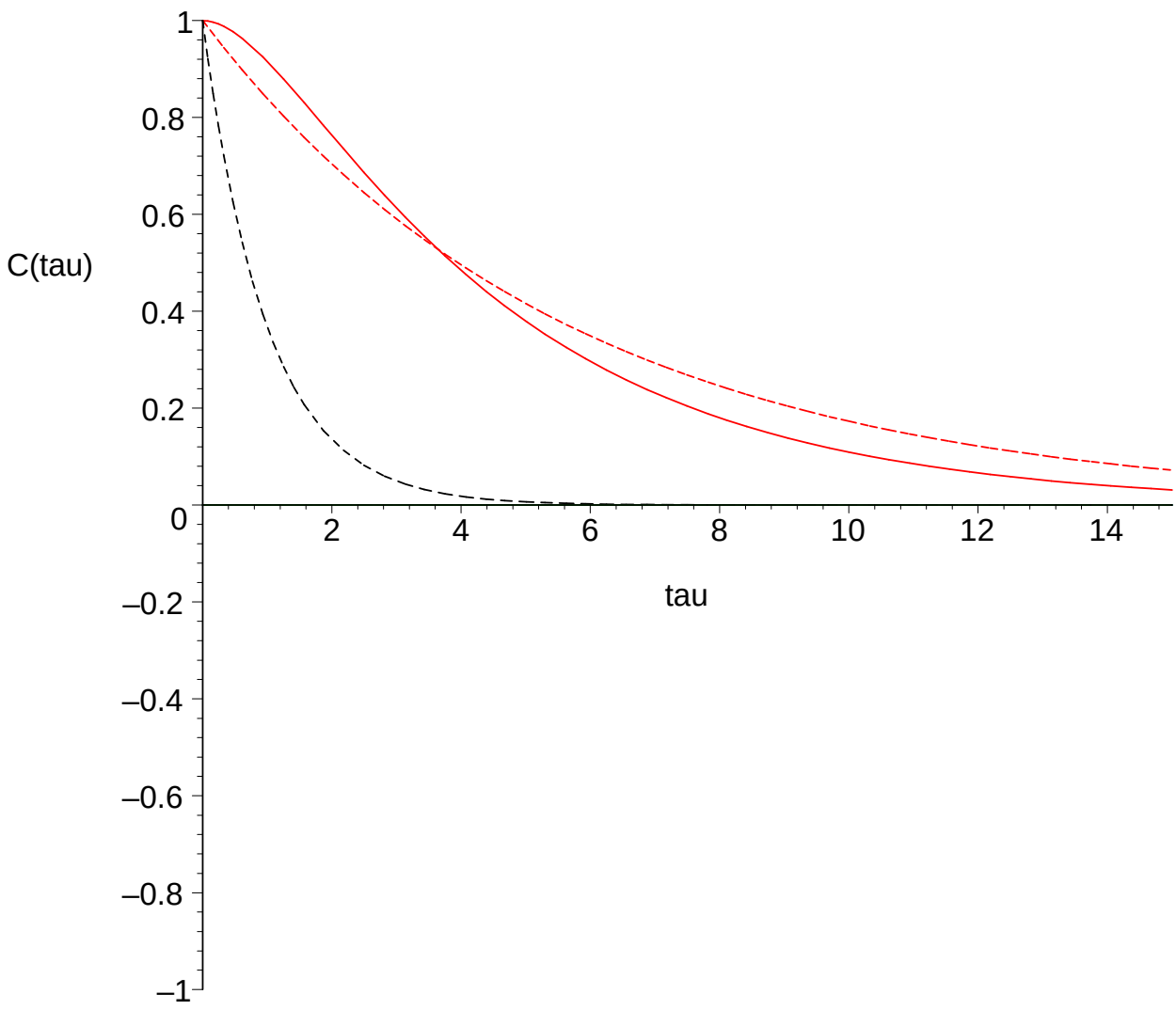

```
> nu:=1 ; nu_f := 0.25;
> nu_eff := 1/tau_eff(nu,nu_f);
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..15, 'C(tau)'=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..15, 'C(tau)'=-1..1,
```

```
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..15, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..15, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..15, 'C(tau)'=-1..1,
  color=black, linestyle=2):
  p2c := plot(Im(exp(-tau*nu)),      tau=0..15, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p2,p1c,p2c,p1b,p2b});
```

$$\nu := 1$$

$$nu\_f := .25$$

$$nu\_eff := .1753905298 - 0.\,I$$



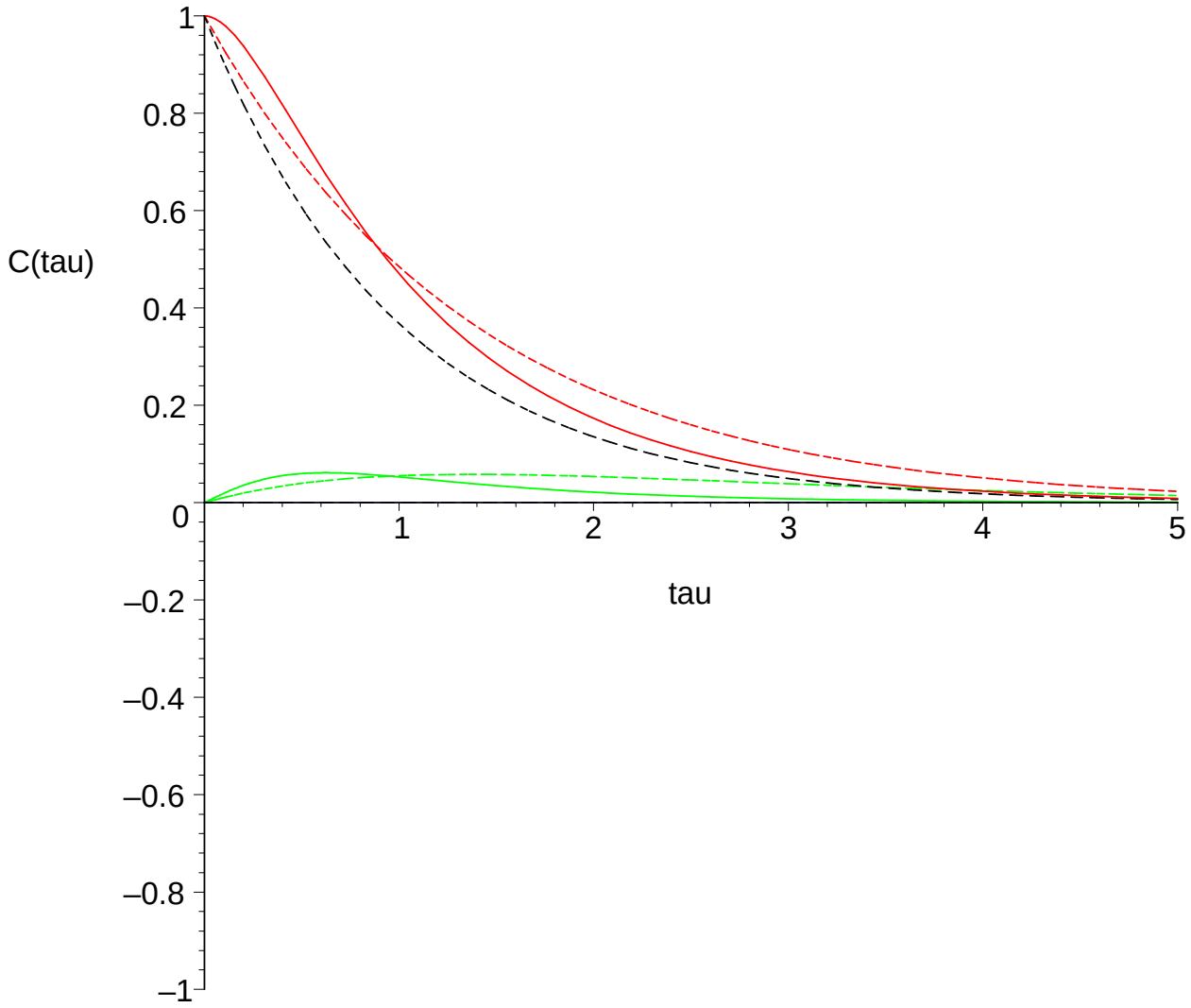

```
> nu:=1 ; nu_f := 4+1*I;
> nu_eff := 1/tau_eff(nu,nu_f);
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=black, linestyle=2):
```

```
  p2c := plot(Im(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p2,p1c,p2c,p1b,p2b});
```

$$\nu := 1$$

$$nu\_f := 4 + I$$

$$nu\_eff := .7172886681 - .1139253599\, I$$



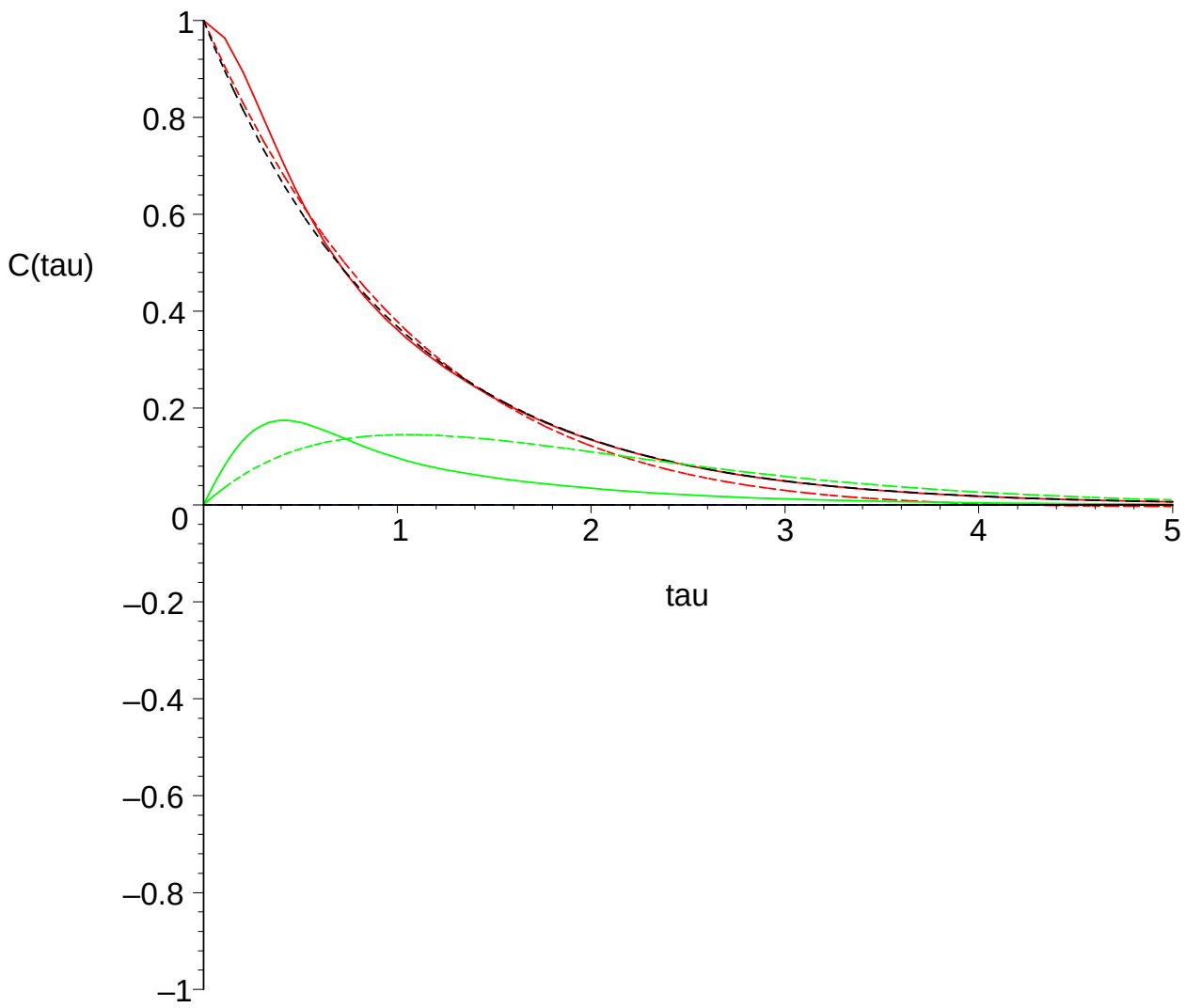

```
> nu:=1 ; nu_f := 4+4*I;
  nu_eff := 1/tau_eff(nu,nu_f);
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
```

```
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=black, linestyle=2):
  p2c := plot(Im(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p2,p1c,p2c,p1b,p2b});
```

$$\nu := 1$$

$$nu\_f := 4 + 4\,I$$

$$nu\_eff := .9031749120 - .3664760420\,I$$



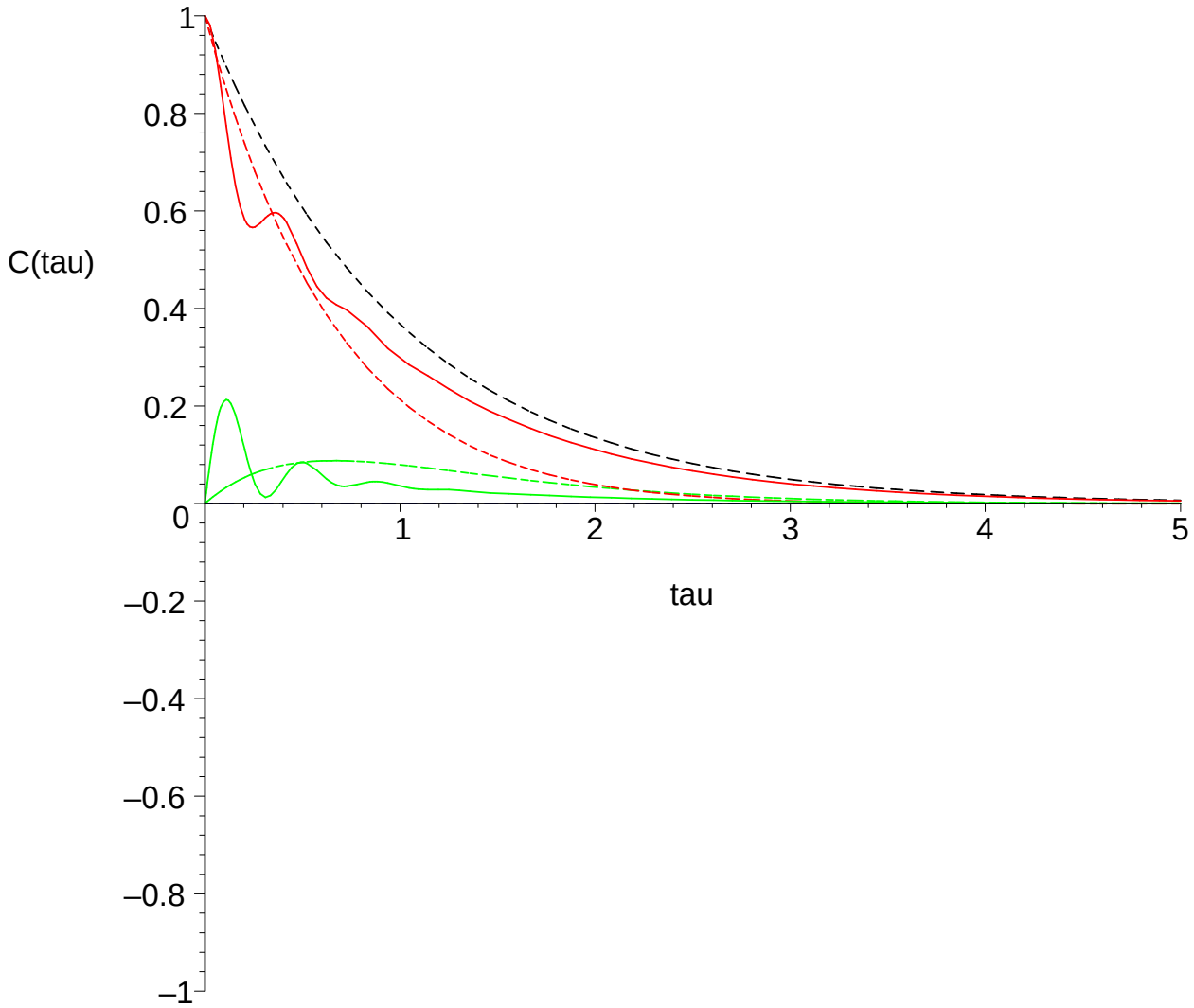

```
> nu:=1 ; nu_f := 4+16*I;
> nu_eff := 1/tau_eff(nu,nu_f);
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=black, linestyle=2):
```

```
  p2c := plot(Im(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p2,p1c,p2c,p1b,p2b});
```

$$\nu := 1$$

$$nu\_f := 4 + 16\,I$$

$$nu\_eff := 1.480416812 - .3575997221\,I$$



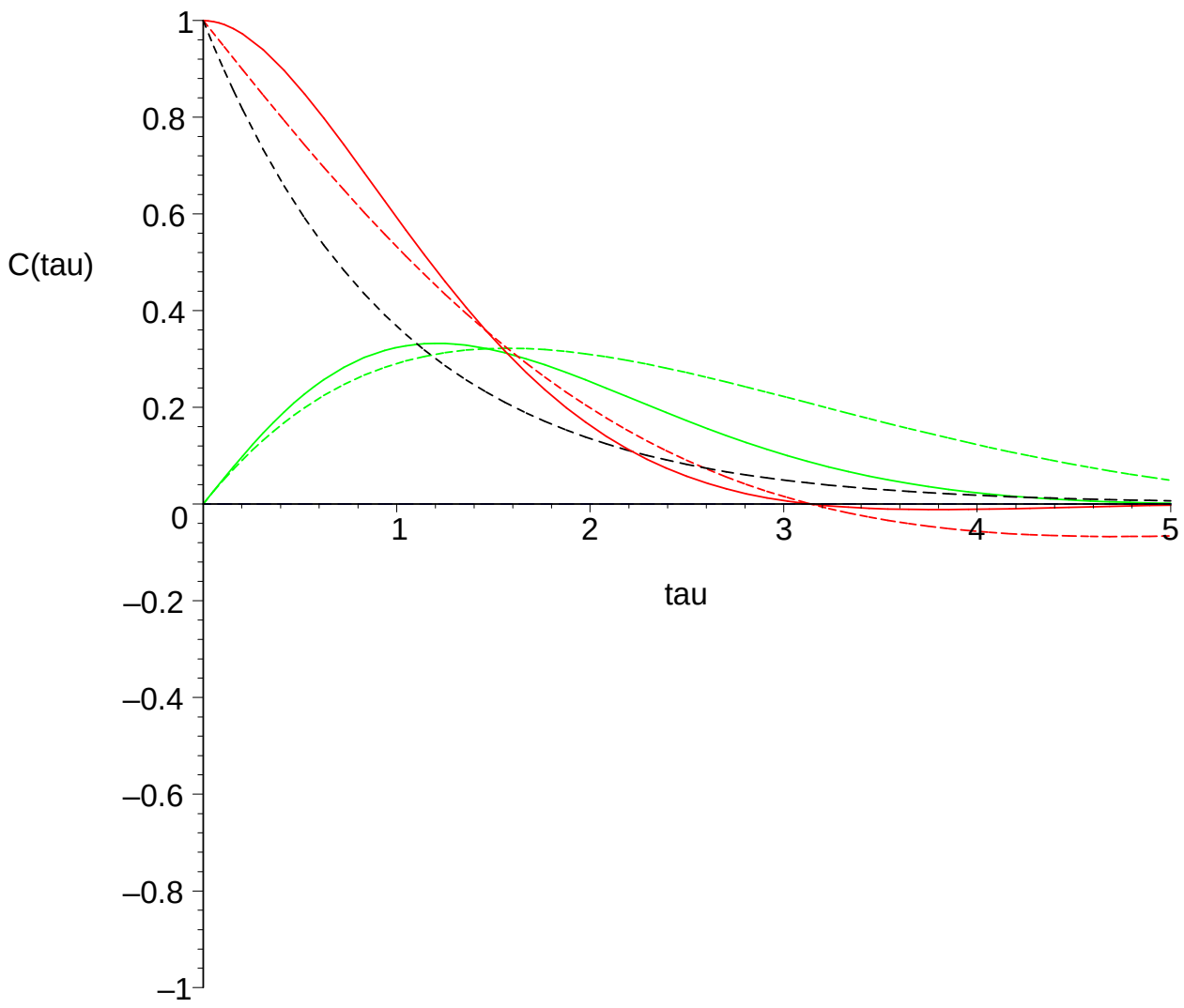

```
> nu:=1 ; nu_f := 1+1*I;
> nu_eff := 1/tau_eff(nu,nu_f);
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
```

```
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=black, linestyle=2):
  p2c := plot(Im(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p2,p1c,p2c,p1b,p2b});
```

$$\nu := 1$$

$$nu\_f := 1 + I$$

$$nu\_eff := .5000000000 - .5000000000\, I$$



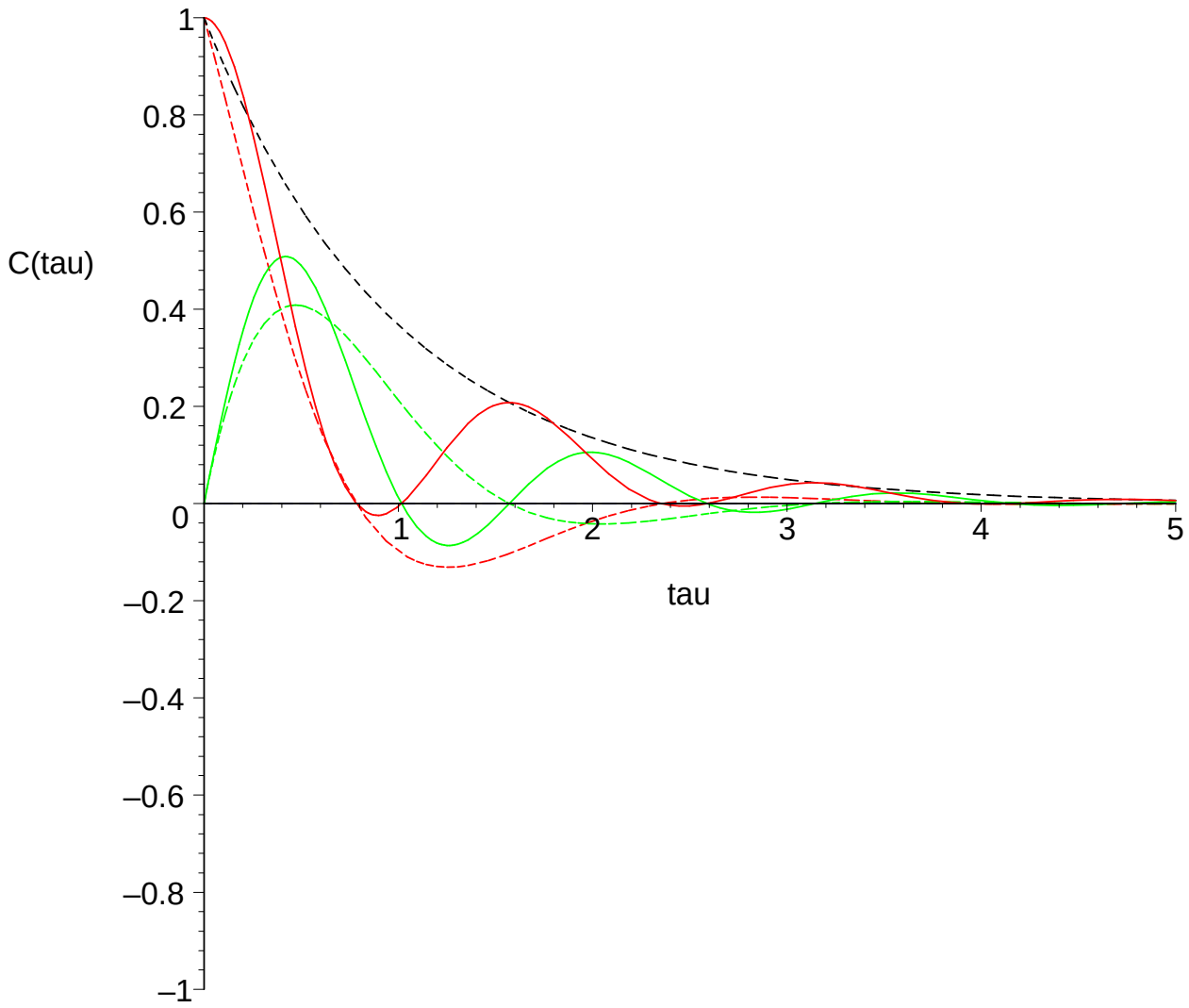

```
> nu:=1 ; nu_f := 1+4*I;
> nu_eff := 1/tau_eff(nu,nu_f);
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=black, linestyle=2):
```

```
  p2c := plot(Im(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p2,p1c,p2c,p1b,p2b});
```

$$\nu := 1$$

$$nu\_f := 1 + 4\,I$$

$$nu\_eff := 1.449489743 - 2.000000000\,I$$



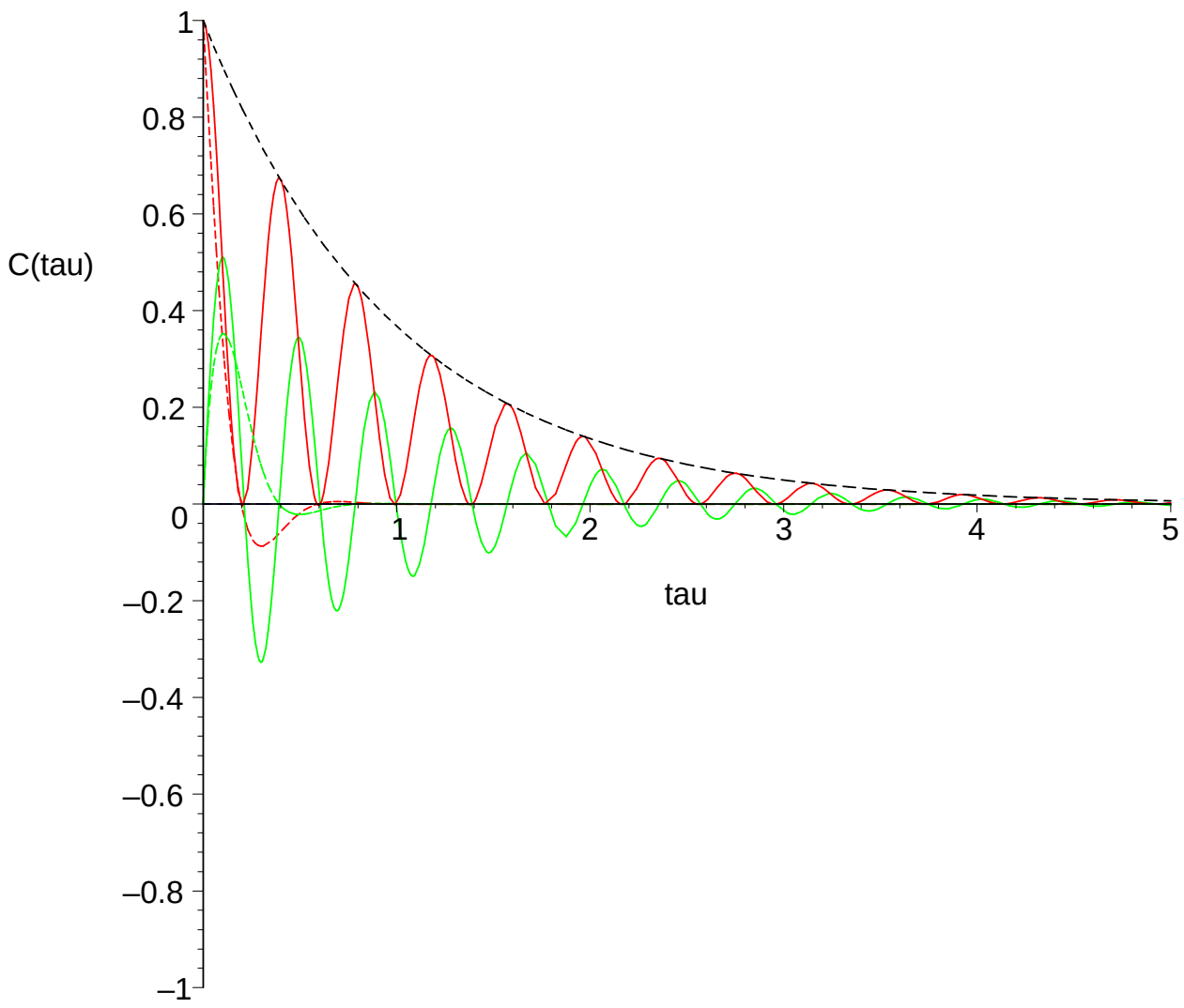

```
> nu:=1 ; nu_f := 1+16*I;
> nu_eff := 1/tau_eff(nu,nu_f);
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
```

```
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=black, linestyle=2):
  p2c := plot(Im(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p2,p1c,p2c,p1b,p2b});
```

$$\nu := 1$$

$$nu\_f := 1 + 16\,I$$

$$nu\_eff := 7.124038406 - 8.000000000\,I$$



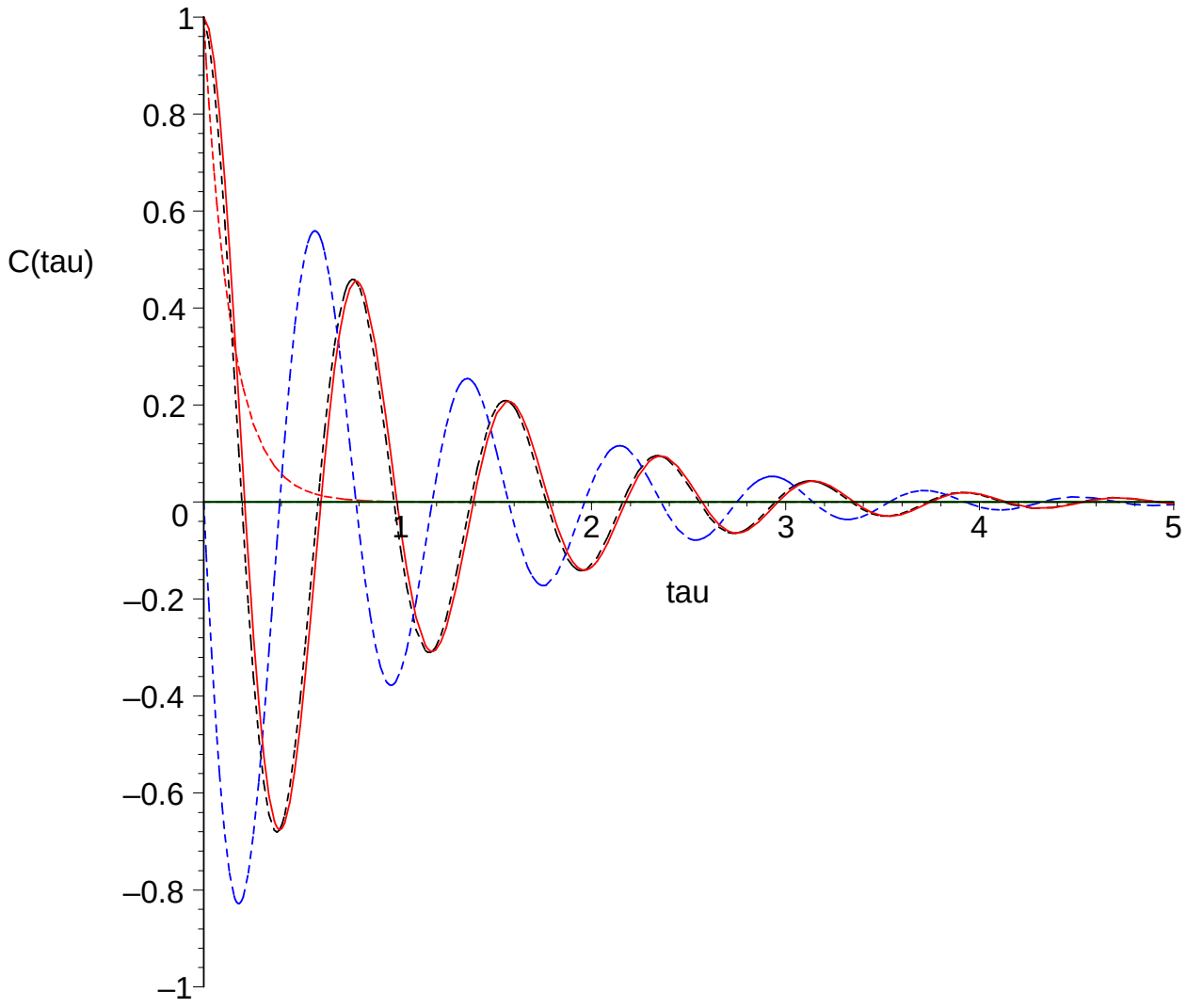

```
> # Look in a frame of motion where nu_eff is real:
  nu:=1+8*I ; nu_f := 1+8*I;
  nu_eff := 1/tau_eff(nu,nu_f);
  p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
```

```
color=black, linestyle=2):
p2c := plot(Im(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
color=blue, linestyle=2):
plots[display]({p1,p2,p1c,p2c,p1b,p2b});
```

$$\nu := 1 + 8\,I$$

$$nu\_f := 1 + 8\,I$$

$$nu\_eff := 7.124038405$$



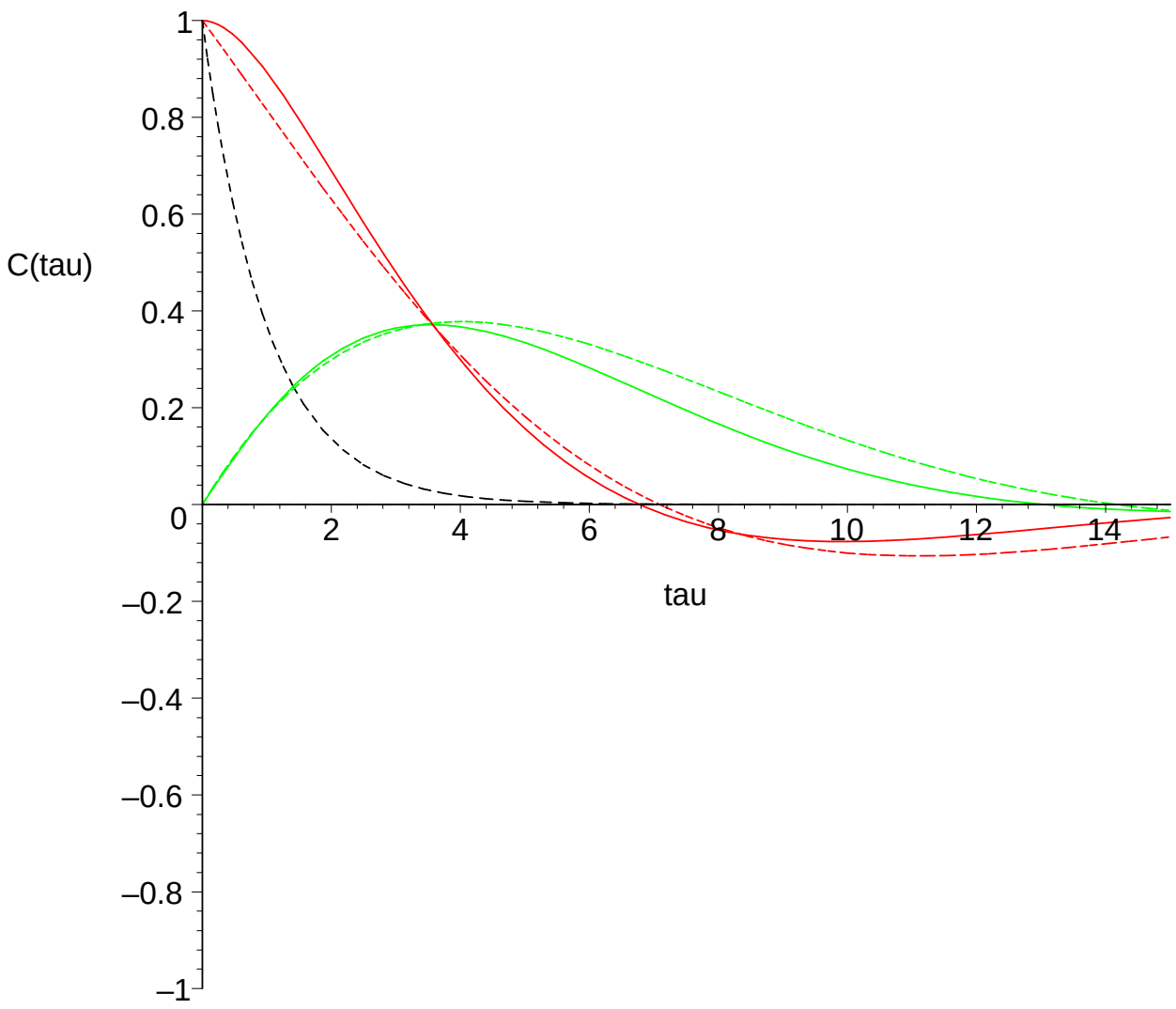

```
> nu:=1 ; nu_f := 0.25+0.25*I;
> nu_eff := 1/tau_eff(nu,nu_f);
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..15, 'C(tau)'=-1..1,
```

```
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..15, 'C(tau)'=-1..1,
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..15, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..15, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..15, 'C(tau)'=-1..1,
  color=black, linestyle=2):
  p2c := plot(Im(exp(-tau*nu)),      tau=0..15, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p2,p1c,p2c,p1b,p2b});
```

$$\nu := 1$$

$$nu\_f := .25 + .25\, I$$

$$nu\_eff := .1793221671 - .2215186601\, I$$



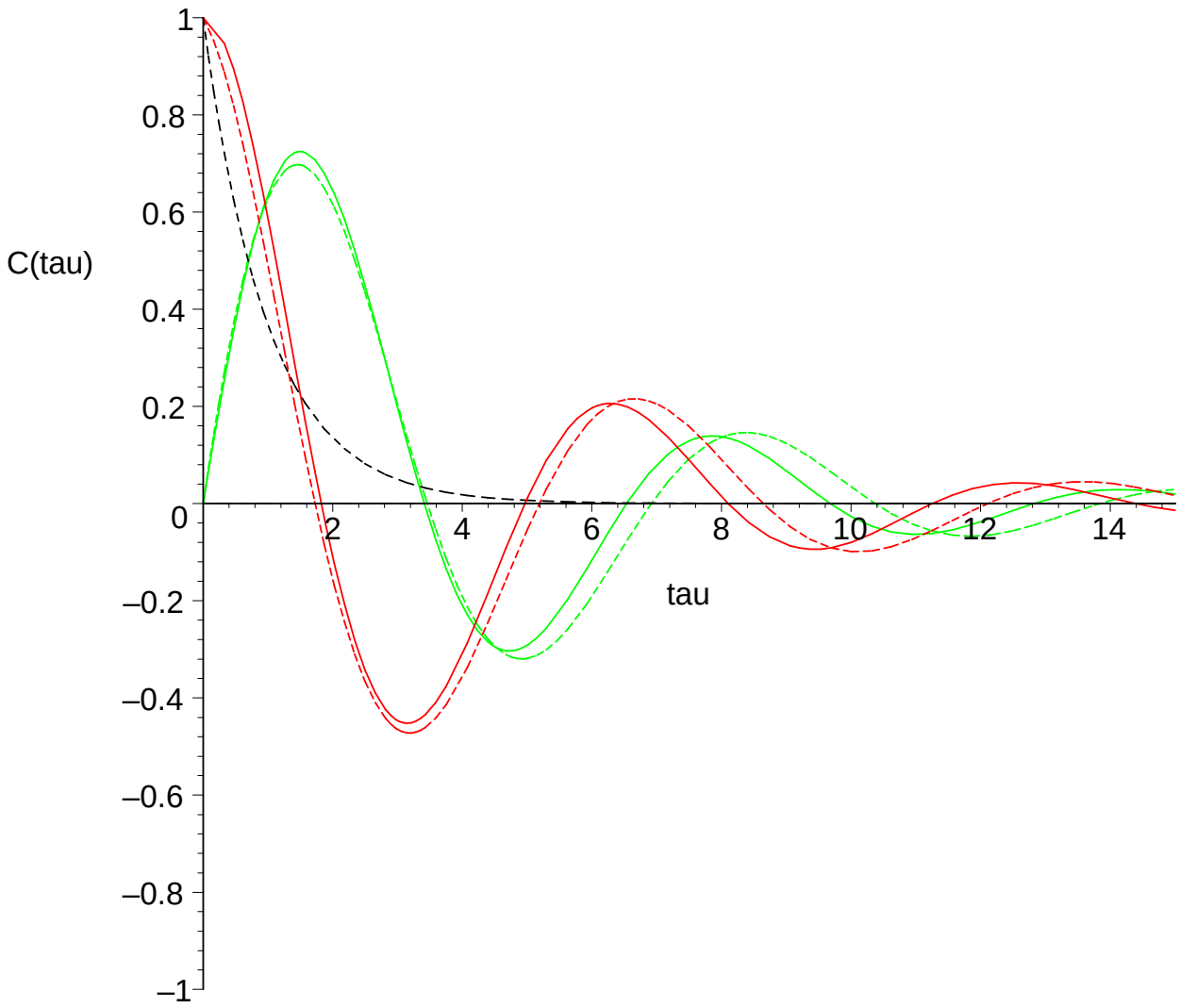

```
> nu:=1 ; nu_f := 0.25+I;
> nu_eff := 1/tau_eff(nu,nu_f);
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..15, ‘C(tau)‘=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..15, ‘C(tau)‘=-1..1,
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..15, ‘C(tau)‘=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..15, ‘C(tau)‘=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..15, ‘C(tau)‘=-1..1,
  color=black, linestyle=2):
```

```
  p2c := plot(Im(exp(-tau*nu)),      tau=0..15, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p2,p1c,p2c,p1b,p2b});
```

$$\nu := 1$$

$$nu\_f := .25 + 1.\,I$$

$$nu\_eff := .2257937283 - .9083809897\,I$$



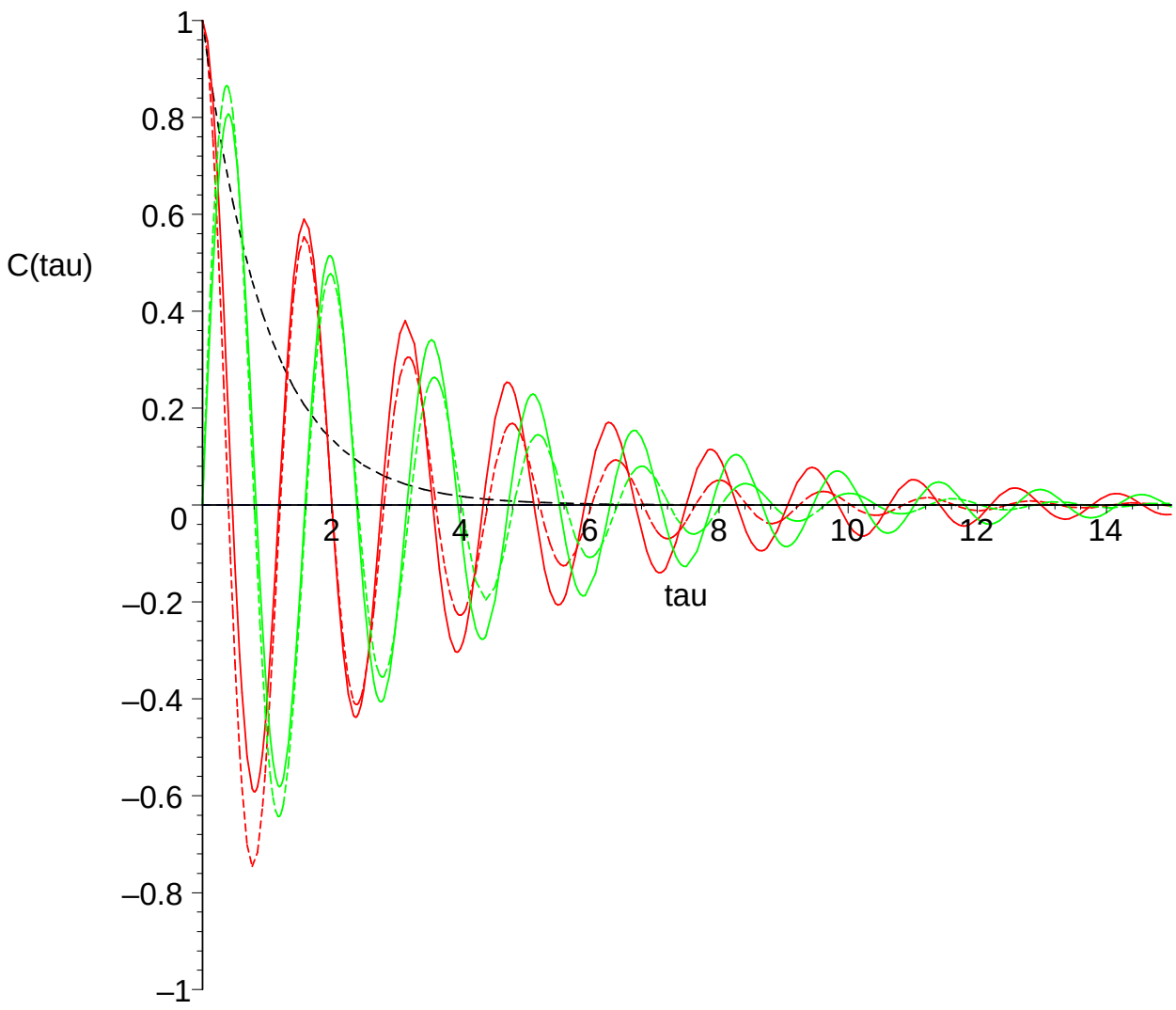

```
> nu:=1 ; nu_f := 0.25+4*I;
> nu_eff := 1/tau_eff(nu,nu_f);
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..15, 'C(tau)'=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..15, 'C(tau)'=-1..1,
```

```
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..15, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..15, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..15, 'C(tau)'=-1..1,
  color=black, linestyle=2):
  p2c := plot(Im(exp(-tau*nu)),      tau=0..15, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p2,p1c,p2c,p1b,p2b});
```

$$\nu := 1$$

$$nu\_f := .25 + 4.\, I$$

$$nu\_eff := .3701042027 - 3.910600069\, I$$



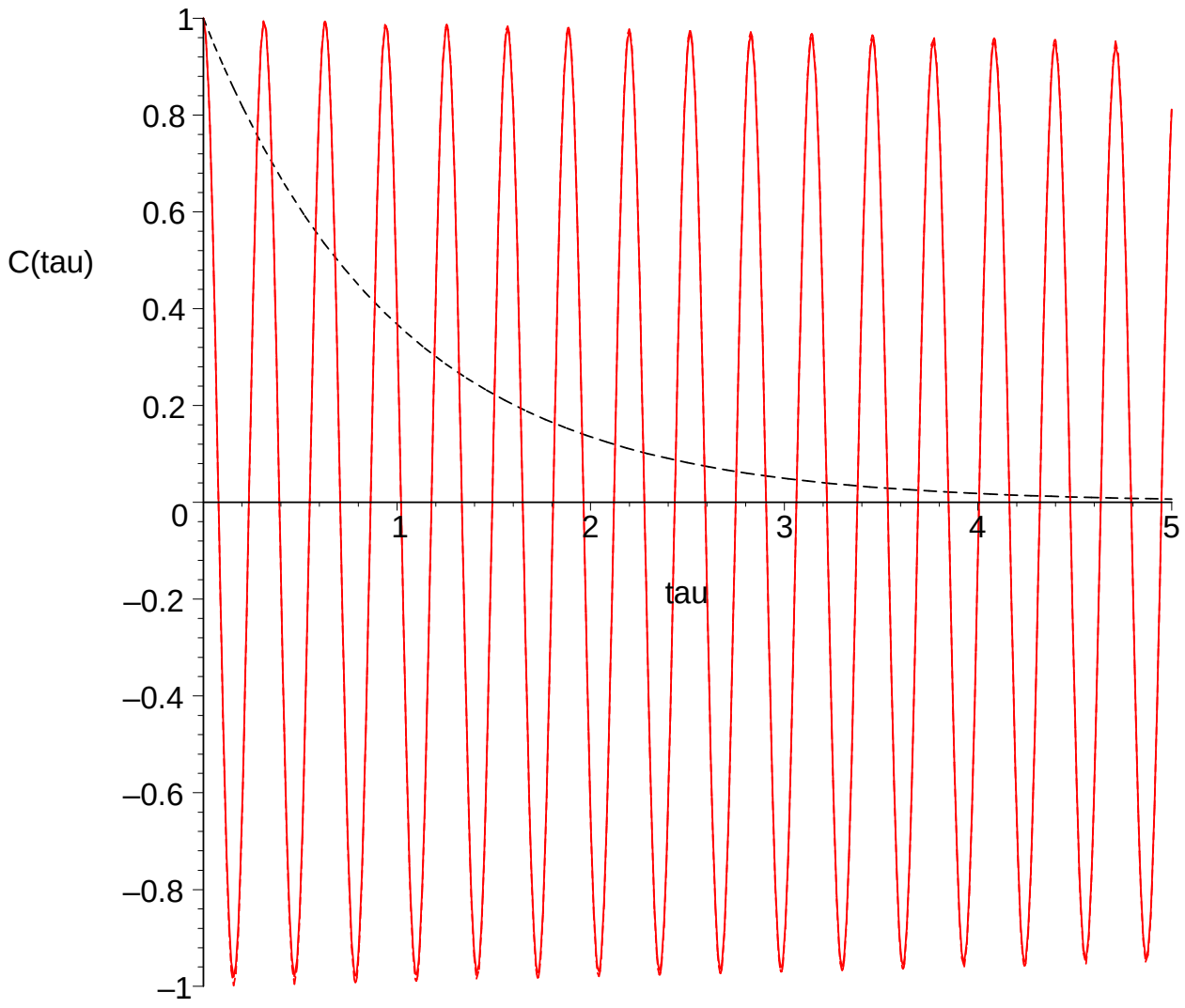

```
> nu:=1+0*I ; nu_f := 0.01000001-20*I ;
> nu_eff := 1/tau_eff(nu,nu_f);
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=black, linestyle=2):
```

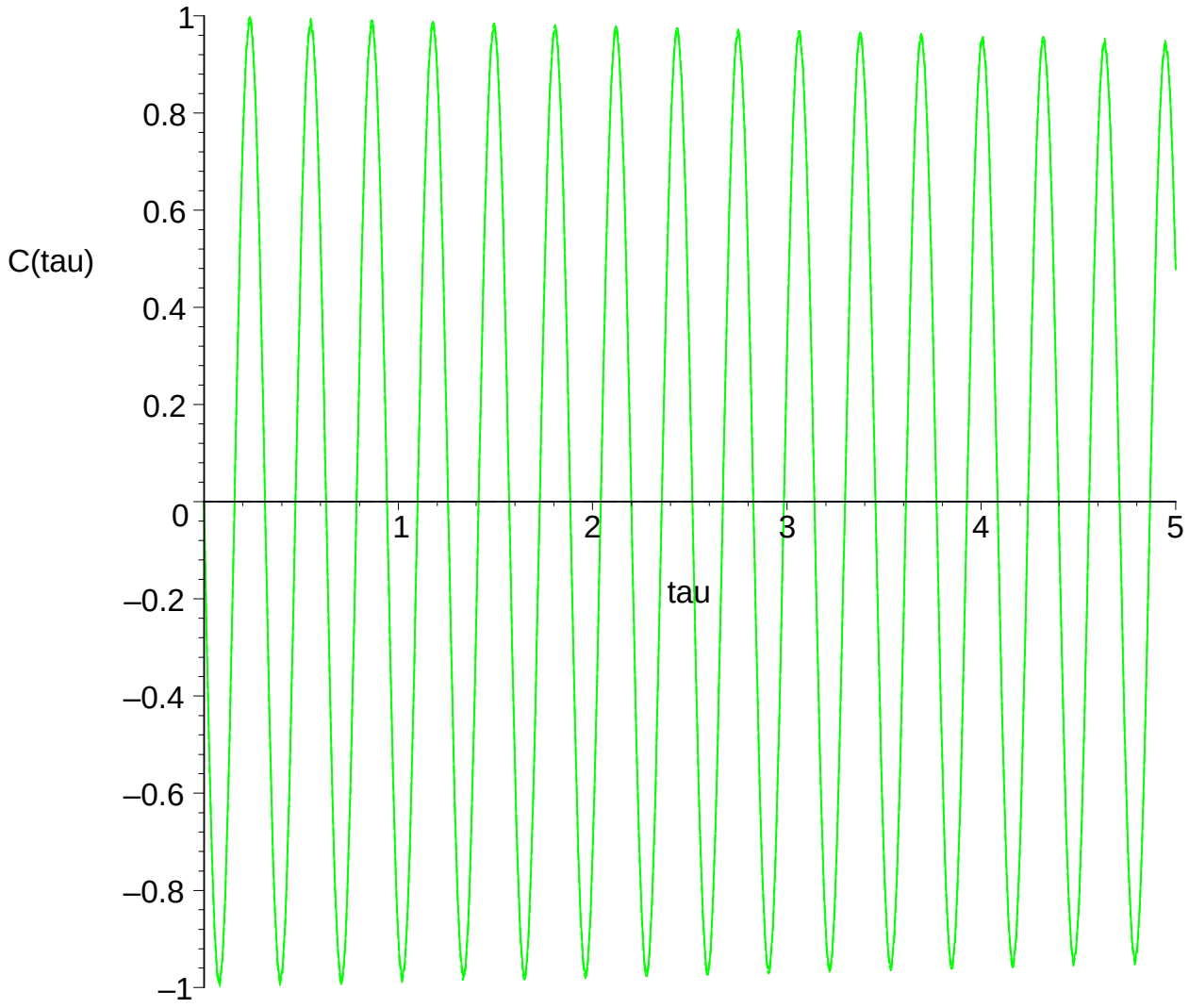
```
  p2c := plot(Im(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p1b, p1c});
  plots[display]({p2,p2b,p2c});
```

$$\nu := 1$$

$$nu\_f := .01000001 - 20.\, I$$

$$nu\_eff := .01020098180 + 19.99997965\, I$$





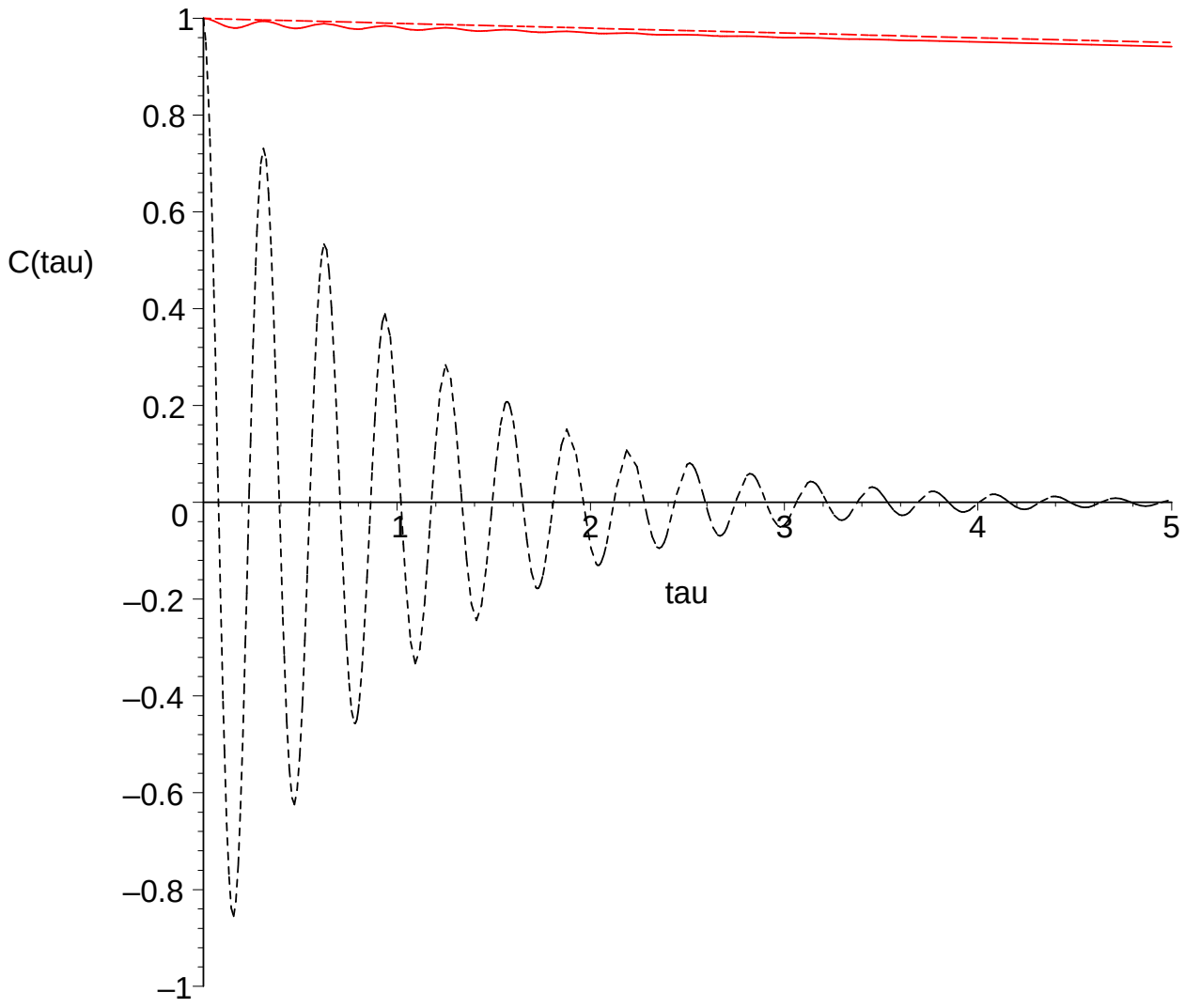

```
> nu:=1+20*I ; nu_f := 0.01000001-0*I ;
> nu_eff := 1/tau_eff(nu,nu_f);
> p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..5, ‘C(tau)‘=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..5, ‘C(tau)‘=-1..1,
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..5, ‘C(tau)‘=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..5, ‘C(tau)‘=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..5, ‘C(tau)‘=-1..1,
  color=black, linestyle=2):
```

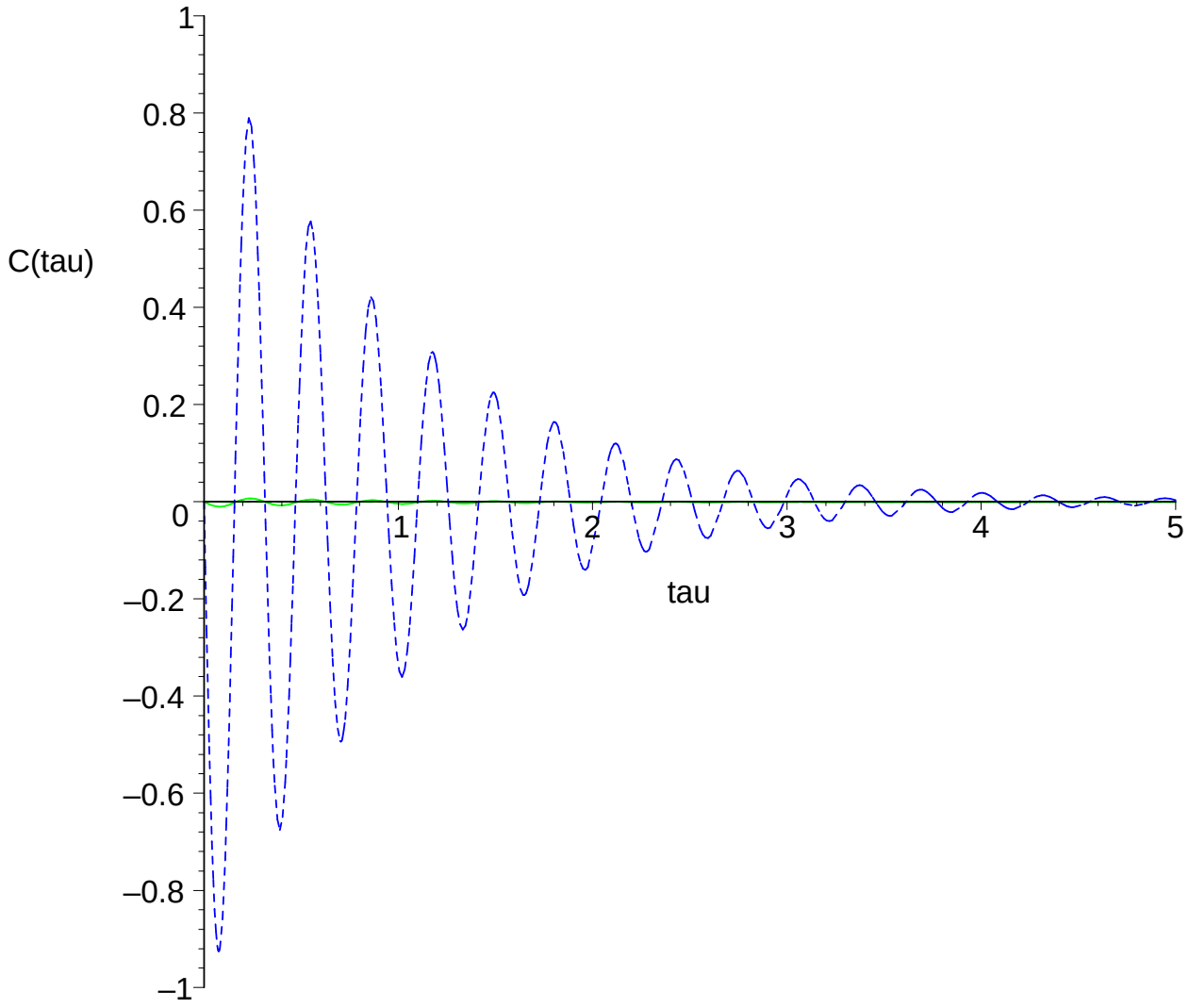
```
  p2c := plot(Im(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=blue, linestyle=2):
> plots[display]({p1,p1b, p1c});
  plots[display]({p2,p2b,p2c});
```

$$\nu := 1 + 20\,I$$

$$nu\_f := .01000001$$

$$nu\_eff := .01020098180 + .00002035721948\,I$$





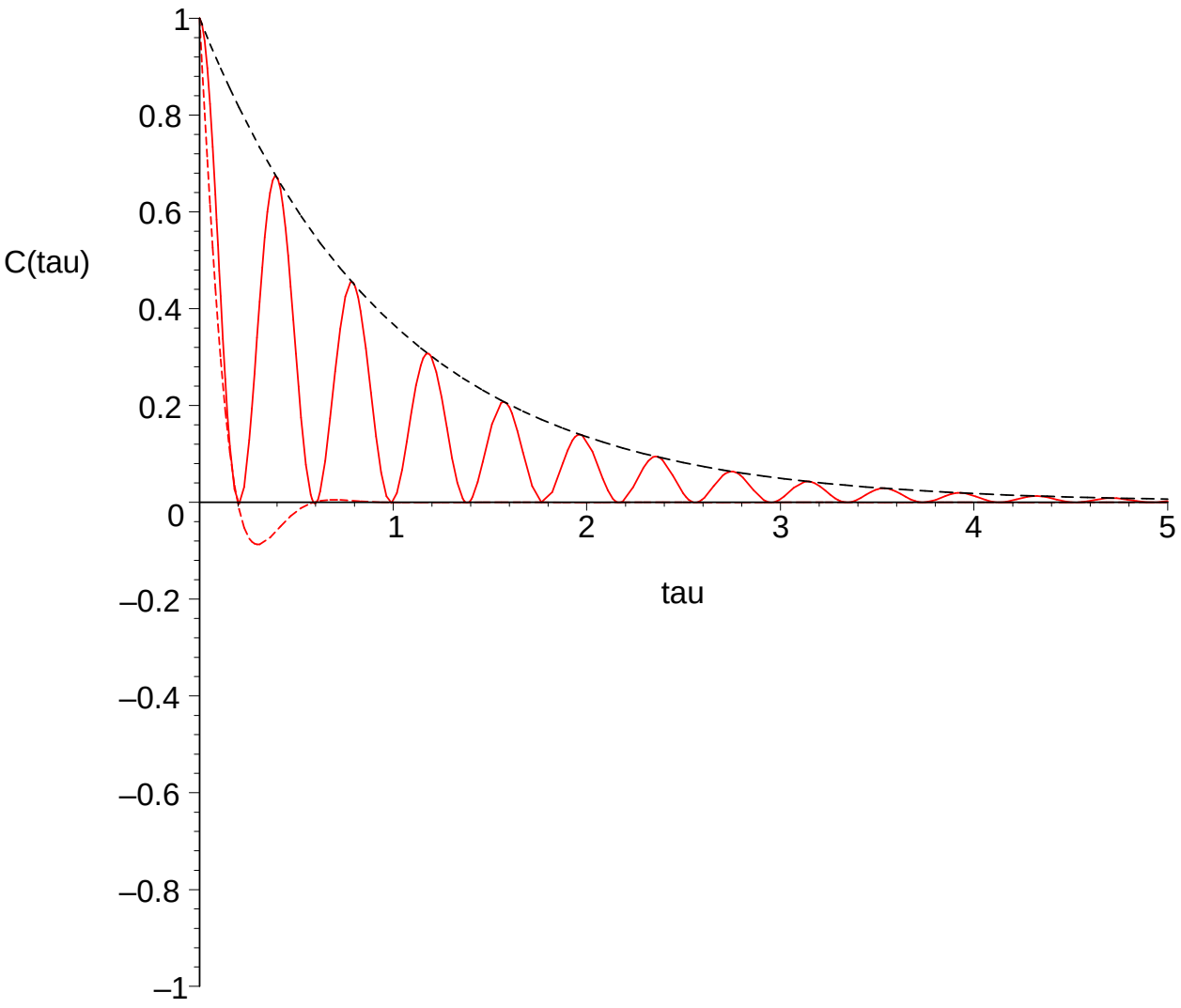

```
> nu:=1 ; nu_f := 1+16*I ;nu_eff := 1/tau_eff(nu,nu_f);
  p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..5, ‘C(tau)‘=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..5, ‘C(tau)‘=-1..1,
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..5, ‘C(tau)‘=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..5, ‘C(tau)‘=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..5, ‘C(tau)‘=-1..1,
  color=black, linestyle=2):
  p2c := plot(Im(exp(-tau*nu)),      tau=0..5, ‘C(tau)‘=-1..1,
```

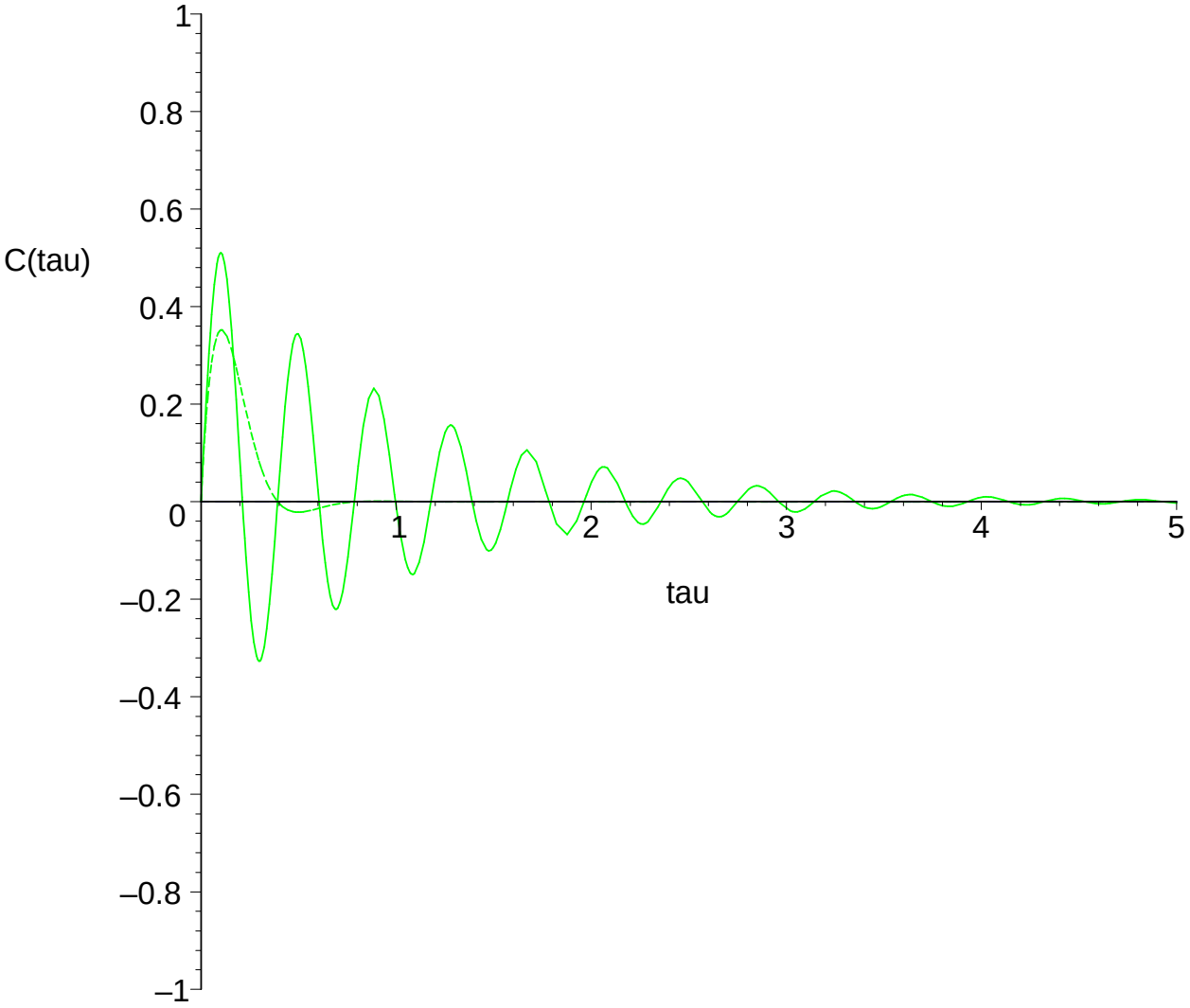
```
color=blue, linestyle=2):
plots[display]({p1,p1b, p1c});
plots[display]({p2,p2b,p2c});
```

$$\nu := 1$$

$$nu\_f := 1 + 16\,I$$

$$nu\_eff := 7.124038406 - 8.000000000\,I$$





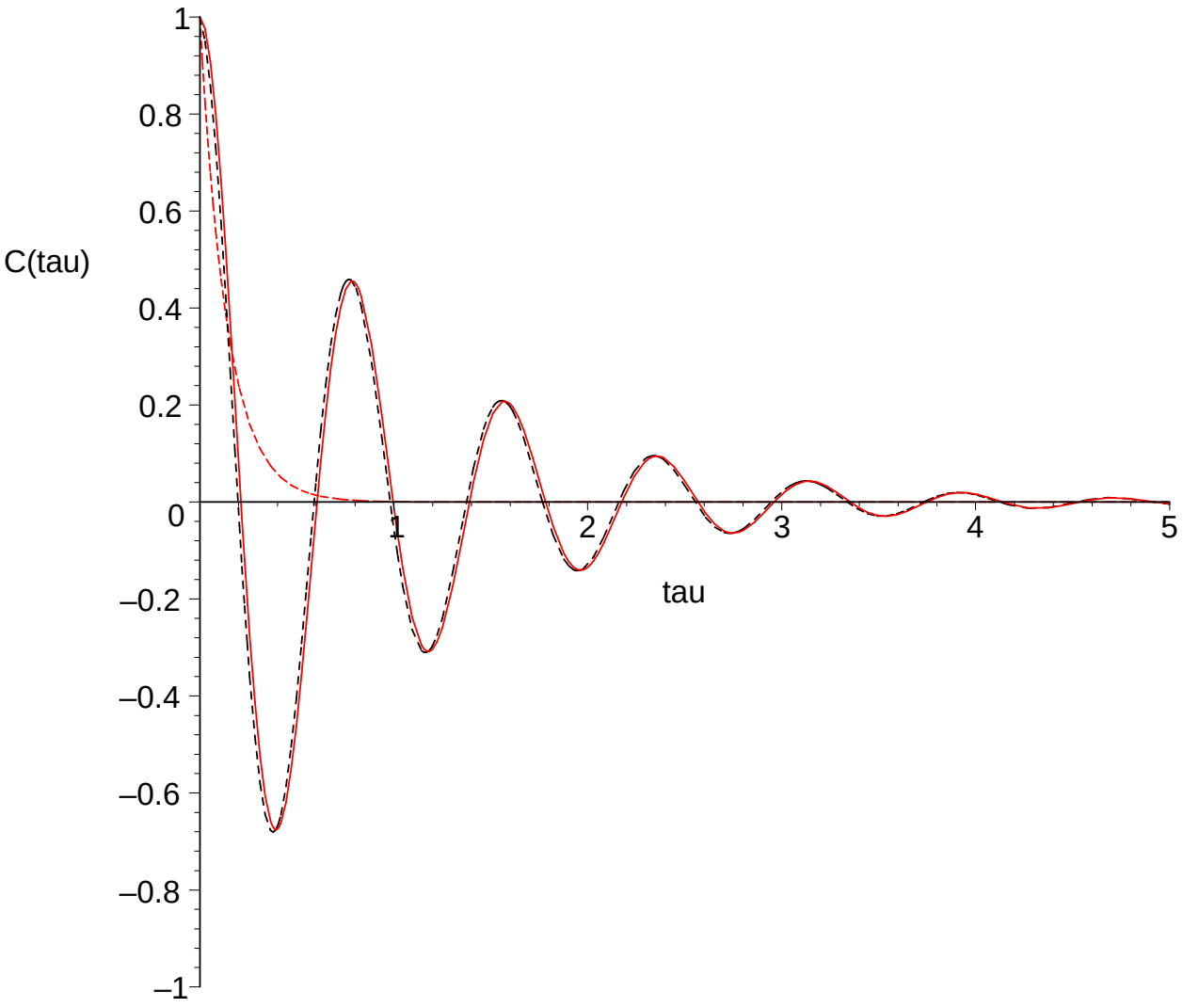

```
> # Redo in frame where wave is stationary (nu_eff is real):
  nu:=1+8*I ; nu_f := 1+8*I ;nu_eff := 1/tau_eff(nu,nu_f);
  p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=red):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..5, 'C(tau)'=-1..1,
  color=green):
  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=red, linestyle=3):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..5, 'C(tau)'=-1..1,
  color=green, linestyle=3):
  p1c := plot(Re(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
  color=black, linestyle=2):
```

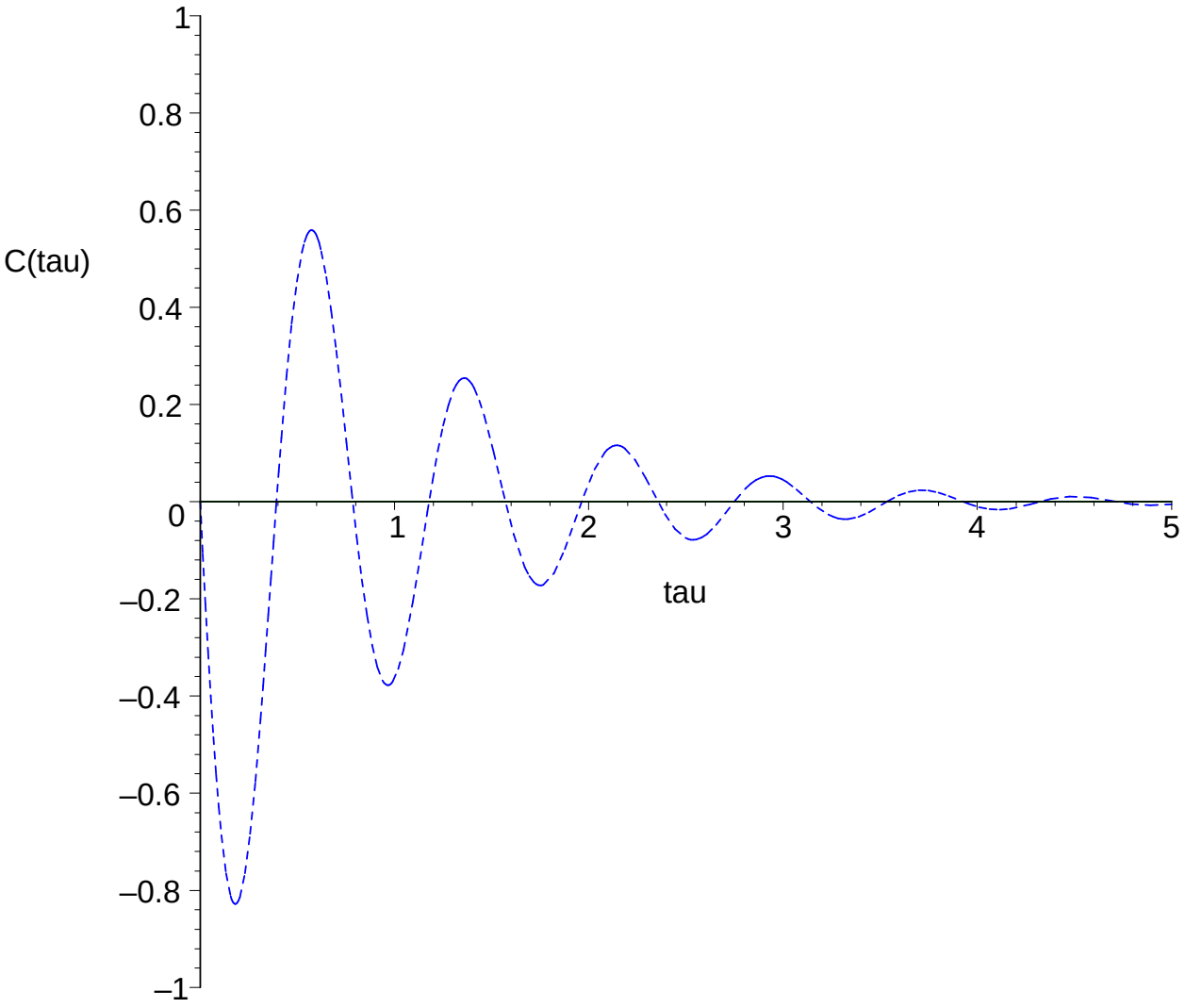
```
p2c := plot(Im(exp(-tau*nu)),      tau=0..5, 'C(tau)'=-1..1,
color=blue, linestyle=2):
plots[display]({p1,p1b, p1c});
plots[display]({p2,p2b,p2c});
```

$$\nu := 1 + 8\,I$$

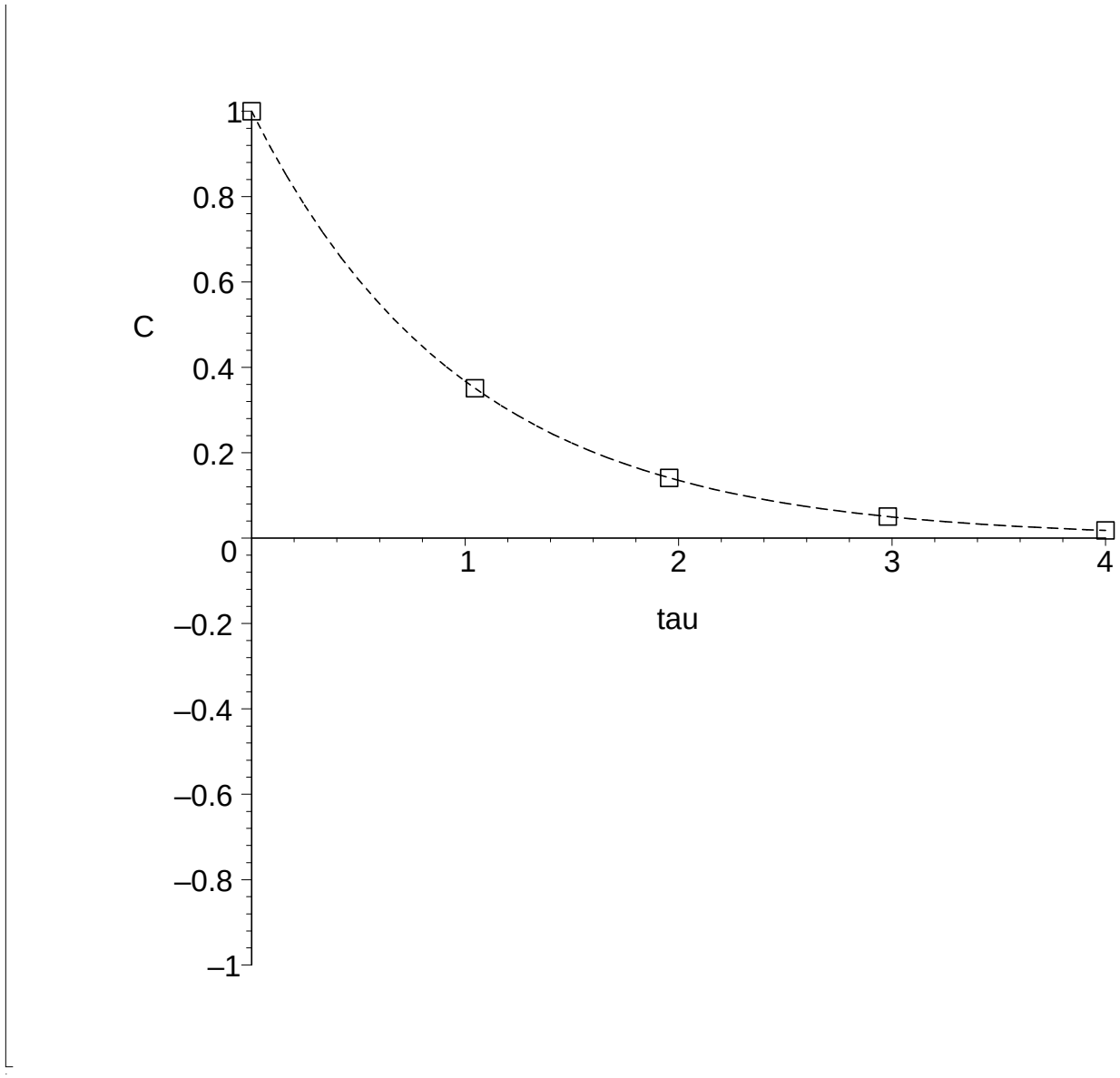
$$nu\_f := 1 + 8\,I$$

$$nu\_eff := 7.124038405$$






```
> # Explore a few plot options:
  plotsetup(default);
  p1 := plot(exp(-tau),tau=0..4, 'C'=-1..1, thickness=2,
  color=black, adaptive=false, numpoints=5, style=point, symbol=box,
  symbolsize=20):
  p2 := plot(exp(-tau),tau=0..4, 'C'=-1..1, thickness=0,
  linestyle=2, color=black):
  plots[display]({p1,p2});
```


C
tau
1
0.8
0.6
0.4
0.2
0
–0.2
–0.4
–0.6
–0.8
–1
1
2
3
4

This final section generates all of the plots (in encapsulated postscript files) that appear in the paper.

```
> mkrplot := proc(nu, nu_f, nu_f_lab, tmax, pfile)
    local p1, p2, p1b, p2b, p1c, p2c, p1cp, p2cp, pr_label,
  pi_label, nu_eff;

  description "Make a Re plot for a single set of parameters";

  nu_eff := 1/tau_eff(nu,nu_f);

  p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..tmax, ‘C‘=-1..1,
  labels=[‘‘,‘‘], thickness=4, color=‘black‘):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..tmax, ‘C‘=-1..1,
  labels=[‘‘,‘‘], thickness=4, color=‘black‘):

  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..tmax, ‘C‘=-1..1,
  labels=[‘‘,‘‘], linestyle=4, thickness=3, color=‘black‘):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..tmax, ‘C‘=-1..1,
  labels=[‘‘,‘‘], linestyle=4, thickness=3, color=‘black‘):

  p1c := plot(Re(exp(-tau*nu)),      tau=0..tmax, ‘C‘=-1..1,
  labels=[‘‘,‘‘], linestyle=3, thickness=2, color=‘black‘):
  p2c := plot(Im(exp(-tau*nu)),      tau=0..tmax, ‘C‘=-1..1,
  labels=[‘‘,‘‘], linestyle=3, thickness=2, color=‘black‘):

  p1cp := plot(Re(exp(-tau*nu)),      tau=0..tmax, ‘C‘=-1..1,
  labels=[‘‘,‘‘], numpoints=6, style=point, symbol=box,
  symbolsize=20, adaptive=false, thickness=2, color=‘black‘):
  p2cp := plot(Im(exp(-tau*nu)),      tau=0..tmax, ‘C‘=-1..1,
  labels=[‘‘,‘‘], numpoints=6, style=point, symbol=box,
  symbolsize=20, adaptive=false, thickness=2, color=‘black‘):

  pr_label := PLOT(TEXT([5,-0.2],‘t-t‘‘,FONT(HELVETICA,BOLD,14)),
              TEXT([-0.2,0.9],‘Re C(t,t’)    ‘, ALIGNABOVE,
  ALIGNLEFT, FONT(HELVETICA, BOLD, 14)),
              CURVES([[0.6*tmax,-0.6],[0.8*tmax,-0.6]],THICKNESS(4)),

                     TEXT([0.82*tmax,-0.6],‘exact‘,
  ALIGNRIGHT,FONT(HELVETICA,BOLD,14)),
              CURVES([[0.6*tmax,-0.7],[0.8*tmax,-0.7]],THICKNESS(3),
  LINESTYLE(4)),
                     TEXT([0.82*tmax,-0.7],‘model‘,
  ALIGNRIGHT,FONT(HELVETICA,BOLD,14)),
              CURVES([[0.6*tmax,-0.8],[0.8*tmax,-0.8]],THICKNESS(2),
  LINESTYLE(3)),
              POINTS([0.7*tmax,-0.8], SYMBOL(BOX,20)),
                     TEXT([0.82*tmax,-0.8],‘wn‘,
```

```
  ALIGNRIGHT,FONT(HELVETICA,BOLD,14)),
                    TEXT([0.65*tmax,-0.91],'h', ALIGNRIGHT,
  FONT(SYMBOL,14)),
                    TEXT([0.68*tmax,-0.93],'f', ALIGNRIGHT,
  FONT(HELVETICA,12)),
                    TEXT([0.7*tmax,-0.9],nu_f_lab, ALIGNRIGHT,
  FONT(HELVETICA,14)),
             VIEW(-0.5..tmax,-1..1)):

  pi_label := PLOT(TEXT([5,-0.2],'t-t'',FONT(HELVETICA,BOLD,14)),
             TEXT([-0.2,0.9],'Im C(t,t')    ', ALIGNABOVE,
  ALIGNLEFT, FONT(HELVETICA, BOLD, 14)),
             CURVES([[0.6*tmax,-0.6],[0.8*tmax,-0.6]],THICKNESS(4)),

                    TEXT([0.82*tmax,-0.6],'exact',
  ALIGNRIGHT,FONT(HELVETICA,BOLD,14)),
             CURVES([[0.6*tmax,-0.7],[0.8*tmax,-0.7]],THICKNESS(3),
  LINESTYLE(4)),
                    TEXT([0.82*tmax,-0.7],'model',
  ALIGNRIGHT,FONT(HELVETICA,BOLD,14)),
             CURVES([[0.6*tmax,-0.8],[0.8*tmax,-0.8]],THICKNESS(2),
  LINESTYLE(3)),
             POINTS([0.7*tmax,-0.8], SYMBOL(BOX,20)),
                    TEXT([0.82*tmax,-0.8],'wn',
  ALIGNRIGHT,FONT(HELVETICA,BOLD,14)),
                    TEXT([0.65*tmax,-0.91],'h', ALIGNRIGHT,
  FONT(SYMBOL,14)),
                    TEXT([0.68*tmax,-0.93],'f', ALIGNRIGHT,
  FONT(HELVETICA,12)),
                    TEXT([0.7*tmax,-0.9],nu_f_lab, ALIGNRIGHT,
  FONT(HELVETICA,14)),
             VIEW(-0.5..tmax,-1..1)):

  plotsetup(ps,plotoutput=pfile,plotoptions='portrait,width=5.0in,he
  ight=5.0in,noborder');
  plots[display]({p1, p1b, p1c, p1cp, pr_label});

  end;
```

mkrplot := **proc**(ν, *nu_f*, *nu_f_lab*, *tmax*, *pfile*)

local *p1*, *p2*, *p1b*, *p2b*, *p1c*, *p2c*, *p1cp*, *p2cp*, *pr_label*, *pi_label*, *nu_eff*;

description "Make a Re plot for a single set of parameters";

nu_eff := 1 / tau_eff(ν, *nu_f*);

p1 := plot(ℜ(c2(τ, ν, *nu_f*)), τ = 0 .. *tmax*, *C* = -1 .. 1, *labels* = [‘‘, ‘‘], *thickness* = 4,
color = *black*);

p2 := plot(ℑ(c2(τ, ν, *nu_f*)), τ = 0 .. *tmax*, *C* = -1 .. 1, *labels* = [‘‘, ‘‘], *thickness* = 4,
color = *black*);

```
p1b := plot(ℜ(exp(–τ*nu_eff)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], linestyle = 4,
    thickness = 3, color = black);
p2b := plot(ℑ(exp(–τ*nu_eff)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], linestyle = 4,
    thickness = 3, color = black);
p1c := plot(ℜ(exp(–τ*ν)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], linestyle = 3, thickness = 2,
    color = black);
p2c := plot(ℑ(exp(–τ*ν)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], linestyle = 3, thickness = 2,
    color = black);
p1cp := plot(ℜ(exp(–τ*ν)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], numpoints = 6,
    style = point, symbol = box, symbolsize = 20, adaptive = false, thickness = 2, color = black)
    ;
p2cp := plot(ℑ(exp(–τ*ν)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], numpoints = 6,
    style = point, symbol = box, symbolsize = 20, adaptive = false, thickness = 2, color = black)
    ;
pr_label := PLOT(TEXT([5, -.2], ‘t-t’‘, FONT(HELVETICA, BOLD, 14)), TEXT([-.2, .9],
    ‘Re C(t,t’)    ‘, ALIGNABOVE, ALIGNLEFT, FONT(HELVETICA, BOLD, 14)),
    CURVES([[.6*tmax, -.6], [.8*tmax, -.6]], THICKNESS(4)),
    TEXT([.82*tmax, -.6], exact, ALIGNRIGHT, FONT(HELVETICA, BOLD, 14)),
    CURVES([[.6*tmax, -.7], [.8*tmax, -.7]], THICKNESS(3), LINESTYLE(4)),
    TEXT([.82*tmax, -.7], model, ALIGNRIGHT, FONT(HELVETICA, BOLD, 14)),
    CURVES([[.6*tmax, -.8], [.8*tmax, -.8]], THICKNESS(2), LINESTYLE(3)),
    POINTS([.7*tmax, -.8], SYMBOL(BOX, 20)),
    TEXT([.82*tmax, -.8], wn, ALIGNRIGHT, FONT(HELVETICA, BOLD, 14)),
    TEXT([.65*tmax, -.91], h, ALIGNRIGHT, FONT(SYMBOL, 14)),
    TEXT([.68*tmax, -.93], f, ALIGNRIGHT, FONT(HELVETICA, 12)),
    TEXT([.7*tmax, -.9], nu_f_lab, ALIGNRIGHT, FONT(HELVETICA, 14)),
    VIEW(-.5 .. tmax, -1 .. 1));
pi_label := PLOT(TEXT([5, -.2], ‘t-t’‘, FONT(HELVETICA, BOLD, 14)), TEXT([-.2, .9],
    ‘Im C(t,t’)    ‘, ALIGNABOVE, ALIGNLEFT, FONT(HELVETICA, BOLD, 14)),
    CURVES([[.6*tmax, -.6], [.8*tmax, -.6]], THICKNESS(4)),
    TEXT([.82*tmax, -.6], exact, ALIGNRIGHT, FONT(HELVETICA, BOLD, 14)),
    CURVES([[.6*tmax, -.7], [.8*tmax, -.7]], THICKNESS(3), LINESTYLE(4)),
    TEXT([.82*tmax, -.7], model, ALIGNRIGHT, FONT(HELVETICA, BOLD, 14)),
    CURVES([[.6*tmax, -.8], [.8*tmax, -.8]], THICKNESS(2), LINESTYLE(3)),
    POINTS([.7*tmax, -.8], SYMBOL(BOX, 20)),
    TEXT([.82*tmax, -.8], wn, ALIGNRIGHT, FONT(HELVETICA, BOLD, 14)),
```

```
        TEXT([.65*tmax, -.91], h, ALIGNRIGHT, FONT(SYMBOL, 14)),
        TEXT([.68*tmax, -.93], f, ALIGNRIGHT, FONT(HELVETICA, 12)),
        TEXT([.7*tmax, -.9], nu_f_lab, ALIGNRIGHT, FONT(HELVETICA, 14)),
        VIEW(-.5 .. tmax, -1 .. 1));
    plotsetup(ps, plotoutput = pfile, plotoptions = 'portrait,width=5.0in,height=5.0in,noborder');
    plots[display]({p1b, pr_label, p1cp, p1c, p1})
end proc
> mkiplot := proc(nu, nu_f, nu_f_lab, tmax, pfile) local p1, p2,
  p1b, p2b, p1c, p2c, p1cp, p2cp, pr_label, pi_label, nu_eff;

  description "Make an Im plot for a single set of parameters";

  # mkiplot is identical to mkrplot, except for the last line,
  # because I can't figure out how to get Maple to create anything
  except
  # the last plot right now.

  nu_eff := 1/tau_eff(nu,nu_f);

  p1  := plot(Re(c2(tau,nu,nu_f)),   tau=0..tmax, 'C'=-1..1,
  labels=['','',], thickness=4, color='black'):
  p2  := plot(Im(c2(tau,nu,nu_f)),   tau=0..tmax, 'C'=-1..1,
  labels=['','',], thickness=4, color='black'):

  p1b := plot(Re(exp(-tau*nu_eff)),  tau=0..tmax, 'C'=-1..1,
  labels=['','',], linestyle=4, thickness=3, color='black'):
  p2b := plot(Im(exp(-tau*nu_eff)),  tau=0..tmax, 'C'=-1..1,
  labels=['','',], linestyle=4, thickness=3, color='black'):

  p1c := plot(Re(exp(-tau*nu)),      tau=0..tmax, 'C'=-1..1,
  labels=['','',], linestyle=3, thickness=2, color='black'):
  p2c := plot(Im(exp(-tau*nu)),      tau=0..tmax, 'C'=-1..1,
  labels=['','',], linestyle=3, thickness=2, color='black'):

  p1cp := plot(Re(exp(-tau*nu)),      tau=0..tmax, 'C'=-1..1,
  labels=['','',], numpoints=6, style=point, symbol=box,
  symbolsize=20, adaptive=false, thickness=2, color='black'):
  p2cp := plot(Im(exp(-tau*nu)),      tau=0..tmax, 'C'=-1..1,
  labels=['','',], numpoints=6, style=point, symbol=box,
  symbolsize=20, adaptive=false, thickness=2, color='black'):

  pr_label := PLOT(TEXT([5,-0.2],'t-t'',FONT(HELVETICA,BOLD,14)),
              TEXT([-0.2,0.9],'Re C(t,t')    ', ALIGNABOVE,
  ALIGNLEFT, FONT(HELVETICA, BOLD, 14)),
              CURVES([[0.6*tmax,-0.6],[0.8*tmax,-0.6]],THICKNESS(4)),

                    TEXT([0.82*tmax,-0.6],'exact',
```

```
  ALIGNRIGHT,FONT(HELVETICA,BOLD,14)),
             CURVES([[0.6*tmax,-0.7],[0.8*tmax,-0.7]],THICKNESS(3),
  LINESTYLE(4)),
                    TEXT([0.82*tmax,-0.7],‘model‘,
  ALIGNRIGHT,FONT(HELVETICA,BOLD,14)),
             CURVES([[0.6*tmax,-0.8],[0.8*tmax,-0.8]],THICKNESS(2),
  LINESTYLE(3)),
             POINTS([0.7*tmax,-0.8], SYMBOL(BOX,20)),
                    TEXT([0.82*tmax,-0.8],‘wn‘,
  ALIGNRIGHT,FONT(HELVETICA,BOLD,14)),
                    TEXT([0.65*tmax,-0.91],‘h‘, ALIGNRIGHT,
  FONT(SYMBOL,14)),
                    TEXT([0.68*tmax,-0.93],‘f‘, ALIGNRIGHT,
  FONT(HELVETICA,12)),
                    TEXT([0.7*tmax,-0.9],nu_f_lab, ALIGNRIGHT,
  FONT(HELVETICA,14)),
             VIEW(-0.5..tmax,-1..1)):

  pi_label := PLOT(TEXT([5,-0.2],‘t-t’‘,FONT(HELVETICA,BOLD,14)),
             TEXT([-0.2,0.9],‘Im C(t,t’)    ‘, ALIGNABOVE,
  ALIGNLEFT, FONT(HELVETICA, BOLD, 14)),
             CURVES([[0.6*tmax,-0.6],[0.8*tmax,-0.6]],THICKNESS(4)),

                    TEXT([0.82*tmax,-0.6],‘exact‘,
  ALIGNRIGHT,FONT(HELVETICA,BOLD,14)),
             CURVES([[0.6*tmax,-0.7],[0.8*tmax,-0.7]],THICKNESS(3),
  LINESTYLE(4)),
                    TEXT([0.82*tmax,-0.7],‘model‘,
  ALIGNRIGHT,FONT(HELVETICA,BOLD,14)),
             CURVES([[0.6*tmax,-0.8],[0.8*tmax,-0.8]],THICKNESS(2),
  LINESTYLE(3)),
             POINTS([0.7*tmax,-0.8], SYMBOL(BOX,20)),
                    TEXT([0.82*tmax,-0.8],‘wn‘,
  ALIGNRIGHT,FONT(HELVETICA,BOLD,14)),
                    TEXT([0.65*tmax,-0.91],‘h‘, ALIGNRIGHT,
  FONT(SYMBOL,14)),
                    TEXT([0.68*tmax,-0.93],‘f‘, ALIGNRIGHT,
  FONT(HELVETICA,12)),
                    TEXT([0.7*tmax,-0.9],nu_f_lab, ALIGNRIGHT,
  FONT(HELVETICA,14)),
             VIEW(-0.5..tmax,-1..1)):

  plotsetup(ps,plotoutput=pfile,plotoptions=‘portrait,width=5.0in,he
  ight=5.0in,noborder‘);
  plots[display]({p2, p2b, p2c, p2cp, pi_label});

  end;
```

mkiplot := **proc**(ν, *nu_f, nu_f_lab, tmax, pfile*)

local *p1, p2, p1b, p2b, p1c, p2c, p1cp, p2cp, pr_label, pi_label, nu_eff*;

```
description "Make an Im plot for a single set of parameters";
    nu_eff := 1 / tau_eff(ν, nu_f);
    p1 := plot(ℜ(c2(τ, ν, nu_f)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], thickness = 4,
        color = black);
    p2 := plot(ℑ(c2(τ, ν, nu_f)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], thickness = 4,
        color = black);
    p1b := plot(ℜ(exp(–τ*nu_eff)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], linestyle = 4,
        thickness = 3, color = black);
    p2b := plot(ℑ(exp(–τ*nu_eff)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], linestyle = 4,
        thickness = 3, color = black);
    p1c := plot(ℜ(exp(–τ*ν)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], linestyle = 3, thickness = 2,
        color = black);
    p2c := plot(ℑ(exp(–τ*ν)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], linestyle = 3, thickness = 2,
        color = black);
    p1cp := plot(ℜ(exp(–τ*ν)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], numpoints = 6,
        style = point, symbol = box, symbolsize = 20, adaptive = false, thickness = 2, color = black)
        ;
    p2cp := plot(ℑ(exp(–τ*ν)), τ = 0 .. tmax, C = -1 .. 1, labels = [‘‘, ‘‘], numpoints = 6,
        style = point, symbol = box, symbolsize = 20, adaptive = false, thickness = 2, color = black)
        ;
    pr_label := PLOT(TEXT([5, -.2], ‘t-t’‘, FONT(HELVETICA, BOLD, 14)), TEXT([-.2, .9],
        ‘Re C(t,t’)    ‘, ALIGNABOVE, ALIGNLEFT, FONT(HELVETICA, BOLD, 14)),
        CURVES([[.6*tmax, -.6], [.8*tmax, -.6]], THICKNESS(4)),
        TEXT([.82*tmax, -.6], exact, ALIGNRIGHT, FONT(HELVETICA, BOLD, 14)),
        CURVES([[.6*tmax, -.7], [.8*tmax, -.7]], THICKNESS(3), LINESTYLE(4)),
        TEXT([.82*tmax, -.7], model, ALIGNRIGHT, FONT(HELVETICA, BOLD, 14)),
        CURVES([[.6*tmax, -.8], [.8*tmax, -.8]], THICKNESS(2), LINESTYLE(3)),
        POINTS([.7*tmax, -.8], SYMBOL(BOX, 20)),
        TEXT([.82*tmax, -.8], wn, ALIGNRIGHT, FONT(HELVETICA, BOLD, 14)),
        TEXT([.65*tmax, -.91], h, ALIGNRIGHT, FONT(SYMBOL, 14)),
        TEXT([.68*tmax, -.93], f, ALIGNRIGHT, FONT(HELVETICA, 12)),
        TEXT([.7*tmax, -.9], nu_f_lab, ALIGNRIGHT, FONT(HELVETICA, 14)),
        VIEW(-.5 .. tmax, -1 .. 1));
    pi_label := PLOT(TEXT([5, -.2], ‘t-t’‘, FONT(HELVETICA, BOLD, 14)), TEXT([-.2, .9],
        ‘Im C(t,t’)    ‘, ALIGNABOVE, ALIGNLEFT, FONT(HELVETICA, BOLD, 14)),
        CURVES([[.6*tmax, -.6], [.8*tmax, -.6]], THICKNESS(4)),
```

```
        TEXT([.82*tmax, -.6], exact, ALIGNRIGHT, FONT(HELVETICA, BOLD, 14)),
        CURVES([[.6*tmax, -.7], [.8*tmax, -.7]], THICKNESS(3), LINESTYLE(4)),
        TEXT([.82*tmax, -.7], model, ALIGNRIGHT, FONT(HELVETICA, BOLD, 14)),
        CURVES([[.6*tmax, -.8], [.8*tmax, -.8]], THICKNESS(2), LINESTYLE(3)),
        POINTS([.7*tmax, -.8], SYMBOL(BOX, 20)),
        TEXT([.82*tmax, -.8], wn, ALIGNRIGHT, FONT(HELVETICA, BOLD, 14)),
        TEXT([.65*tmax, -.91], h, ALIGNRIGHT, FONT(SYMBOL, 14)),
        TEXT([.68*tmax, -.93], f, ALIGNRIGHT, FONT(HELVETICA, 12)),
        TEXT([.7*tmax, -.9], nu_f_lab, ALIGNRIGHT, FONT(HELVETICA, 14)),
        VIEW(-.5 .. tmax, -1 .. 1));
    plotsetup(ps, plotoutput = pfile, plotoptions = 'portrait,width=5.0in,height=5.0in,noborder');
    plots[display]({p2, pi_label, p2cp, p2b, p2c})
end proc
> mkrplot(1+0*I, 4.0-0*I, "= 4", 5, "cnr4000.ps");
  mkiplot(1+0*I, 4.0-0*I, "= 4", 5, "cni4000.ps");

> # Avoid a slight problem exactly at nu_f=1:
  mkrplot(1+0*I, 1.0000001-0*I, "= 1", 5, "cnr1000.ps");
  mkiplot(1+0*I, 1.0000001-0*I, "= 1", 5, "cni1000.ps");

  mkrplot(1+0*I, 0.25-0*I, "= 0.25", 15, "cnr0200.ps");
  mkiplot(1+0*I, 0.25-0*I, "= 0.25", 15, "cni0200.ps");

> mkrplot(1+0*I, 0.25-I*4, "= 0.25 - 4 i", 5, "cnr0204.ps");
  mkiplot(1+0*I, 0.25-I*4, "= 0.25 - 4 i", 5, "cni0204.ps");

  mkrplot(1+0*I, 1.0-I, "= 1 - i", 5, "cnr1001.ps");
  mkiplot(1+0*I, 1.0-I, "= 1 - i", 5, "cni1001.ps");

  mkrplot(1+0*I, 1.0-I*4, "= 1 - 4 i", 5, "cnr1004.ps");
  mkiplot(1+0*I, 1.0-I*4, "= 1 - 4 i", 5, "cni1004.ps");

  mkrplot(1+0*I, 1.0-I*16, "= 1 - 16 i", 5, "cnr1016.ps");
  mkiplot(1+0*I, 1.0-I*16, "= 1 - 16 i", 5, "cni1016.ps");

> mkrplot(1+0*I, 4.0-I*16, "= 4 - 16 i", 5, "cnr4016.ps");
  mkiplot(1+0*I, 4.0-I*16, "= 4 - 16 i", 5, "cni4016.ps");

>
> nu_effd(1.0,1.0);
                              .4142135625 - 0. I
> tau_eff(1.0,1.0);
                              2.414213562 + 0. I
>
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