Proving the ~~F~~ISI algorithm is symplectic for all iterations

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With Maple Script, we show the argument of the

$\sqrt{}$ in the eigenvalue is!

$$\mathrm{arg} = (4 r L \omega^2 \Delta t^2 - r \omega^4 \Delta t^4 + 16 + 16L)(4rL - r\omega^2 \Delta t^2 - 4 - 4L)$$

$$= \mathrm{arg}_1 \, \mathrm{arg}_2$$

If we can show $\mathrm{arg}_1 > 0$ & $\mathrm{arg}_2 > 0$, then $\mathrm{arg} < 0$,

& the eigenvalue is of the form

$$\lambda = c_1 \pm i\sqrt{|\mathrm{arg}|}$$

& it turns out $|\lambda|^2 = c_1^2 + |\mathrm{arg}| = 1$

Start with arg_1!

$$\mathrm{arg}_1 = \cancel{4} r \cancel{\tfrac{1}{4}} \hat{\omega}^2 \Delta t^2 \omega^2 \Delta t^2 - r \omega^4 \Delta t^4 + 16 + 4 \hat{\omega}^2 \Delta t^2$$

$r = \lambda_2^p$ & $-1 < \lambda_2 < 1$ for convergence.

$$\lambda_2 = \frac{(\hat{\omega}^2 - \omega^2)\Delta t^2}{4 + \hat{\omega}^2 \Delta t^2}$$

$$-1 < \lambda_2 < 1$$ becomes

$$-1 < \frac{(\hat{\omega}^2 - \omega^2)\Delta t^2}{4 + \hat{\omega}^2 \Delta t^2} < 1$$

$$\underbrace{-(4+\hat{\omega}^2\Delta t^2) < (\hat{\omega}^2-\omega^2)\Delta t^2}_{\text{This condition becomes!}} \quad\quad \underbrace{(\hat{\omega}^2-\omega^2)\Delta t^2 < 4 + \hat{\omega}^2\Delta t^2}_{\text{This condition becomes!}}$$

$$-\omega^2 \Delta t^2 < 4$$

which is always satisfied.

$$0 < 4 + (\hat{\omega}^2 - \omega^2)\Delta t^2 + \hat{\omega}^2 \Delta t^2$$

$$\boxed{0 < P + \hat{\omega}^2 \Delta t^2}$$

Therefore the condition that the 1st iteration be stables

$$\boxed{P = 4 + (\hat{\omega}^2 - \omega^2)\Delta t^2 > 0}$$

also ensures $|\lambda_2| < 1$ & $|r| < 1$

$$\arg_1 = r\hat{\omega}^2\Delta t^2\,\omega^2\Delta t^2 - r\omega^4\Delta t^4 + 16 + 4\hat{\omega}^2\Delta t^2$$

$$= r\omega^2\Delta t^2\left(\hat{\omega}^2\Delta t^2 - \omega^2\Delta t^2\right) + 16 + 4\hat{\omega}^2\Delta t^2$$

$$= r\omega^2\Delta t^2(P-4) + 16 + 4\hat{\omega}^2\Delta t^2$$

$$= r\omega^2\Delta t^2 P + 16 + 4\left(\hat{\omega}^2\Delta t^2 - r\omega^2\Delta t^2\right)$$

$$= r\omega^2\Delta t^2 P + \cancel{16} + 4\left[P - \cancel{4} + \omega^2\Delta t^2 - r\omega^2\Delta t^2\right]$$

$$= r\omega^2\Delta t^2 P + 4P + 4(1-r)\omega^2\Delta t^2$$

for $0<r<1$, this is clear that $\arg_1 > 0$.
$(+\ P>0)$

For $-1<r<0$, we have:

$$\arg_1 = -|r|\,\omega^2\Delta t^2 P + 4P + 4(1+|r|)\,\omega^2\Delta t^2$$

However note that for $r<0$, $\lambda_2 < 0$ there is a limit to how large P can be...

$$\cancel{\lambda_2 = (\hat{\omega}^2 - \omega^2)\Delta t^2}$$

$$\lambda_2 = \frac{(\hat{\omega}^2 - \omega^2)\Delta t^2}{4 + \hat{\omega}^2 \Delta t^2} = \frac{P - 4}{P + \omega^2 \Delta t^2}$$

So if $\lambda_2 < 0$, the maximum value of P is 4!

This upper bound on P can be used to provide a lower bound on arg_1 when $r < 0$:

~~lower bound~~ $arg_1 \geq -|r|\,\omega^2 \Delta t^2 \cancel{4} + 4P + 4(\cancel{1}+|r|)\,\omega^2 \Delta t^2$

$$\boxed{arg_1 > 4P + 4|r|\,\omega^2 \Delta t^2}$$

So we have shown $\boxed{P > 0}$ is a sufficient condition for $arg_1 > 1$ no matter what the sign of r

Now have to prove ~~$arg_2 \le 0$ always.~~
sufficient conditions for which $arg_2 < 0$.

$$arg_2 = 4rL - r\omega^2\Delta t^2 - 4 - 4L$$

$$= r\hat{\omega}^2\Delta t^2 - r\omega^2\Delta t^2 - 4 - \hat{\omega}^2\Delta t^2$$

$$= r(P-4) - 4 - \hat{\omega}^2\Delta t^2$$

$$= r(P-4) \cancel{-4} - [P \cancel{+4} + \omega^2\Delta t^2]$$

$$= r(P-4) - P - \omega^2\Delta t^2$$

$$= -(1-r)P - 4r - \omega^2\Delta t^2$$

Thus, if $0 < r < 1$, $P > 0$, then $arg_2 < 0$.
However, For $-1 < r < 0$, we have:

$$arg_2 = -(1+|r|)P + 4|r| - \omega^2\Delta t^2$$

Since $|r| = |\lambda_2|^p \to 0$ as $p \to \infty$,
if there are enough iterations then $arg_2 < 0$.
But we want a proof for arbitrary p....

write arg_2 as:

$$arg_2 = -p - |r| \left[\cancel{4} + \hat{\omega}^2 \Delta t^2 - \omega^2 \Delta t^2 \right] + \cancel{4|r|} - \omega^2 \Delta t^2$$

$$= -p - |r| \hat{\omega}^2 \Delta t^2 - (1 - |r|)\, \omega^2 \Delta t^2$$

Thus $arg_2 < 0$ as long as $|r| < 1$

Q.E.D.

The condition that the 1st iteration is symplectic, $4 + (\hat{\omega}^2 - \omega^2)\Delta t^2 \geq 0$ is also a sufficient condition that all additional iterations are symplectic.