Simplifying Complexity: Reduced Fluid Models of Low-Collisionality, Long-Mean-Free-Path Systems

(APS-AAPT, Washington, April 20, 1995)

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Thanks to the TFTR team for experimental support, and to Gary Kerbel for turbulence visualizations. Supported by the U.S. Dept. of Energy via: contract no. DE-AC02-76CH03073 et.al., the TFTR Experiment, the Numerical Tokamak Project, and the National Energy Research Supercomputer Center at Lawrence Livermore National Laboratory.

Simplifying Complexity: Reduced Fluid Models of Landau-damping and Phase-mixing

- Paradigm Problem: Failure of Taylor Series Approximations to

$$e^{-1/\epsilon^2}$$

- Importance of Mixing Processes in Many Physical Systems (Fusion Plasma Physics, Space and Astrophysics, Semiconductors and Solid-State Physics, Fluid Dynamics, and other many-body/chaotic systems...)
- Physics of Landau-damping and Phase-mixing
- Derivation of Reduced "Landau-Fluid" Models

  (Successful application to fusion plasma turbulence in the next talk by Dorland.)
- Major Progress in Fusion Energy Research

$\boxed{\text{Essential Concept:}}$

Improved methods of averaging over some dimensions of a problem (simplifying complexity) while modelling the *mixing* introduced by that dimension

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Fluid moment equations provide averages over some dimensions of a problem, thereby ignoring the complexities of fine scale details in those dimensions.

Standard fluid equations are known to be valid only in the high-collisionality limit, and are missing important resonant phenomena such as Landau-damping and phase-mixing. Improved by recent closure approximations which introduce damping $\nu \sim v_t|k|$ (a non-local operator in real space).

Clearest Intro: Hammett, Dorland, Perkins, Physics of Fluids **B4**, 2052 (1992).

Caveats: Hammett, et.al., Plasma Phys. Control. Fusion **35**, 973 (1993).

General Geometry CGL $p_\perp, p_\|$ (useful for space and astrophysics): Ph.D. Theses by Dorland (1994) & Beer (1995).

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Paradigm Problem: Approximating e^{-1/ϵ^2}
(the Universal Language of Mathematics)

Taylor-series expansion:

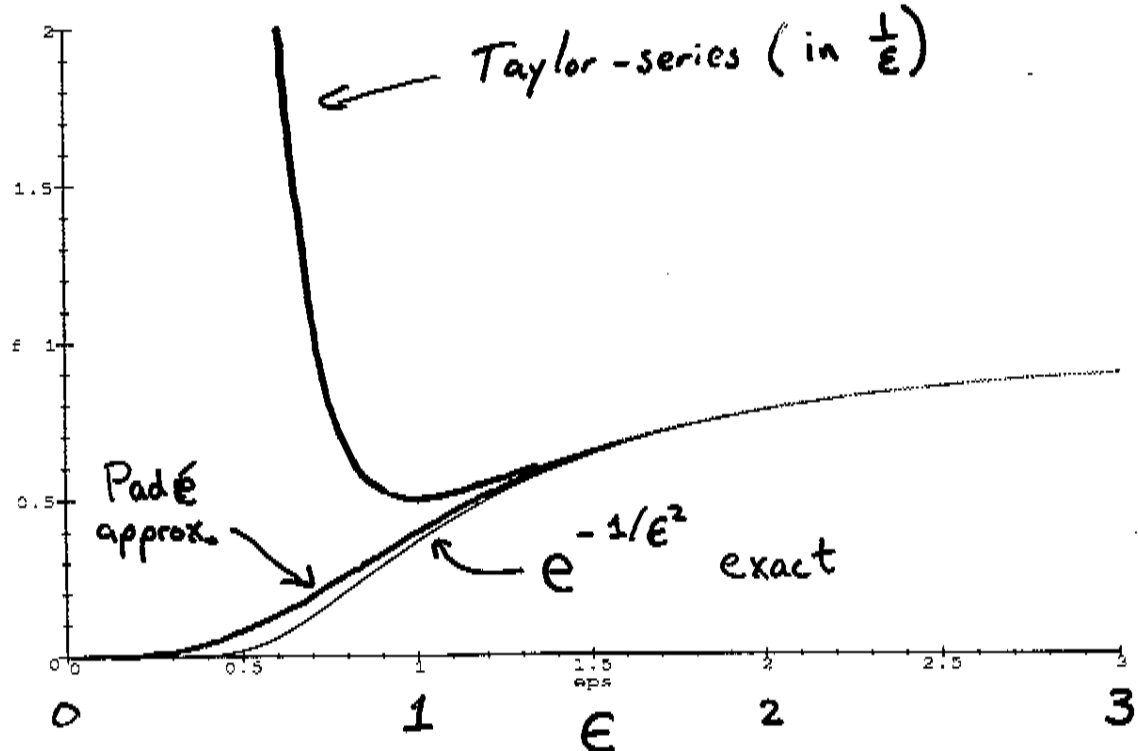
$$
\begin{aligned}
f(x) &\approx f(0) + x f'(0) + \frac{1}{2} x^2 f''(0) + \ldots \\
e^{-1/\epsilon^2} &\approx 0 \quad + \epsilon\, 0 \quad + \frac{1}{2}\epsilon^2\, 0 \quad + \ldots
\end{aligned}
$$

or substitute $x = 1/\epsilon^2$, and try expanding around $x = 0$:

$$e^{-x} \approx 1 - x + \frac{1}{2}x^2 = 1 - \frac{1}{\epsilon^2} + \frac{1}{2}\frac{1}{\epsilon^4} + \ldots$$

Diverges as $\epsilon \to 0$!!!

(Reminiscent of "renormalization" problems in quantum mechanics, etc., where perturbation expansions must be summed to all orders...)



Padé Approximations of e^{-1/ϵ^2}

Padé Approximations are Ratio of Polynomials.
(Taylor series are a simple polynomial)

Example:

$$e^{-1/\epsilon^2} = \frac{1}{e^{+1/\epsilon^2}} \approx \frac{1}{1 + \frac{1}{\epsilon^2} + \frac{1}{2}\frac{1}{\epsilon^4} \dots} = \frac{2\epsilon^4}{1 + 2\epsilon^2 + 2\epsilon^4}$$

"4-pole approximation"

Padé approximations are often much more robust (they have bounded errors) than Taylor series.

The trick for more complicated equations is finding the equivalent of a Padé approximation.

"Landau-Fluid" Models of Phase-Mixing / Landau-Damping

Clearest Intro: Hammett, Dorland, Perkins, Phys. Fluids **B4**, 2052 (1992)

$$\mathrm{Im}(Z) \propto e^{-\frac{1}{\cancel{\left(\frac{\omega}{k_\parallel v_t}\right)}\left(\frac{k_\parallel v_t}{\omega}\right)^2}} = e^{-\frac{1}{\epsilon^2}} \rightarrow 0$$

faster than any power of ϵ.

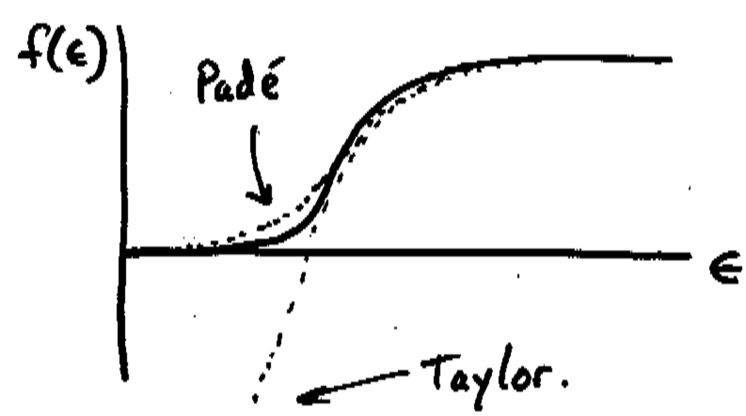
Oberman (1960) pointed out (traditional) fluid closure approximations assume $\left(\frac{\omega}{k_\parallel v_t}\right)^{-1} = \epsilon \ll 1$ and $\therefore$ fail to get Landau damping

Key is to find the equivalent of a Padé approximation:

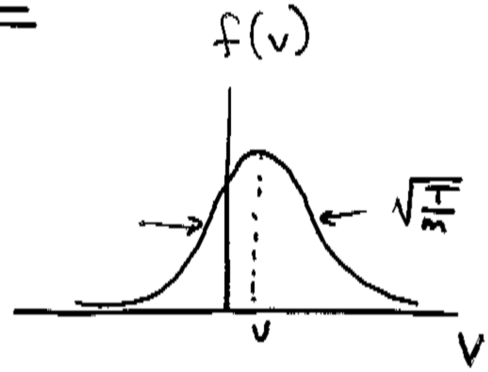
$$e^{-\frac{1}{\epsilon^2}} = \frac{1}{e^{+\frac{1}{\epsilon^2}}} \approx \frac{1}{1+\frac{1}{\epsilon^2}} = \frac{\epsilon^2}{1+\epsilon^2} \xrightarrow{} 0 \text{ as } \epsilon \rightarrow 0$$

Padé approx. are often more robust than simple Taylor series. Error is bounded, unlike Taylor series:

$$e^{-\frac{1}{\epsilon^2}} = 1 - \frac{1}{\epsilon^2}$$



Classical Fluids are Highly Collisional

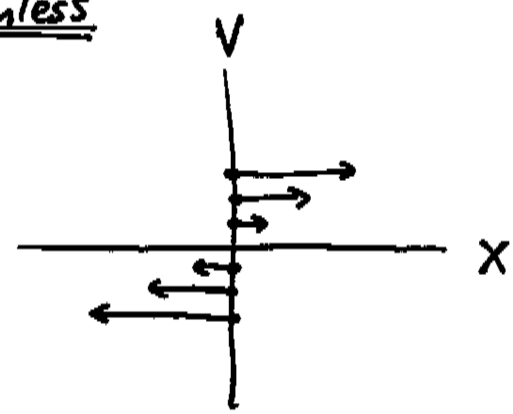


$\lambda_{mfp} \sim 10^{-4}$ cm in air

Particles move together in LTE
  Local Maxwellian specified by n, v, T

Particles slowly diffuse apart.

Fusion Plasmas are often Highly Collisionless



$\lambda_{mfp} \sim$ kilometers $\gg$ tokamak size

Particles quickly separate + "phase-mix"
  $\Rightarrow$ collisionless Landau-damping

Other Low-Collisionality Systems:

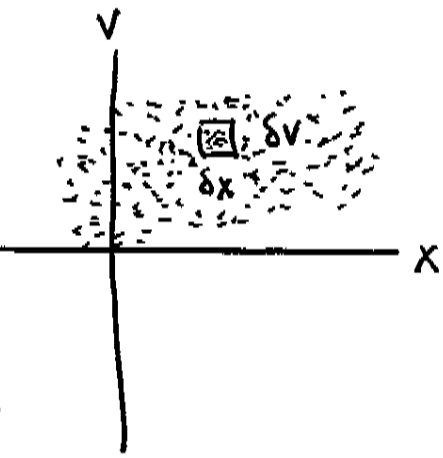
* Often occur in space + astrophysical plasmas.
* The "gas" of stars evolving as a galaxy.

* Computer chips getting so small that feature sizes are approaching λ_{mfp}.

Generic Vlasov / Boltzmann Eq.

Describes Hamiltonian interaction of a classical many-body system.
in 2n-dimensional phase space (q_i, p_i)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{q}_i \frac{\partial f}{\partial q_i} + \dot{p}_i \frac{\partial f}{\partial p_i} = 0 \qquad (\text{or} = C(f))$$



Consider 2-D:

$f(x,v,t)$ = phase-space density of particles

$f\,\delta x\,\delta v$ = # of particles with $\begin{cases} \text{position } x \pm \frac{\delta x}{2} \\ \text{velocity } v \pm \frac{\delta v}{2} \end{cases}$

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(v f) + \frac{\partial}{\partial v}(\dot{v} f) = C(f) \approx 0$$

$$\dot{v} = \frac{dv}{dt} = \frac{e}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\right)$$

or $= -\frac{e}{m}\nabla\Phi$ Electrostatic potential in plasmas (fusion, astro, accelerators...)

gravitational potential (galactic, stellar dynamics, ...)

quantum potential in solid state (semiconductors)

or $= 0$ for classical fluid dynamics (but then strong collisions)

2-D incompressible hydrodynamics (x, y) isomorphic to 2-D Vlasov (x, v)

(Wigner transform of Schrödinger Eq. reduces to Vlasov Eq.
in the classical limit $\hbar \to 0$.)

Widely thought that fluid simulations of plasma turbulence were inherently unable to model important kinetic effects.

Classical Chapman-Enskog closure procedure used by Braginskii fails for collisionless plasmas and misses kinetic effects such as Landau damping. (Improved by Chang + Callen)

"When collisions are infrequent, ... (the) heat flow depends on the detailed nature of the velocity distribution function, and cannot be determined in any simple way from the macroscopic (fluid) equations."

Spitzer, Physics of Fully Ionized Gases, 1962, p. 25 (see also p. 159).

"A property of Langmuir waves that is predicted by the Vlasov theory but which is completely outside the scope of fluid theory is the collisionless damping of electrostatic potentials..."

Krall and Trivelpiece, Principles of Plasma Physics, 1973, p. 386.

"What has happened to the Landau damping? One cannot expect the Landau damping to manifest itself in such a procedure, a power series expansion in $(k\, v_{th} / \omega)$, for in the Landau problem, in this limit, the damping goes as

$$\mathrm{Im}\,(Z) \sim \exp\left(-\frac{1}{(k\, v_{th} / \omega)^2}\right) = e^{-\frac{1}{\varepsilon^2}}$$

i.e., the damping goes to zero faster than any power of $(k\, v_{th} / \omega)$."

Oberman, Matt-57, Project Matterhorn, Princeton, 1960, p.8.

Nevertheless, some hardy souls have charged in:

Similon, Princeton Ph.D. 1981

Terry...

Lee & Diamond...

Waltz

Introduced phenomenological damping at high $k_\perp$ or high $k_\parallel$ to model ion Landau damping.

Unstable modes at some $\underline{k}$'s $\Longleftarrow$ Nonlinear Coupling $\Longrightarrow$ Damped modes at other $\underline{k}$'s.

Kadomtsev & Pogutse '84 IAEA

Use a resistivity $\propto |k_\parallel| v_t$ (estimated from Z-function) in "resistive" MHD turbulence simulations....

Chang & Callen, Phys. Fluids B $\underline{4}$ 1167 (1992)

Chapman-Enskog-like derivation of fluid Eqs.,

$$\chi_\parallel = \chi_\parallel\left(Z\left(\frac{\omega}{|k_\parallel| v_t}\right)\right) \qquad \text{etc....}$$

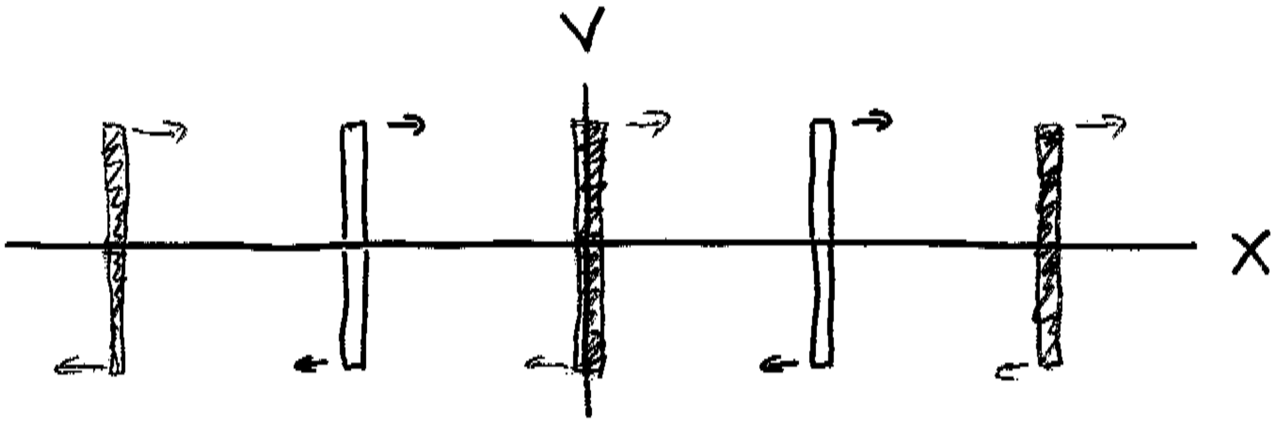
Hammett & Perkins, PRL $\underline{64}$ 3019 (1990)

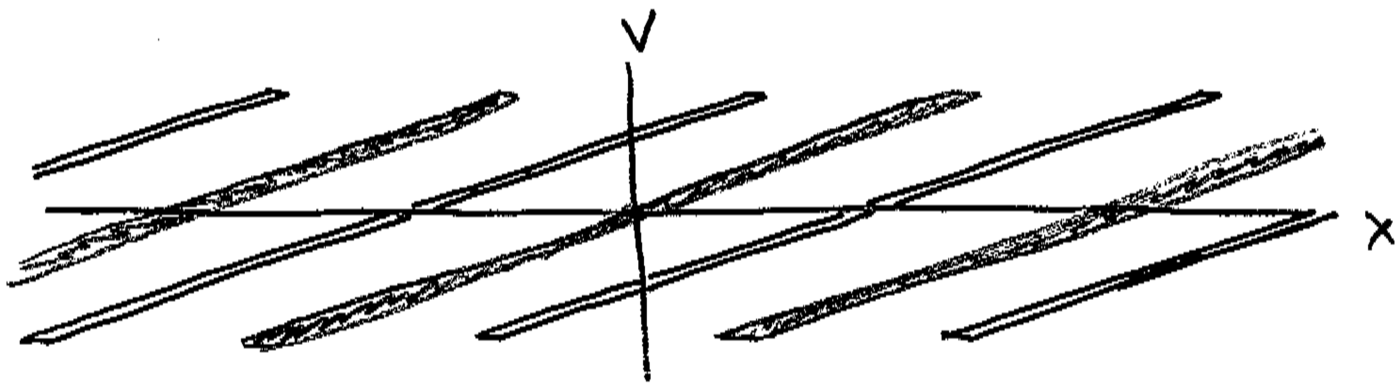
Hammett, Dorland, Perkins, Phys. Fluids $\underline{B}$ $\underline{4}$ 2052 (1992) (clearer)

$$\chi_\parallel \sim \frac{v_t}{|k_\parallel|} \qquad = \text{ n-pole approximation the Z-function...}$$

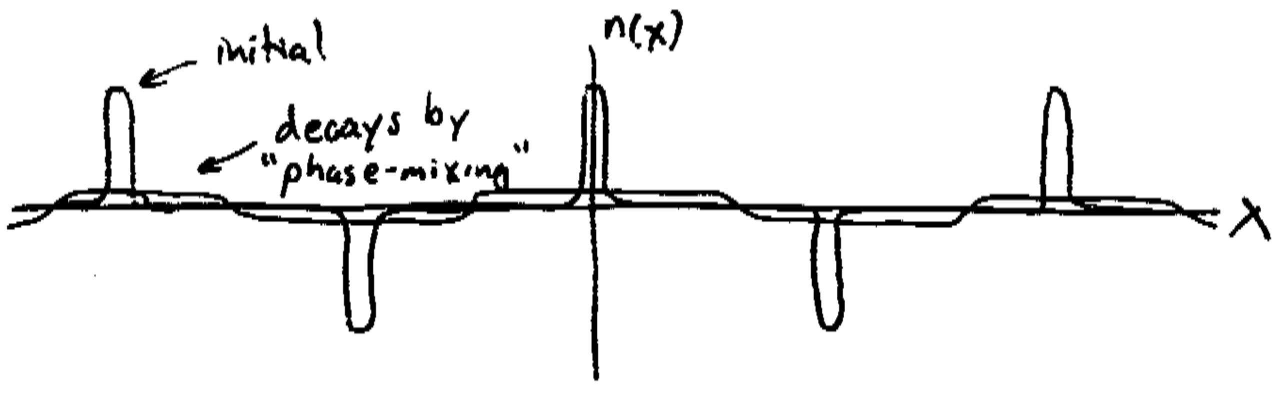
$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} = 0$$

Contours of $f(x,v)$:



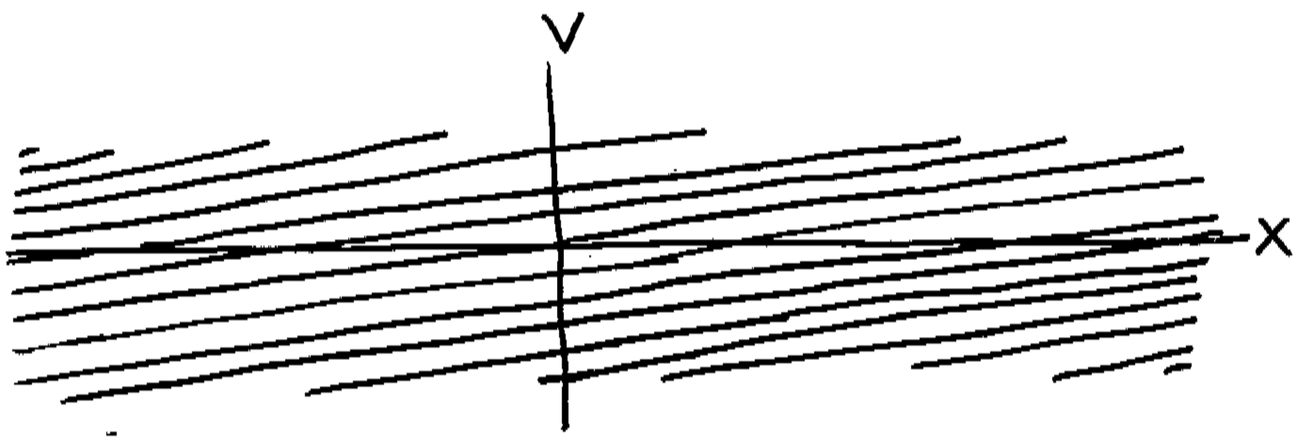


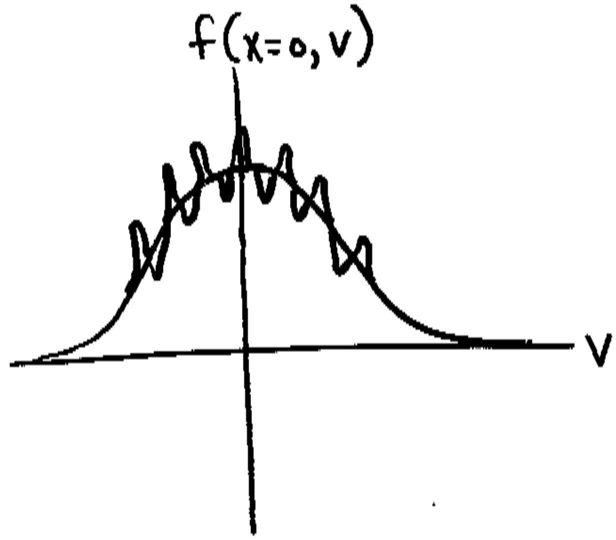
$$n(x) = \int dv\, f$$



Phase-mixing damping rate $\sim |k|\, v_t$, roughly analogous to shearing rate $|\nabla u|$ or eddy-turnover time $|k| u$ in Navier-Stoke

Long time limit:





Even weak collisions will wipe out such small scale velocity features.

$$C(f) = \nu_{col} V_t^2 \frac{\partial^2 f}{\partial v^2}$$

$$f \propto e^{-ikvt}$$

$$\frac{\partial^2 f}{\partial v^2} \approx -k^2 t^2 f$$

Collisions dominate at $\tau \propto \nu_{col}^{1/3}$

A Simple Phase-Mixing Paradigm

(Carl Oberman reminded me of this view of Landau damping.)

Consider a 1-D kinetic Eq. for $f(z,v,t)$, with no $\underset{\sim}{E}$ field:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} = 0$$

The exact solution is $f(z,v,t) = f_0(z-vt, v)$

Consider a single Fourier mode in z with a Maxwellian distribution in v:

$$f_0 = n_0 e^{ikz} f_M(v)$$

$$f = n_0 e^{ik(z-vt)} \frac{1}{\sqrt{2\pi v_t^2}} e^{-v^2/(2v_t^2)}$$

At any fixed v, f oscillates in time with $\omega = kv$ & no damping.
However, any v-moment of f will exponentially decay in time:

$$n(z,t) = \int dv\, f = n_0 \frac{e^{ikz}}{\sqrt{2\pi v_t^2}} \underbrace{\int dv}_{\text{mixing}} \underbrace{e^{-ikvt}}_{\text{phases}} e^{-v^2/(2v_t^2)}$$

$$\boxed{n(z,t) = n_0 e^{ikz} e^{-k^2 v_t^2 t^2/2}}$$

The Closure Problem in the Fluid Moment Hierarchy

$$\int dv\, v^\ell * \qquad \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} + \frac{e}{m}E_\parallel \frac{\partial f}{\partial v} = 0$$

$$\frac{\partial}{\partial t}\int dv\, v^\ell f + \frac{\partial}{\partial z}\int dv\, v^{\ell+1} f - \frac{e}{m}E_\parallel \int dv\, \ell v^{\ell-1} f = 0$$

$$\ell=0 \Rightarrow \qquad \frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nu) = 0$$

$$\ell=1 \Rightarrow \qquad \frac{\partial}{\partial t}(mnu) + \frac{\partial}{\partial z}(u\, mnu) = -\frac{\partial p}{\partial z} + enE_\parallel$$

$$\ell=2 \Rightarrow \qquad \frac{\partial p}{\partial t} + \frac{\partial}{\partial z}(up) = -2p\frac{\partial u}{\partial z} - \frac{\partial q}{\partial z}$$

etc.

Exact, nonlinear, conservation laws (particles, momentum, energy, ...)

But each $\langle v^\ell \rangle$ Eq. requires knowledge of $\langle v^{\ell+1} \rangle$

= Infinite Hierarchy ⇒ Closure problem

Usual approximations: neglect a higher moment

$p=0$ cold plasma approximation. No phase-mixing.

$q=0$, or $q=\infty$ (isothermal $\nabla T=0$) still no phase-mixing.

Seemingly "better" approximation: $\langle v^{10} \rangle = \alpha \langle v^8 \rangle \dfrac{\langle v^8 \rangle}{\langle v^6 \rangle}$

still fails to reproduce phase-mixing.

Traditional fluid eqns. fail to reproduce phase-mixing
(even if higher-order moments are kept.)

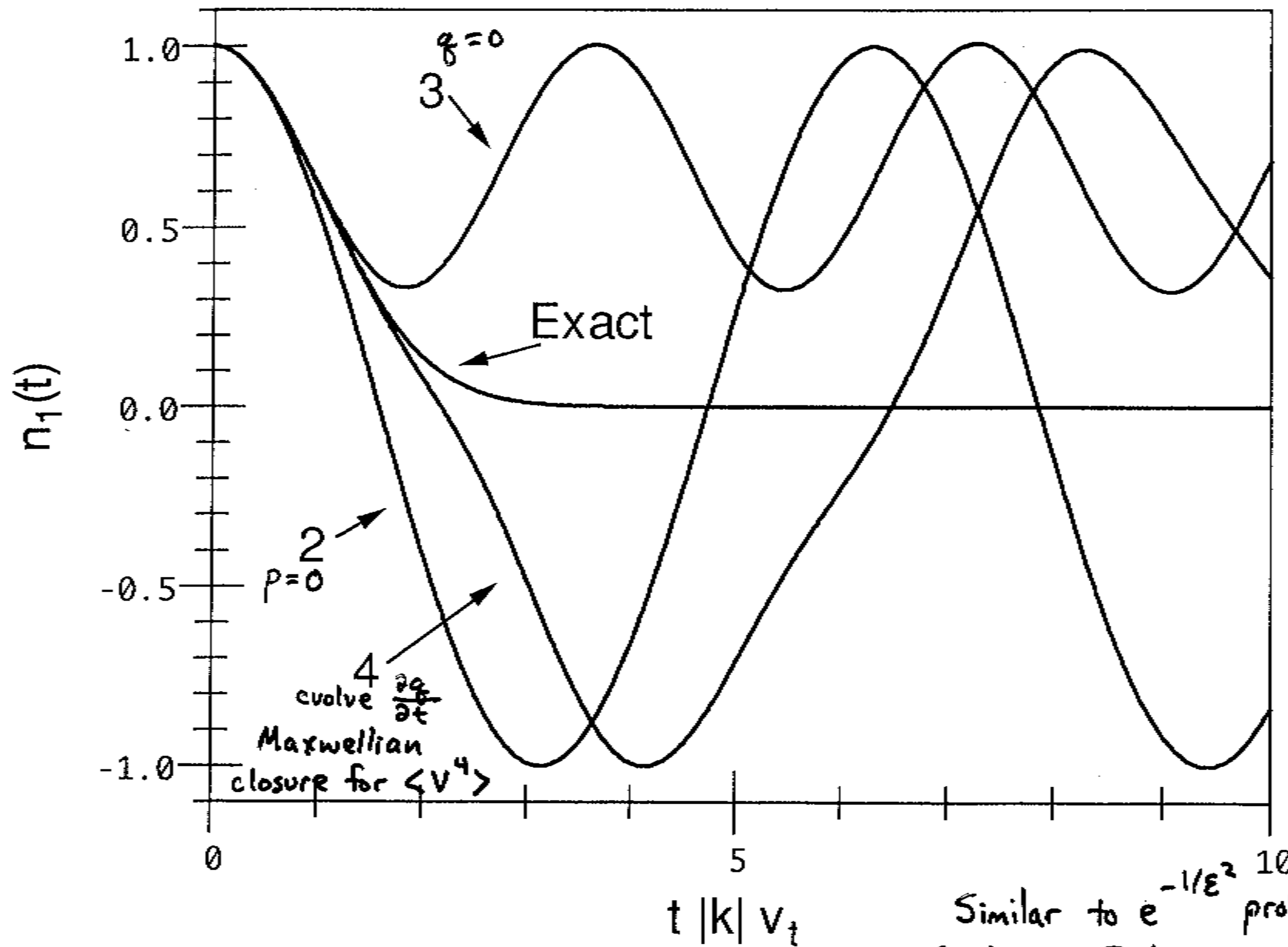


Fig. 1

Similar to e^{-1/ε^2} problem!
Good as a Taylor-series for short time,
breaks down for long time...

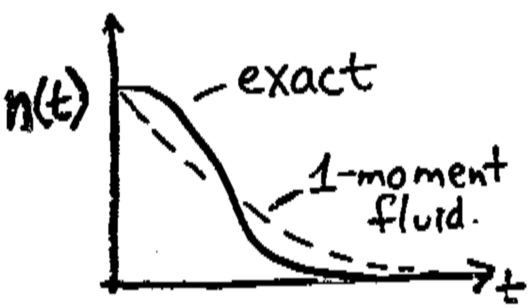
Diffusive-type Closure needed to model Collisionless Phase-Mixing.

Simplest possible 1-moment fluid model (too simple for most purposes). Start with exact density conservation Eq.:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nv) = 0$$

Approximate higher moment (nv) in terms of lower moments (n), with a diffusive-type term to model phase-mixing damping:

$$nv \approx -D\frac{\partial n}{\partial z} \quad \Rightarrow \quad \frac{\partial n_k}{\partial t} + Dk^2 n_k = 0$$



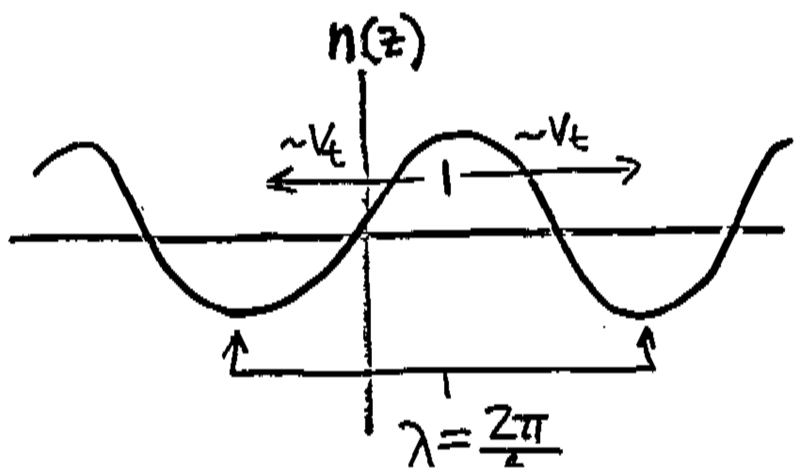
Fluid Model: $n_k(t) \propto e^{-Dk^2 t}$

Exact Kinetic: $\propto e^{-k^2 v_t^2 t^2/2}$

$$\boxed{D = \sqrt{\frac{2}{\pi}}\,\frac{v_t}{|k|}}$$

Damping rate $\nu \sim |k| v_t \sim Dk^2$

(D is an integral operator in real space.)



New closure approximation for fluid moments successfully models long-time-scale phase-mixing.

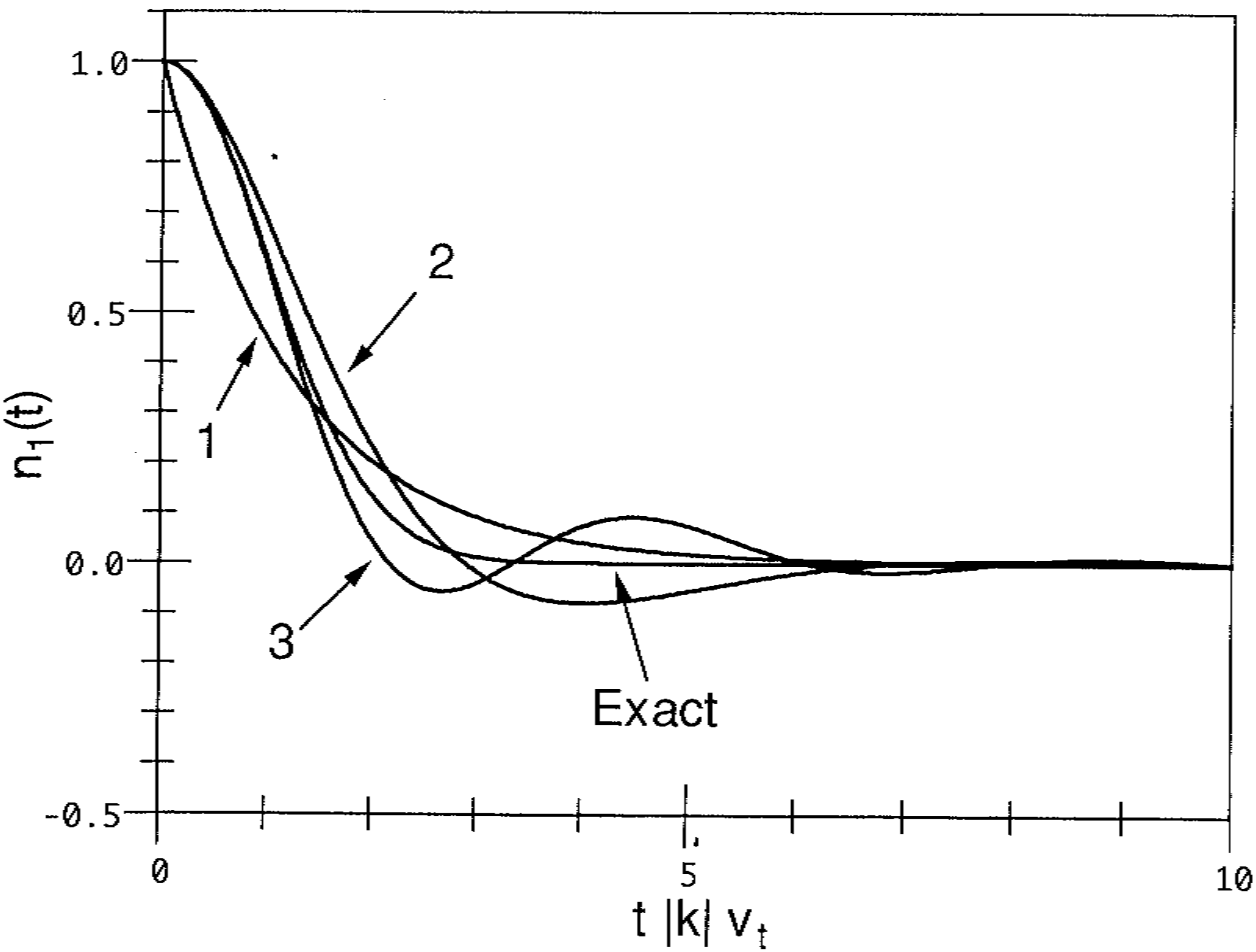
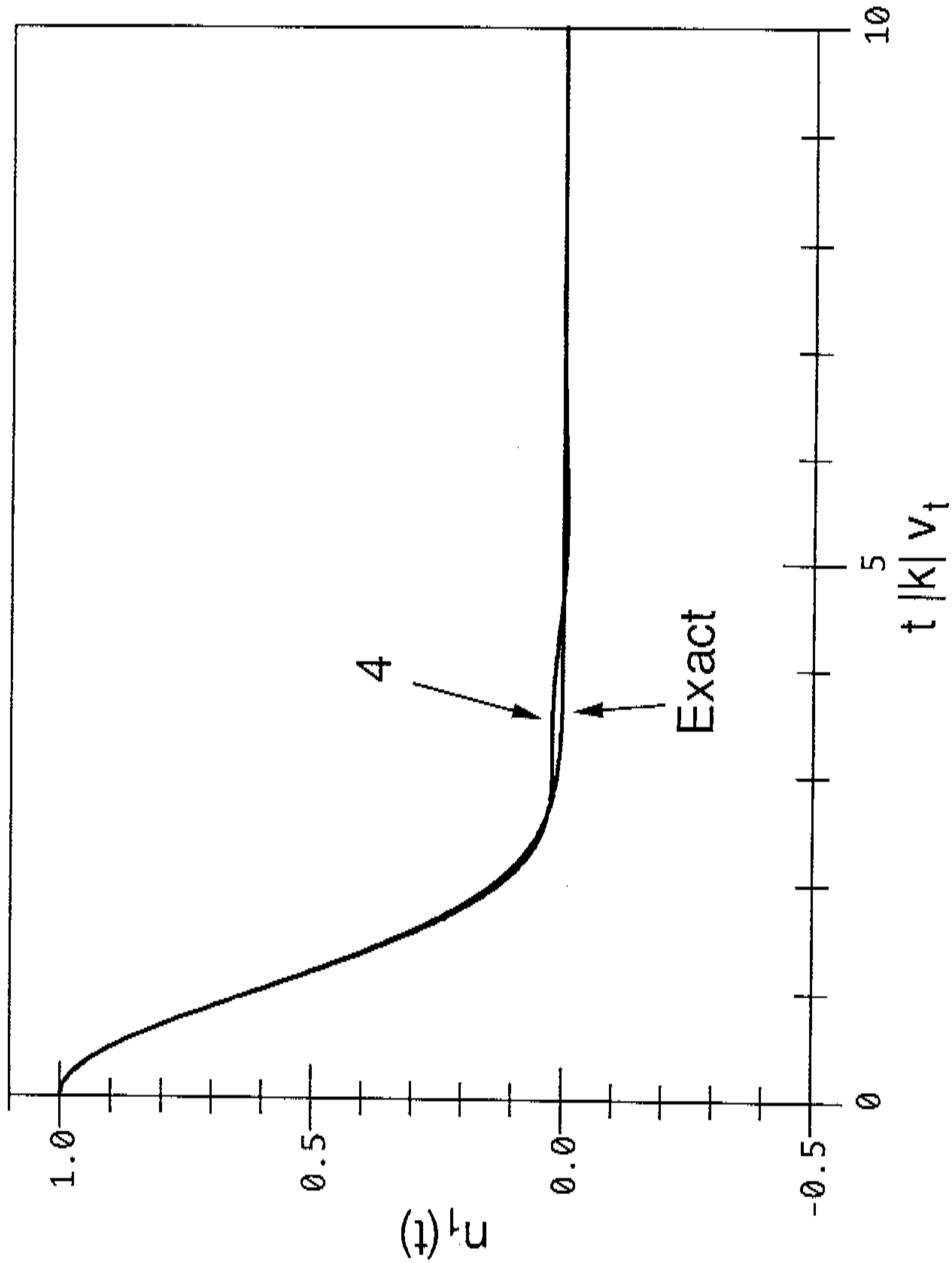


Fig. 2

Fig. 3

New Landau-fluid closure approximation
Converges rapidly as more moments are kept.



$$q = -n\chi\nabla T$$

Thermal Diffusivity (Conductivity) $\chi \cong \dfrac{v_t^2}{\nu_{coll} + |k| v_t}$

Collisional limit
Braginskii
Chapman-Enskog

Landau-damping
Phase-mixing limit on heat flux.
related to "flux limiters" in laser-plasma problems.

$\chi \simeq \dfrac{v_t}{|k|} \Rightarrow$ non-local operator in real space, not simple diffusion.

We developed these Landau-closure approximations for tokamak drift-wave turbulence. Other applications:

* α-particle driven TAE instability (Spong, Hedrick, et.al.)
* Resistive-wall MHD (Bondeson & Ward)
* Laser-plasma filamentation (Kaiser et.al.)
* Extend Zakharov Eqs. of Langmuir turbulence, ionospheric applications (Goldman & Newman, H. Rose et.al.)

Probably other useful applications.

Phase-Mixing = Landau Damping

We've been solving $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = S = f_0(x,v)\,\delta(t)$

initial condition

But just sum over lots of sources at different times, $S(t)$.

Green's function (Linear propagator) applicable to other $S(t)$.

Landau damping problem:

$$\frac{\partial \tilde{f}}{\partial t} + v\frac{\partial \tilde{f}}{\partial x} = -\frac{e}{m} E_{\parallel 0}\, e^{ikx - i\omega t} \frac{\partial F_0}{\partial v}$$

Useful trick:

$$Z\left(\frac{\omega}{|k| v_t \sqrt{2}}\right) \propto \int dv \frac{e^{-\frac{v^2}{2v_t^2}}}{\omega - kv}$$

$$= -i\int dv\, e^{-\frac{v^2}{2v_t^2}} \int_0^\infty d\tau\, e^{i(\omega + i\varepsilon - kv)\tau}$$

$$= -i\int_0^\infty d\tau\, e^{i(\omega+i\varepsilon)\tau} \underbrace{\int dv\, e^{-\frac{v^2}{2v_t^2}}\, e^{-ikv\tau}}_{\text{Just like before !!}}$$

$$= -i\int_0^\infty d\tau\, e^{i\omega\tau}\, e^{-k^2 v_t^2 \tau^2/2}$$

Landau damping is Fourier-tranform of Phase-Mixing
(ω-space) (or Laplace-transform) (t-space)

<u>Caveats:</u> (Hammett et.al., Plasma Physics Contr. Fus. <u>35</u>, 973 (1993))

There are some cases where the Landau-fluid approximations don't work well (or require many moments to converge), where it is necessary to follow full details of full $f(x,v,t)$. (plasma echoes, quasilinear flattening near narrow resonances).
(Mattor, Phys. Fluids B<u>4</u>, 3952 (1992)).

But works well as a model of rate at which f phase-mixes to small velocity scales which can then be ignored.

(Philisophically similar to subgrid turbulence models, or to the EDQNM simplified vs. of the DIA)

[Usually there are many processes which quickly wipe out small velocity scales (collisions) or make them incoherent (turbulence).]

Should be well-suited to typical strong-turbulence tokamak regimes.

Provide accurate linear growth rates for instabilities which drive the turbulence, + for damped modes.

E×B nonlinearities in fluid Eqs. couple these together.

Fluid moment Eqs. express important conservation laws (particles, momentum, magnetic moment, parallel energy, ...) which constrain the nonlinear dynamics of the turbulence.

Why Research Fusion Energy?

- Energy Fuels Our Economy: economic shocks from the energy crisis of the 1970's...
- Increased demand from China, India, ... bid up oil prices, $\uparrow$ pollution (acid rain, greenhouse?)
- U.S. spends $450 Billion/year on Energy.
  $<$ 1% for alternative energy research
  $<$ 0.1% for fusion research ($370 Million)
- Japan and Europe each spend 50% more on fusion than U.S. Relative to their GNP, they spend 2-4 times U.S.
- Other renewable energy sources also worth R&D, but can't meet full energy demand unless there are large cutbacks. Diversification (in stocks and energy) is good.
- Fusion is also fascinating science, interacts with astrophysical and space plasmas, plasma processing, high power microwaves, accelerators, lasers, solid-state, atomic physics, computational science, general physics of nonlinear, many-body systems, chaos theory, ...

Major Progress is Being Made in Fusion Research

- Fusion power production in experiments has risen by a factor of 10^{11} since the the first tokamaks outside of Russia in 1970.
- TFTR tritium-deuterium experiments have made > 10 MW of Fusion Power, demonstrating reactor-like fusion power densities.

To make fusion cheaper, pursuing advanced tokamak concepts (negative magnetic shear, stabilizing sheared flows, detached divertor regimes, etc.) in present experiments (TFTR and smaller tokamaks) and in proposed future experiments. Trying to

- further improve the magnetic bottle (decreasing the turbulent leakage rate, and improving the plasma pressure limits)
- improving the divertor (plasma-wall interface) to handle higher power loads and temperatures

Summary: Simplifying Complexity: Reduced Fluid Models of Landau-damping and Phase-mixing

- Paradigm Problem of approximating e^{-1/ϵ^2}
- Importance of Mixing Processes in Many Physical Systems (Fusion Plasma Physics, Space and Astrophysics, Semiconductors and Solid-State Physics, Fluid Dynamics, and other many-body/chaotic systems...)
- Physics of Landau-damping and Phase-mixing
- Derivation of Reduced "Landau-Fluid" Models

  (Successful application to fusion plasma turbulence in the next talk by Dorland.)
- Major Progress in Fusion Energy Research