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Fluid Moment Models for Landau Damping with Application to the Ion-Temperature-Gradient Instability

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A closed set of fluid moment equations is developed which represents kinetic Landau damping physics and which takes a simple form in wave-number space. The linear-response function corresponds to a three-pole (or four-pole) approximation to the plasma dispersion function Z. Alternatively, the response is exact for a distribution function which is close to Maxwellian, but which decreases asymptotically as $1/v^4$ (or $1/v^6$). Among other applications, these equations should be useful for nonlinear studies of turbulence driven by the ion-temperature-gradient instability or other drift-wave microinstabilities.

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Because of their relative simplicity, fluid moment equations have been used in a number of recent nonlinear studies of turbulence driven by the ion-temperature-gradient (ITG) instability[1-3] and other microinstabilities. This turbulence is of interest because it can cause transport in tokamaks and other plasmas. Moment equations must be closed by an approximation scheme. The classic method of Braginskii[4] is rigorous in the short-mean-free-path regime, but inapplicable to collisionless plasmas. This Letter proposes a closure method which (1) ensures particle, momentum, and energy conservation, (2) takes on a simple form in wave-number space, and (3) has a linear-response function very close to that of a collisionless, Maxwellian plasma. This closure method successfully models kinetic resonances (such as Landau damping) not only in one dimension but also in slab geometry where it reproduces the correct marginal stability behavior of the ITG mode.

Several authors have suggested that the effects of kinetic Landau damping may be modeled in fluid moment equations by adding dissipative terms. Lee and Diamond[1] set the parallel momentum viscosity to $\mu_\parallel \approx v_{ti}^2/|\omega|$, where ω is the mode frequency, and $v_{ti}=(T_i/m_i)^{1/2}$ is the thermal ion speed. Hamaguchi and Horton[2] suggest modifying both $\mu_\parallel$ and the parallel heat conductivity $\chi_\parallel$, although their simulations use values which are constant for all modes independent of wave frequency and wavelength. Waltz[3] has proposed setting $\mu_\parallel=\chi_\parallel=\min(2^{1/2}v_{ti}/|k_\parallel|, 2v_{ti}^2/|\omega_r|)$, where ω_r is the real part of an instantaneous estimate of the mode frequency. However, no comparison of any of these models with exact kinetic Landau damping has been published. We shall show that any model with a nonzero $\mu_\parallel$ faces difficulties of interpretation and yields inaccurate thresholds and growth rates for the ITG instability.

It has been suggested[1] that Landau damping for ITG modes can be ignored well above marginal stability so that Braginskii-based fluid equations can be used. However, kinetic effects cannot be ignored for higher radial eigenmodes which appear to cause more transport.[2,5] Furthermore, most experiments[6-8] find that the observed ion temperature gradient is usually within a factor of 2 or less of marginal stability, so that kinetic effects are always important. Recent experimental counterexamples[9] to ITG marginal stability cast some doubt on the accuracy of present ITG theories.

We first discuss the simplest possible case of linear one-dimensional electrostatic waves. The exact kinetic response, to which we will compare our fluid approximations, is governed by the one-dimensional Vlasov equation

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial z}+\frac{e}{m}E\frac{\partial f}{\partial v}=0\,, \tag{1}$$

where $f(z,v,t)$ is the particle distribution function as a function of position z, velocity v, and time t. Consider the linear response $f=f_0(v)+\tilde{f}(z,v,t)$ to a small driving electric field $E=-\partial\tilde{\phi}/\partial z$. After the standard Fourier-Laplace transforms where perturbed quantities vary as $\exp(ikz-i\omega t)$, the exact linear response is

$$\tilde{n}=\int dv\,\tilde{f}\equiv -n_0\frac{e\tilde{\phi}}{T_0}R(\zeta)=\frac{e\tilde{\phi}}{T_0}kv_t^2\int dv\frac{\partial f_0/\partial v}{kv-\omega}\,. \tag{2}$$

We have defined a normalized response function $R(\zeta)$, a normalized frequency $\zeta=\omega/|k|v_t\sqrt{2}$, and a generalized "temperature" $T_0=mv_t^2=m\int dv\,f_0v^2/\int dv\,f_0$. Following Landau's prescription to insure casuality,[10] the velocity integral in Eq. (2) is along the real axis only for $\mathrm{Im}(\omega)>0$, otherwise the integral must be analytically continued for $\mathrm{Im}(\omega)\le 0$ (i.e., the velocity contour integral must be deformed around the pole at $v=\omega/k$). The response function for a Maxwellian f_0 is shown in Fig. 1 and can be written as $R(\zeta)=1+\zeta Z(\zeta)$, where $Z(\zeta)=\pi^{-1/2}\int dt\exp(-t^2)/(t-\zeta)$ is the usual plasma dispersion function. Note that for a general f_0 and real ζ, the imaginary part of R is related to f_0 by

$$\mathrm{Im}\,R(\zeta)=-\frac{\pi v_t^2}{n_0}\frac{\partial f_0}{\partial v}\bigg|_{v=\zeta 2^{1/2}v_t}. \tag{3}$$

Thus, when we develop a fluid approximation for the

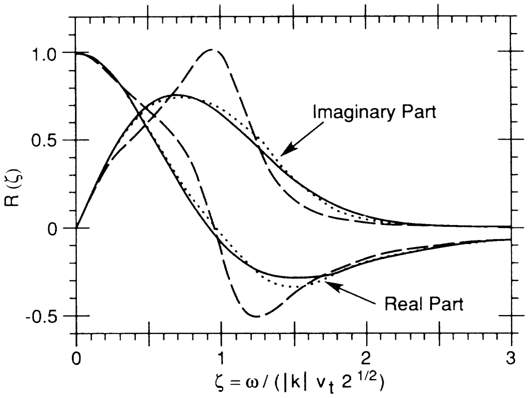

FIG. 1. The real and imaginary parts of the normalized response function $R(\zeta) = -\tilde{n}T_0/n_0 e\tilde{\phi}$ vs the normalized frequency ζ. The solid lines are the exact kinetic result for a Maxwellian, $R(\zeta) = 1 + \zeta Z(\zeta)$. The dashed lines are from the three-moment fluid model with $\Gamma = 3$, $\mu_1 = 0$, and $\chi_1 = 2/\sqrt{\pi}$. The dotted lines are from the four-moment model.

Maxwellian $R(\zeta)$, we can find an f_0 which will give an identical kinetic response by integrating $\mathrm{Im}(R(\zeta))$.

Consider the following generalized set of fluid equations for the particle density $n = \int dv\, f$, the momentum density $mnu = m\int dv\, fv$, and the pressure $p = m\int dv\, f(v-u)^2$:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(un) = 0\,, \tag{4}$$

$$\frac{\partial}{\partial t}(mnu) + \frac{\partial}{\partial z}(umnu) = -\frac{\partial p}{\partial z} + enE - \frac{\partial S}{\partial z}\,, \tag{5}$$

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial z}(up) = -(\Gamma - 1)(p + S)\frac{\partial u}{\partial z} - \frac{\partial q}{\partial z}\,. \tag{6}$$

The heat flux moment is $q = m\int dv\, f(v-u)^3$. In the $\Gamma = 3$ and $S = 0$ limit, Eqs. (4)–(6) are exact moments of Eq. (1) and therefore represent particle, momentum, and energy conservation. We have introduced a dissipative momentum flux S and an adjustable ratio of specific heats Γ in order to compare with previously suggested Landau damping models, although we will conclude that it is best to set $\Gamma = 3$ (as expected for a collisionless one-dimensional gas where $p_\parallel$ and $p_\perp$ decouple) and $S = 0$. Previous models based on Braginskii's collisional equations assume $p_\parallel \approx p_\perp$ so that $\Gamma = \frac{5}{3}$. Kinetic effects are modeled by judicious closure approximations for q and S which are most straightforward in wave-number space. We will assume linear closure approximations of the form

$$\tilde{q}_k = -n_0\chi_1 \frac{2^{1/2} v_t}{|k|} ik\tilde{T}_k\,, \tag{7}$$

and, similarly, $\tilde{S}_k = -mn_0\mu_1 2^{1/2}(v_t/|k|)ik\tilde{u}_k$, where χ_1 and μ_1 are dimensionless coefficients, and $\tilde{T} = (\tilde{p} - T_0\tilde{n})/n_0$ is the perturbed temperature. These closure *Ansätzes* can be written symbolically in the standard Fick's law forms $q = -n\chi\partial T/\partial z$ and $S = mn\mu\partial u/\partial z$, except that χ and μ are integral operators in physical space because they are proportional to $v_t/|k|$ in wave-number space. For example, by performing the inverse Fourier transform of $\tilde{q}_k$, we find that the real-space representation of $\tilde{q}(z)$ is

$$\tilde{q}(z) = \frac{1}{(2\pi)^{1/2}} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dk\, e^{ik} e^{-|k|\epsilon} \tilde{q}_k$$
$$= -\frac{n_0\chi_1 2^{1/2} v_t}{\pi} \int_0^\infty dz' \frac{\tilde{T}(z+z') - \tilde{T}(z-z')}{z'}\,, \tag{8}$$

There is a missing factor of z in the exponential, it should be e^{i k z}.

where we have used the convolution theorem, and the factor of $\exp(-|k|\epsilon)$ was added to define infinite integrals. Equation (8) shows that the parallel heat flux $q(z)$ is driven by an average nonlocal temperature difference. Many numerical codes use a spectral representation in the magnetic-field direction and so can use the simple Fourier representation for $\tilde{q}_k$ rather than the convolution form for $\tilde{q}(z)$.

Let us turn to the choice of the free parameters χ_1, μ_1, and Γ. Linearizing Eqs. (4)–(6) leads to the following three-moment fluid model for the response function:

$$R_3 = \frac{\chi_1 - i\zeta}{\chi_1 - i\Gamma\zeta - 2i\chi_1\mu_1\zeta - 2\chi_1\zeta^2 - 2\mu_1\zeta^2 + 2i\zeta^3}\,. \tag{9}$$

Our *Ansatz* for q and S has led to a response function which depends only on ζ, as it should. Note that the three-moment fluid model yields a three-pole approximation for the response function (and therefore for the plasma dispersion function Z). The asymptotic expansion of R_3 in the cold-plasma limit $|\zeta| \gg 1$ is $R_3 \sim -1/2\zeta^2 + i\mu_1/2\zeta^3 + \cdots$. According to Eq. (3), this implies that R_3 is equivalent to an $f_0(v) \sim \mathrm{const}/v^2$ and hence has an infinite-pressure moment. Therefore we choose $\mu_1 = 0$. (Because the ITG instability is driven by gradients in the pressure moment, one may question the applicability of any $\mu_1 \neq 0$ model.) Carrying the asymptotic expansion to higher order then gives us $R_3 \sim -1/2\zeta^2 - \Gamma/4\zeta^4 + i(\Gamma - 1)\chi_1/4\zeta^5 + \cdots$, which is equivalent to a more physical $f_0(v) \sim 1/v^4$. Setting $\Gamma = 3$ puts the proper amount of compressional $p\partial u/\partial z$ heating into Eq. (6) to conserve total energy. Expanding R_3 for small ζ leads to $R_3 \approx 1 + i2\zeta/\chi_1$. Requiring this to match the Maxwellian R for small ζ leads to the condition $\chi_1 = 2/\sqrt{\pi}$. Although χ_1 is chosen to fit the low-frequency limit, the closure is used in fluid equations which are automatically valid in the high-frequency limit, and the resulting R_3 does a fair job of approximating the Maxwellian R over the full frequency range (Fig. 1), and is equivalent to an $f_0(v)$ which is fairly close to Maxwellian (Fig. 2).

We obtain more accurate results by applying our closure to a four-moment fluid model. For the first three moments we use Eqs. (4)–(6) with $\Gamma = 3$ and $S = 0$. The heat flux q is then found from the next moment of the Vlasov equation:

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial z}(uq) = -3q\frac{\partial u}{\partial z} + 3\frac{p}{mn}\frac{\partial p}{\partial z} - \frac{\partial r}{\partial z}\,, \tag{10}$$

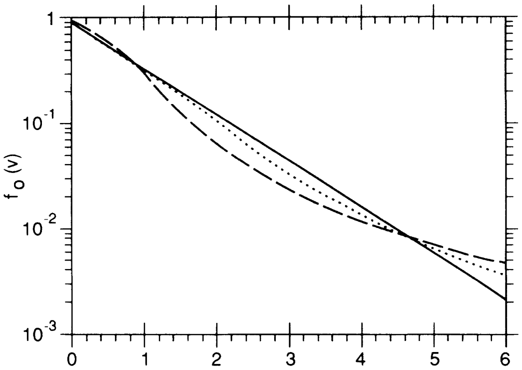

FIG. 2. The $f_0(v)$'s equivalent to our three-moment model (dashed line) and four-moment model (dotted line) compared with a Maxwellian (solid line).

where $r \equiv m \int dv\, f(v-u)^4 \equiv 3p^2/mn + \delta r$. Using our linear low-frequency closure method, δr can be expressed in terms of lower moments as

$$\delta \tilde{r}_k \approx -D_1 \frac{\sqrt{2} v_t}{|k|} ik\tilde{q}_k + \beta_1 n_0 2 v_t^2 \tilde{T}_k ,$$

where $D_1 = 2\sqrt{\pi}/(3\pi - 8)$ and $\beta_1 = (32-9\pi)/(6\pi-16)$. This is accurate through second order in ζ, while our closure for $\tilde{q}_k$ was only first-order accurate. The resulting response function is an excellent four-pole approximation of the function Z, equivalent to an $f_0(v) \sim 1/v^6$ (Figs. 1 and 2).

Having shown the effectiveness of our fluid model of Landau damping in the simple one-dimensional case, we now show that the same model continues to work in a more complicated system which describes the ITG instability. Consider electrostatic perturbations in an unsheared slab geometry governed by the zero-gyroradius drift-kinetic equation:

$$\frac{\partial f}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla f + \frac{e}{m} E_\parallel \frac{\partial f}{\partial v_\parallel} = 0 , \tag{11}$$

where $f = f(v_\parallel, v_\perp, \mathbf{x}, t)$ is the gyroaveraged distribution function, $\mathbf{v}_E = c(\mathbf{E}\times\hat{\mathbf{b}})/B$ is the $\mathbf{E}\times\mathbf{B}$ drift, $\hat{\mathbf{b}} = \hat{\mathbf{z}} = \mathbf{B}/B$ is the direction of the magnetic field, and $v_\parallel$ is the velocity parallel to $\hat{\mathbf{b}}$. (While the zero-gyroradius limit describes some basic properties of the ITG instability, one must include second-order gyroradius corrections in order to get the radial eigenmode equation in a sheared slab.) We linearize and Fourier transform as before, except now f_0 is Maxwellian with density and temperature gradients in the $\hat{\mathbf{x}}$ direction. The resulting slab response function is

$$R_s \equiv \frac{-\tilde{n} T_0}{n_0 e \tilde{\phi}} = 1 - \eta\zeta\zeta_* + [\zeta - \zeta_*(1 - \tfrac{1}{2}\eta + \eta\zeta^2)] Z(\zeta) . \tag{12}$$

We have made use of the following definitions: $\zeta = \omega / |k_\parallel| v_t \sqrt{2}$, $\zeta_* = \omega_* / |k_\parallel| v_t \sqrt{2}$, $\omega_* = (cT_0/eB) k_y / L_n$, $1/L_n = (\partial n_0/\partial x)/n_0$, $1/L_T = (\partial T_0/\partial x)/T_0$, and $\eta = L_n/L_T$. Note that Eq. (12) reduces to Eq. (2) in the one-dimensional $\omega_* \to 0$ limit.

Extending the three- and four-moment models from one dimension to slab geometry is straightforward. Taking parallel velocity moments of Eq. (11) leads to a set of fluid equations for the density n, parallel momentum density $mnu_\parallel = m\int d^3v\, f v_\parallel$, parallel pressure $p_\parallel = m \int d^3v \times f(v_\parallel - u_\parallel)^2$, and parallel heat flux $q_\parallel = m\int d^3 v f \times (v_\parallel - u_\parallel)^3$, which are identical to the previous one-dimensional equations [(4)-(6) and (10)] for n, u, p, and q, except that $\partial/\partial t$ is replaced by $\partial/\partial t + \mathbf{v}_E \cdot \nabla$. The Braginskii-type fluid equations used in previous turbulence studies[1-3] can be written in this form in the zero-gyroradius limit. We have written the fluid equations in their fully nonlinear form as exact moments of the kinetic equation, though general drift-wave turbulence scaling arguments[1,2] indicate that only the $\mathbf{v}_E \cdot \nabla$ nonlinearities (and some finite-gyroradius nonlinearities when present) need to be kept.

We make the same closure approximations as in the one-dimensional case. For example, after linearizing and Fourier transforming, the three-moment closure approximations are $\tilde{q}_\mathbf{k} = -ik_\parallel n_0 \chi_1 2^{1/2} v_t \tilde{T}_\mathbf{k} / |k_\parallel|$ and $\tilde{S}_\mathbf{k} = -i \times k_\parallel m n_0 \mu_1 2^{1/2} v_t \tilde{u}_\mathbf{k} / |k_\parallel|$. This leads to the following three-moment response functions for an unsheared slab:

$$R_{s,3} = \frac{\chi_1 - i\zeta + i\zeta_* [1 + \eta - \Gamma + 2(i\chi_1\zeta + i\mu_1\zeta - \chi_1\mu_1 + \zeta^2)]}{\chi_1 - i\Gamma\zeta - 2i\chi_1\mu_1\zeta - 2\chi_1\zeta^2 - 2\mu_1\zeta^2 + 2i\zeta^3} . \tag{13}$$

In order that $R_{s,3}$ be uniformly good approximation of the exact kinetic R_s for all values of ζ_* and η, we would like to be able to show that Eq. (13) can be rewritten in the form of Eq. (12) but with $Z(\zeta)$ replaced by its three-pole fluid approximation $Z_3(\zeta) = [R_3(\zeta) - 1]/\zeta$. It is straightforward to show that this is possible only if $\mu_1 = 0$ and $\Gamma = 3$, and we come to the conclusion that the same three-moment model which worked in one dimension ($\mu_1 = 0$, $\Gamma = 3$, and $\chi_1 = 2/\sqrt{\pi}$) continues to work just as well in slab geometry, while previously suggested models will have errors proportional to $\mu_1 \eta$. (These errors vanish in the $\eta = 0$ limit, so a viscosity-based two-moment model might be useful for $\eta = 0$ drift waves.) Our four-moment model leads to a slab response function which automatically factors into the form of Eq. (12) with the following four-pole approximation for the function Z:

$$Z_4(\zeta) = \frac{i4D_{q1} + (10 + 4\beta_1)\zeta - i4D_{q1}\zeta^2 - 4\zeta^3}{3 + 2\beta_1 - i6D_{q1}\zeta - (12 + 4\beta_1)\zeta^2 + i4D_{q1}\zeta^3 + 4\zeta^4} .$$

We will now investigate the dispersion relation for the ITG instability which results from using Eq. (13) for the ions, and a Boltzmann response for the electrons, $\tilde{n}_e = n_0 |e| \tilde{\phi}/T_{e0}$. (Subscripts i or e indicate ion or electron species.) Imposing quasineutrality $\tilde{n}_i = \tilde{n}_e$, the dispersion relation is found from the roots of $R_{s,3} = -T_{i0}/T_{e0}$, which is a cubic equation for ζ and so will have three roots. In the 1D limit $\zeta_{*i}=0$, and with $T_{i0} \ll T_{e0}$, the three roots are $\omega = \pm k_{\parallel} c_s$ (i.e., the usual ion acoustic waves with $c_s^2 = T_{e0}/m_i$) and $\omega = -i2(2/\pi)^{1/2} |k_{\parallel}| v_{ti}$. This strongly damped "heat wave" corresponds to phase mixing which takes a purely exponential form for our $f_0(v)$. Consider the dispersion relation with $\zeta_{*i} \gg 1$ (or $\omega_{*i} \gg |k_{\parallel}| v_{ti}\sqrt{2}$). One of the roots is $\omega = -(T_{e0}/T_{i0})\omega_{*i} = \omega_{*e}$ (the familiar electron drift wave). Another root is always damped for $\eta_i > 0$. Near marginal stability, $\zeta \ll 1$, the third root is given by

$$\omega = i \frac{|k_{\parallel}| v_{ti}}{\sqrt{2}} \frac{\eta_i - (\Gamma - 1 + 2\chi_1 \mu_1)}{\chi_1 + \mu_1}.$$

This is the ITG-mode dispersion relation near marginal stability. Inserting our recommended values of $\Gamma = 3$, $\mu_1 = 0$, and $\chi_1 = 2/\sqrt{\pi}$, we are able to reproduce the exact kinetic results[11,12] in the small-$k_{\perp}\rho$ limit for the threshold and growth rate [$\omega = i|k_{\parallel}| v_{ti}(\eta_i - 2)\sqrt{2\pi}/4$ is obtained from Eq. (12) in the $\zeta_{*i} \gg 1$, $\zeta \ll 1$ limit]. Previously suggested models all result in errors in the marginal-stability dispersion relation, confirming our original concern about $\mu_1 \neq 0$ models because they correspond to an $f_0(v)$ which has an infinite-energy moment while the ITG instability is driven by gradients in the energy moment. (Another possible choice for $\Gamma = 3$ might appear to be $\chi_i = 0$ and $\mu_1 = 2/\sqrt{\pi}$, but then R_3 fails to recover the fundamental Boltzmann limit for $\zeta \ll 1$ and $\zeta_* = 0$.)

There are some similarities of our closure method with Kadomtsev and Pogutse's derivation of reduced MHD-like turbulence,[13] but they use a viscosity-based model which will not work for ITG modes. Wang, Callen, and Chang[14] are applying a related closure procedure to the more ambitious problem of deriving fluid equations from the full three-dimensional Boltzmann equation including collisions. In the drift-kinetic ordering, their closure is expressed as complicated functions of the function Z and so is linearly exact (but difficult to implement in a nonlinear initial-value code where the frequencies are not known). Their closure reduces to ours in the low-frequency limit $\omega \ll kv_t$. We show that this much simpler low-frequency closure leads to an approximation for R which is fairly good for all frequencies. Also, their present formulation is missing some nonlinear parallel pressure terms which we have kept.

Our fluid models of kinetic resonances should improve the accuracy of future nonlinear calculations of ITG and other microinstability turbulence. Future work should also add finite-gyroradius and toroidal effects to these equations. In the course of thinking about these issues, we (and, in particular, G.W.H.) have benefited from discussions with Dr. Liu Chen, Dr. John Krommes, Dr. Taik Soo Hahm, Dr. Hamid Biglari, Dr. W. W. Lee, and Dr. Andris Dimits. Many of the calculations for this paper were done with the very useful *Mathematica* computer program written by Wolfram.[15] This work was supported by U.S. DOE Contract No. DE-AC02-76CH03073.

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