(DRAFT) Comment on "Dual cascade and its possible variations in magnetized kinetic plasma turbulence"

Note added Nov. 28, 2011: This draft comment is in response to version 2 of http://arxiv.org/abs/1008.0330v2
I have not yet had time to look at version 3, whose title and contents have changed to some extent. -GWH

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A recent paper, "Dual cascade and its possible variations in magnetized kinetic plasmas turbulence" (Ref. 1, hereafter Zhu10), tries to carry out calculations of the statistical absolute equilibrium of fluctuations in the 2-D gyrokinetic equations for plasmas in the continuum velocity limit, i.e., without making a Galerkin truncation in the velocity dimension. This paper misapplies some results from an earlier paper by that author and myself (Ref. 2, hereafter Zhu-Hammett10). In particular, the paper Zhu10 appears to contain some mathematical mistakes (or contains assumptions that are not explained), as I explain below. Unless otherwise specified, all equation and page numbers below will refer to equations in Zhu10.

The heart of the matter is that Zhu10 replaces a singularity that would result from evaluating a Dirac delta function $\delta(u-v)$ at $u = v$ with just $\delta(0) = 1$, which is incorrect. There are at least two different, straightforward ways to see the problem:

(1) Start with his definition of $G(v)$ from the first displayed equation on p.3, at the beginning of Sec. II:

$$G(v) = \int \frac{d^2\mathbf{R}}{2V} g^2(\mathbf{R}, v, t)$$

where V is the integration volume. The function $G(v)$ is a conserved quantity, so we can equivalently replace the RHS with its ensemble-averaged value in statistical equilibrium: $G(v) = \int d^2\mathbf{R}\langle g^2(\mathbf{R}, v)\rangle/(2V)$. Expanding $g(\mathbf{R}, v) = \sum_{\mathbf{k}} \hat{g}(\mathbf{k}, v) \exp(i\mathbf{k} \cdot \mathbf{R})$ gives

$$G(v) = \sum_{\mathbf{k}} \langle \hat{g}(\mathbf{k}, v)\hat{g}^*(\mathbf{k}, v)\rangle/2,$$

Equating this with the definition of the spectral representation $G(v) = \widetilde{\sum}_{\mathbf{k}} G(\mathbf{k}, v) = \sum_{\mathbf{k}} G(\mathbf{k}, v)/2$ (the Fourier conventions are described in more detail in Zhu-Hammett10, but in brief, $\widetilde{\sum}_{\mathbf{k}}$ is a sum over the upper half $\vec{k}$ plane, while $\sum_{\mathbf{k}}$ is a sum over all Fourier modes) leads to

$$G(\mathbf{k}, v) = \langle \hat{g}(\mathbf{k}, v)\hat{g}^*(\mathbf{k}, v)\rangle,$$

Since the correlation function is given by $C(\mathbf{k}, u, v) = \langle \hat{g}(\mathbf{k}, v)\hat{g}^*(\mathbf{k}, u)\rangle$, according to Eq. 13, this means

$$G(\mathbf{k}, v) = C(\mathbf{k}, v, v)$$

However, $C(\mathbf{k}, u, v) = \delta(u-v)/\alpha(v)+...$, as given in Eq. 12, which goes to infinity if one evaluates this at $u = v$ (ignoring for the moment the second term in $C(\mathbf{k}, u, v)$). One gets his Eq. 14 only by ignoring this problem and setting $\delta(v - v) = \delta(0) = 1$, which is incorrect.

(2) Another way to see the problem is to note that his continuous result in Eq. 12 rigorously follows from his discrete result in Eq. 16 (which agrees with our earlier results for the discrete case in Zhu-Hammett10) with the substitution $\alpha_i = \alpha(v_i)m_i = \alpha(v_i)\Delta v_i$ (as he gives after Eq. 16, where $m_i = \Delta v_i$ is the velocity lattice spacing) and taking the continuous limit $\Delta v_i \to 0$. For this to work, as he says, he must use $\delta_{i,j}/\Delta v_i \leftrightarrow \delta(u - v)$, which is correct. But if he were to apply the same prescription to go from the discrete result for $G_i(\mathbf{k})$ in Eq. 18 to the continuous result, then the first term would give $G(\mathbf{k}, v_i) = 1/(\alpha(v_i)\Delta v)$ (if the other terms can be neglected), which blows up as $\Delta v \to 0$ and disagrees with the claim in Eq. 14 that the continuous result is $G(\mathbf{k}, v) = 1/\alpha(v)$.

Zhu10 tries to justify his treatment of the delta functions in Appendix B. That appendix seems to be confusing this case with other cases where delta functions can arise (such as in Fourier spectra for statistically homogeneous fluctuations on an infinite domain), but the problem can always be rigorously reformulated in consistent ways to handle this. There is no justification for introducing additional integrations over infinitesimal regions du, as tried in Appendix B. Something similar to what the author is trying to do might be okay if $G(\mathbf{k}, v)$ was defined by $G(\mathbf{k}, v) = \int_{-\infty}^{\infty} du C(\mathbf{k}, u, v)$ or defined through $C(\mathbf{k}, u, v) = G(\mathbf{k}, v)\delta(u-v) + \ldots$, but neither of these is true.

Taking the continuous limit of the velocity coordinate (i.e., an infinite number of velocity grid points) is physically very subtle. It is similar to trying to take the limit of an infinite number of Fourier modes when calculating absolute equilibrium in fluids, instead of the standard approach of a Galerkin truncation of a finite set of Fourier modes (the standard approach is explained further in Refs. [3 and 4]). The discrete set of velocity grid points in the gyrokinetic calculation of Zhu-Hammett10 is the analog of the finite set of Fourier basis functions in the standard fluid approach. There might be some academic reasons why one would want to consider absolute equilibrium for continuum cases where $G(v)$ is infinite. On the other hand, the $\Delta v \to 0$ continuum limit has been considered while holding $G(v)$ finite, which leads to particular scalings for the α_i and α_0 coefficients as the continuum limit is approached. This is discussed briefly at the end of Appendix D of Zhu-Hammett10. Since $G(v)$ is related to the entropy and generalized free energy of the system, the physical implications of finite vs. infinite $G(v)$ needs to be understood before drawing any strong conclusions. It is not clear how much new is to be learned from the absolute equilibrium of the continuum case relative to what was learned from the previous discrete velocity case. A higher priority seems to be to consider more realistic non-equilibrium turbulence with forcing and dissipation, which introduces natural cutoffs at small scales due to dissipation.

[1] Jian-Zhou Zhu, "Dual cascade and its possible variations in magnetized kinetic plasma turbulence" (2010, v. 2) `http://arxiv.org/abs/1008.0330v2`
[2] Jian-Zhou Zhu and G. W. Hammett, "Gyrokinetic statistical absolute equilibrium and turbulence", Phys. Plasmas **17**, 122307 (2010).
[3] T.-D. Lee, Q. Appl. Math. **10**, 69 (1952).
[4] R. H. Kraichnan and D. Montgomery, Rep. Prog. Phys. **43**, 35 (1980).