The effects of non-white noise on decorrelation in Markovian statistical theories of turbulence.

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Abstract

Markovian models of turbulence are often derived by starting from the renormalized statistical closure equations of the direct-interaction approximation (DIA). Various simplifications are introduced, including an assumption that the two-time correlation function is proportional to the infinitesimal propagator (the Green's function), i.e. that the decorrelation rate is equal to the decay rate for small perturbations. In particular, in the context of a Langevin equation which underlies the realizable class of Markovian models, the assumption that the decorrelation rate and the Green's function decay rate are equal is only strictly valid if the nonlinear driving terms are treated as white noise. Here we show a way to build on previous work on realizable Markovian closures to allow for non-white noise in a more self-consistent way, allowing the decorrelation rate to differ from the decay rate while retaining the computational advantages of a Markovian approximation. While some Markovian approximations differ only in the initial transient phase, the non-white-noise Markovian model presented here will give different steady state spectra as well. The present derivation is restricted to the 1-field 2-dimensional limit but could be generalized. The DIA and related turbulence theories are generic to a class of quadraticly nonlinear equations, and so are relevant to both Navier-Stokes fluid turbulence and plasma turbulence. Markovian models can be used by themselves in studying turbulence issues such as zonal flows, or they may be a useful starting point for deriving sub-grid turbulence models for computer simulations.

I. INTRODUCTION

Our derivation builds on and closely follows the work by Bowman, Krommes, and Ottaviani[1], Phys. Fluids B**5**, 3558 (1993) (we will frequently refer to this paper as BKO), on systematic methods of deriving realizable Markovian closures from Kraichnan's DIA.

Motivation:

- Study zonal flows: strong turbulence theory needed to investigate finiteness of Dimits shift
  (however Dorland and Rogers approach of analyzing secondary and tertiary instabilities may provide a more tractable approach)

- General turbulence interest

- Markovian closures might be a good starting point for deriving sub-grid turbulence models.

In thermodynamic equilibrium, the Fluctuation-Dissipation theorem in fact says that the 2-time correlation function $C(t,t') = \langle \psi(t)\psi^*(t') \rangle$ is proportional to the infinitesimal response function $R(t,t')$. Standard Markovian theories use this to assume that the decorrelation rate and the infinitesimal response decay rate are equal. However, the Kolmogorov spectrum $E_k \sim 1/k^{5/3}$ differs from the thermodynamic equilibrium spectrum. In the context of a Langevin equation, the assumption of $C(t,t') \propto R(t,t')$ restricts the noise to be white.

II. SIMPLE EXAMPLES BASED ON THE LANGEVIN EQUATION

Since realizable Markovian closure approximations to the DIA can be shown to correspond exactly to an underlying set of coupled Langevin equations, the analogy is quite relevant.

Consider the simple Langevin equation

$$\left(\frac{\partial}{\partial t} + \eta\right)\psi = f^*(t) \qquad (1)$$

where η is a damping rate and f^* is a random forcing or stirring term (a.k.a. "noise"). [We use the complex conjugate of f for consistency with the form of the equations used later for the Navier-Stokes/DIA.]

If f is white noise, then the decorrelation rate η_c for ψ is $\eta_c = \eta$.

$$C(t,t') = \langle\psi(t)\psi(t')\rangle = C_0 \exp(-\eta_c|t-t'|)$$

But if $f(t)$ varies slowly compared to η (the "red noise" limit), then

$$\psi(t) \approx \frac{f^*(t)}{\eta}$$

and the decorrelation rate for ψ is the same as the decorrelation rate for f^*, $\eta_c = \eta_f^* \ll \eta$.

Summary of 3 Important Rates:

- η = infinitesimal response function decay rate
$$R(t,t') = \exp(-\eta(t-t'))$$
- η_f = Noise (or forcing) decorrelation rate
$$\langle f(t)f^*(t')\rangle = C_{f0}\exp(-\eta_f(t-t'))$$
- η_c = two-time decorrelation rate ψ
$$C(t,t') = \langle\psi(t)\psi^*(t')\rangle = C_0\exp(-\eta_c(t-t'))$$

One might guess that a simple Padé-type formula to roughly interpolate between the white-noise limit $\eta_f \gg \eta$ and the opposite "red-noise" limit $\eta_f \ll \eta$, would be something like $1/\eta_c \approx 1/\eta + 1/\eta_f^*$, or

$$\eta_c = \frac{\eta\, \eta_f^*}{\eta + \eta_f^*} \tag{2}$$

We will find that the formulas are more complicated in the presence of wave behavior with complex η and η_f.

We note that in many cases of interest, the noise decorrelation rate η_f turns out to be of comparable magnitude to η (for example, if the dominant interactions occur between modes of comparable scale). In this case, while the white-noise approximation is not rigorously valid, the corrections to the decorrelation rate considered in this paper might turn out to be quantitatively modest.

III. DETAILED DEMONSTRATION OF THE NON-WHITE NOISE MARKOVIAN METHOD WITH THE LANGEVIN EQUATION

In this section, we present a more complete demonstration of our approach of a Markovian approximation including the effects of non-white noise, while still starting with a simple Langevin equation. The steps in the derivation are quite similar to the steps that will be taken in the next section for the case of the more complete DIA for more complicated nonlinear problems, and so helps build insight and familiarity. In this section, we will be introducing various approximations that may seem unnecessary for the simple Langevin problem, which can be solved exactly in many cases (for simple forms for the noise correlation function). But these are the same approximations that will be used later in deriving Markovian approximations to the DIA, and so it is useful to be able to test their accuracy in the Langevin case.

The response function (or Green's function or propagator) for the Langevin equation satisfies

$$\left(\frac{\partial}{\partial t} + \eta\right) R(t,t') = \delta(t-t') \tag{3}$$

The solution is

$$R(t,t') = \exp(-\int_{t'}^{t} d\bar{t}\,\eta(t))H(t-t') \tag{4}$$

The solution to the Langevin equation is

$$\psi(t) = R(t,0)\psi(0) + \int_0^t d\bar{t}R(t,\bar{t})f(\bar{t}) \tag{5}$$

In principle it is possible to directly calculate two-time statistics like $C(t,t') = \langle\psi(t)\psi^*(t')\rangle$ from this, but in practice it is often convenient to consider instead the differential equation for $\partial C(t,t')/\partial t$, which from Eq. (1) and the above is

$$\left(\frac{\partial}{\partial t}+\eta\right)C(t,t') = \langle f^*(t)\psi^*(t')\rangle$$

$$= \int_0^{t'} d\bar{t}\, R^*(t',\bar{t})C_f^*(t,\bar{t}), \qquad (6)$$

where the noise correlation function is defined as $C_f(t,t') = \langle(f(t)f^*(t')\rangle$, and we have assumed the initial condition $\psi(0)$ has a random phase. This equation is the analog of the DIA equations (Eqs. (24) and Eqs. (25a)) for the 2-time correlation function, though with an integral only over noise and no nonlinear modification of the damping term.

We define the *equal-time correlation function* $C(t)$ in terms of the *two-time correlation function* $C(t,t')$ as $C(t) = C(t,t) = \langle\psi(t)\psi^*(t')\rangle$ (note that the two different functions are distinguished only by the number of arguments). Then

$$\frac{\partial C(t)}{\partial t} + 2\,\mathrm{Re}\,\eta C(t) = 2\,\mathrm{Re}\int_0^t d\bar{t}R^*(t,\bar{t})C_f^*(t,\bar{t}) \qquad (7)$$

This is the analog of the DIA equal-time covariance equation.

A. Langevin statistics in the steady-state limit

Consider the steady state limit, $t, t' \to \infty$ (but finite time separation $t-t'$), and assume the noise correlation function has the simple form $C_f(t,t') = C_{f0}\exp[-\eta_f(t-t')]$ for $t > t'$. η and η_f are time-independent constants in this section. The response function reduces back to its steady state form $R(t,t') = \exp[-\eta(t-t')]H(t-t')$. Then Eq. (7) in steady state gives $C_0 \doteq \lim_{t\to\infty} C(t) = C_{f0}\,\mathrm{Re}(\eta+\eta_f)/[\mathrm{Re}(\eta)(\eta+\eta_f)(\eta^*+\eta_f^*)]$. Writing $\eta = \nu + i\omega$ and $\eta_f = \nu_f + i\omega_f$ in terms of their real and imaginary components, and denoting the frequency mismatch $\Delta\omega = \omega + \omega_f$ (remember, because the complex conjugate f^* is used as the forcing term, resonance occurs when $\mathrm{Im}(\eta) = \mathrm{Im}(\eta_f^*)$) this can be written as

$$C_0 = \frac{C_{f0}}{\nu} \frac{(\nu + \nu_f)}{(\nu + \nu_f)^2 + (\Delta\omega)^2} \tag{8}$$

This has a familiar form characteristic of resonances.

To find the two-time correlation function, the time integral in Eq. (6) can be evaluated for $t > t'$ to give

$$\left(\frac{\partial}{\partial t} + \eta\right) C(t,t') = \frac{C_{f0}}{\eta^* + \eta_f^*} \exp[-\eta_f^*(t - t')] \tag{9}$$

With the boundary condition $C(t = t', t') = C_0$, this can be solved to give

$$\begin{aligned} C(t,t') = C_0 & \left[1 - \frac{\mathrm{Re}(\eta)(\eta + \eta_f)}{\mathrm{Re}(\eta + \eta_f)(\eta - \eta_f^*)}\right] \exp[-\eta(t - t')] \\ & + C_0 \frac{\mathrm{Re}(\eta)(\eta + \eta_f)}{\mathrm{Re}(\eta + \eta_f)(\eta - \eta_f^*)} \exp[-\eta_f^*(t - t')] \end{aligned} \tag{10}$$

In the white-noise limit, $|\eta_f| \gg |\eta|$, this reduces to the standard simple result $C(t,t') = C_0 \exp[-\eta(t - t')]$. But in the more general case of non-white noise, the two-time correlation function is more complicated. [Despite the apparent singularity in the denominator, it is cancelled by the exponentials so that $C(t,t')$ is well-behaved in the limit $\eta \to \eta_f$.] Even if the noise correlation function has a simple exponential dependence $C_f(t,t') \propto \exp[-\eta_f(t-t')]$, we see that the resulting correlation function for ψ is more complicated.

Consider the task of fitting this $C(t,t')$ with a simpler model of the form $C_{mod}(t,t') = C_0 \exp[-\eta_c(t - t')]$ (for $t > t'$). One way to determine the effective decorrelation rate η_c might be from the area under the time integral,

$$\frac{C_0}{\eta_c} = \int_{-\infty'}^{t} dt'\, C(t,t') \tag{11}$$

This can be evaluated either by directly plugging in Eq. (10), or by taking a time average of Eq. (9); the same answer results either way. It turns out that in the later versions of this calculation it is easier to determine η_c by operating on the governing differential equation with a time integral. Operating on Eq. (9) with $\int_{-\infty}^{t} dt'$, and using

$$\int_{-\infty}^{t} dt' \, \frac{\partial C(t,t')}{\partial t} = \frac{\partial}{\partial t} \int_{-\infty}^{t} dt'\, C(t,t') \, - C(t,t) \tag{12}$$

we can solve to find

$$\frac{1}{\eta_c} = \frac{1}{\eta} + \frac{\mathrm{Re}(\eta)(\eta+\eta_f)}{\mathrm{Re}(\eta+\eta_f)\, \eta\, \eta_f^*} \tag{13}$$

This recovers the white-noise limit $\eta_f \gg \eta$ and the red-noise limit $\eta_f \ll \eta$ discussed in the introduction in Sec II. In the limit of real η and real η_f it simplifies to the Padé approximation $\eta_c = \eta\eta_f/(\eta+\eta_f)$ also suggested in the introduction. However, there is a problem with Eq. (13) related to Galilean invariance. Suppose we make the substitutions $\psi = \hat{\psi} \exp[i\omega_2 t]$ and $f^* = \hat{f}^* \exp[i\omega_2 t]$ into the Langevin equation, Eq. (1). Then it can be written as

$$\left(\frac{\partial}{\partial t} + \hat{\eta} \right) \hat{\psi} = \hat{f}^*(t) \tag{14}$$

where $\hat{\eta} = \eta + i\omega_2$, and so all results should be the same if written in terms of the transformed variables. In particular, the correlation function should transform as $\langle \hat{\psi}(t)\hat{\psi}^*(t')\rangle = \exp[-i\omega_2(t - t')]\langle \psi(t)\psi^*(t')\rangle = \exp[-i\omega_2(t-t')]C(t,t')$. Thus the decorrelation rate $\hat{\eta}_c$ for $\hat{\psi}$ should be related to the decorrelation rate η_c for ψ by $\hat{\eta}_c = \eta_C + i\omega_2$. The decorrelation rate for the transformed noise term $\hat{f}^*$ also transforms as $\hat{\eta}_f^* = \eta_f^* + i\omega_2$. In the case of fluid or plasma turbulence where ψ represents the amplitude of a Fourier mode $\propto \exp[i\boldsymbol{k}\mathbf{x}]$ and f^* represents the amplitude of two modes beating together with $\boldsymbol{k}_2 + \boldsymbol{k}_3 = \boldsymbol{k}$, these transformations correspond to a Galilean transformation to a moving frame $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{v}t$, with $\omega_2 = \boldsymbol{k} \cdot \mathbf{v}$.

So all results should be independent of ω_2 under the transformation $\eta = \hat{\eta} - i\omega_2$, $\eta_f^* = \hat{\eta}_f^* - i\omega_2$, (thus $\eta_f = \hat{\eta}_f + i\omega_2$), $\eta_c = \hat{\eta}_c - i\omega_2$. Eq. (8) satisfies this, but Eq. (13) fails this test. This problem and its solution is described in the review paper by Krommes[2], who shows it is related to other differences in various previous Markovian closures. The problem can be traced to the definition of Eq. (11) which doesn't satisfy the invariance for general forms of $C(t,t')$. For example, we could have multiplied by an arbitrary weight function (such as $\exp[-\omega_2(t-t'))$ before taking the time average and the results would have changed. The way to fix this problem is to do the time-average in a natural frame of reference for ψ that accounts for the frequency dependence. This leads us to the definition:

$$\frac{C_0^2}{\eta_c + \eta_c^*} \doteq \int_{-\infty}^{t} dt'\, C_{mod}^*(t,t')C(t,t') \tag{15}$$

This corresponds to fitting $C_{mod}(t,t')$ to $C(t,t')$ by requiring that both effectively have the same projection onto the function $C_{mod}(t,t')$. [As Krommes[2] points out, using the invariant definition Eq. (15) instead of Eq. (11) is a non-trivial point needed to insure realizability and avoid spurious nonphysical solutions in some cases.]

Operating on Eq. (9) with $\int_{-\infty}^{t} dt'\, C_{mod}^*(t,t')$, using a generalization of Eq. (12), and doing a little rearranging yields

$$\eta_c = \eta - \frac{C_{f0}(\eta_c + \eta_c^*)}{C_0(\eta^* + \eta_f^*)(\eta_c^* + \eta_f^*)} \tag{16}$$

This is properly invariant to the transformation described in the previous paragraph. Solving for η_c while leaving η_c^* on the other side of the equation, eventually leads to

$$\eta_c = \frac{\eta\eta_f^* \operatorname{Re}(\eta + \eta_f) + \eta_c^* \operatorname{Im}(\eta\eta_f)}{(\eta + \eta_c^*) \operatorname{Re}(\eta + \eta_f) + (\eta_f^* + \eta^*) \operatorname{Re}(\eta_f)} \tag{17}$$

Some of the intermediate algebra in deriving this can be quite laborious, so we made significant use of the computer-aided mathematics package Maple[3]. Maple worksheets which prove this and other main results in this paper are available online[4]. If we consider the limit where η, η_f and thus η_c are all real, this simplifies to the form

$$\eta_c = \frac{\eta\eta_f}{\eta + \eta_f + \eta_c} \tag{18}$$

This is similar to (but more accurate than) the rough interpolation formula Eq. (2) suggested in the introduction. This kind of recursive definition, with η_c appearing on both sides, is a common feature of the steady-state limit of theories based on the renormalized DIA equations, and can be solved in practice by iteration, or by considering the time-dependent versions of the theories. In Eq. (18) with real coefficients, one can easily solve this equation for η_c, but the solution is much more difficult in the case of complex coefficients. With the help of Maple, looking at the real and imaginary parts of Eq. (17) separately eventually leads to a quadratic equation and a linear equation to determine the real and imaginary parts of η_c. Unfortunately it takes 16 lines of code to write down the resulting closed-form solution. This is tedious for humans but easy to evaluate in Fortran, C, or other computer language. But this is only helpful for the simple Langevin problem anyway, since direct solution is not really practical for the full nonlinear problem considered by the DIA, where the noise term of the Langevin equation is replaced by a sum over many other modes. In many cases of interest, the noise decorrelation rate η_f turns out to be of comparable magnitude to η, so Eq. (17) usually converges well in a few iterations. The other option is to consider the time-dependent problem, the topic of the next section, which effectively iterates for you as it approach steady state.

B. Comparison of non-white model with exact Langevin result

In this section we compare results from the exact Langevin solution for $C(t,t')/C_0$ given by Eq. (10), with the non-white model $C_{mod}(t,t')/C_0 = \exp(-\eta_c|t-t'|)$, where η_c is given by solving Eq. (17), and with a simple white-noise assumption $C(t,t')/C_0 = \exp(-\eta|t-t'|)$. In general this shows that this formula for η_c does fairly well.

Discuss the normalizations, $C(t,t')/C_0$ is plotted, and the frame of reference is chosen so that $\mathrm{Im}\,\eta = 0$, and the frequency mismatch between the oscillator and the driving term is absorbed into $\mathrm{Im}\,\eta_c$. $\mathrm{Re}\,\eta = 1$ is chosen as a normalization.

All formulas of course agree well in the white noise limit of $\mathrm{Re}\,\eta_c \gg \mathrm{Re}\,\eta$. The non-white model does better, particularly for $\mathrm{Re}\,\eta_c < \mathrm{Re}\,\eta$, and moderate frequency mismatch $\mathrm{Im}(\eta + \eta_c)$. As seen in Fig. 5, the non-white model does have some difficulties with regimes with large frequency mismatch and $\mathrm{Re}\,\eta \sim \mathrm{Re}\,\eta_c$. However, Eq. (8) shows that this non-resonant region corresponds to a very small amplitude C_0, and so perhaps doesn't matter too much. The white-noise model has no imaginary part in all of the figures, and so misses any frequency shifts due to frequency differences between ν and the driving noise term ν_f.

FIGURES

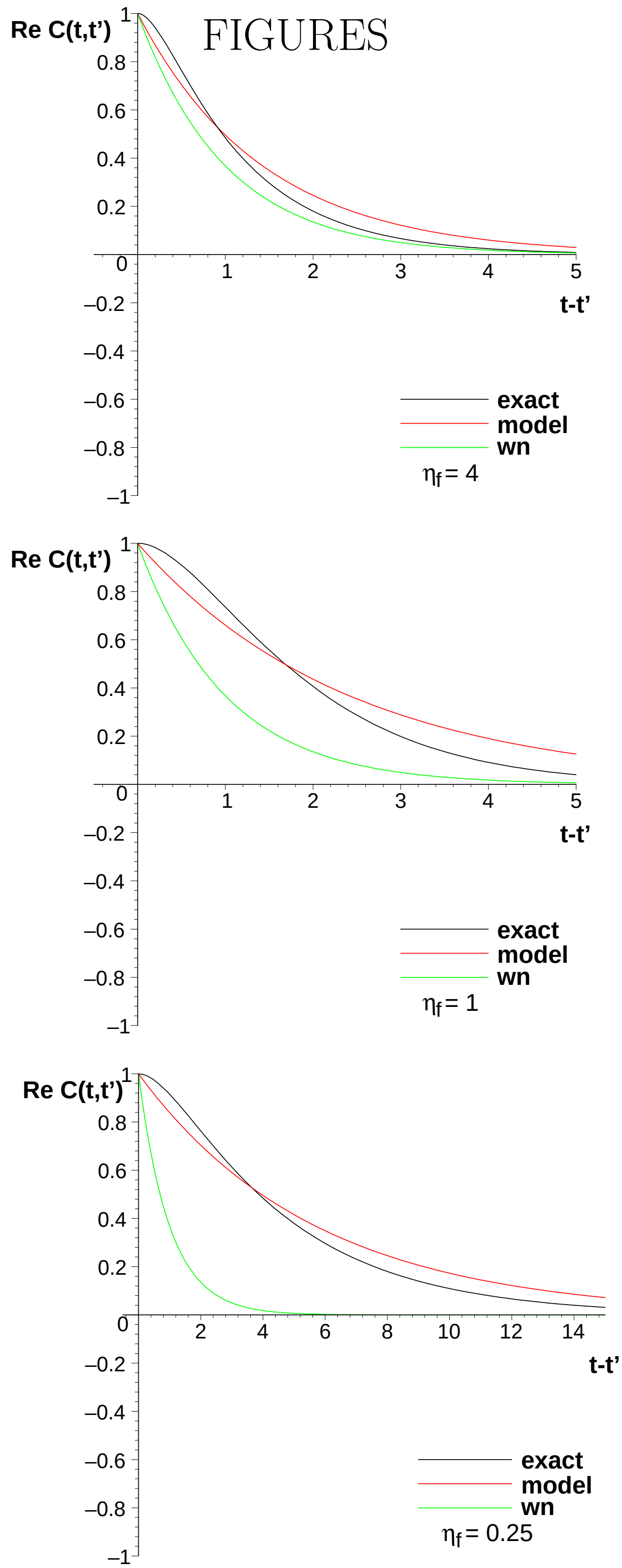

FIG. 1. $\mathrm{Re}\, C(t,t')/C_0$ vs $t-t'$, for the exact Langevin result of Eq. (10), for the non-white-noise model with decorrelation rate η_c given by Eq. (17), and for a simple white-noise assumption $C(t,t') = C_0 \exp(-\eta|t-t'|)$. $\eta = 1$. η_f is noted in each figure.

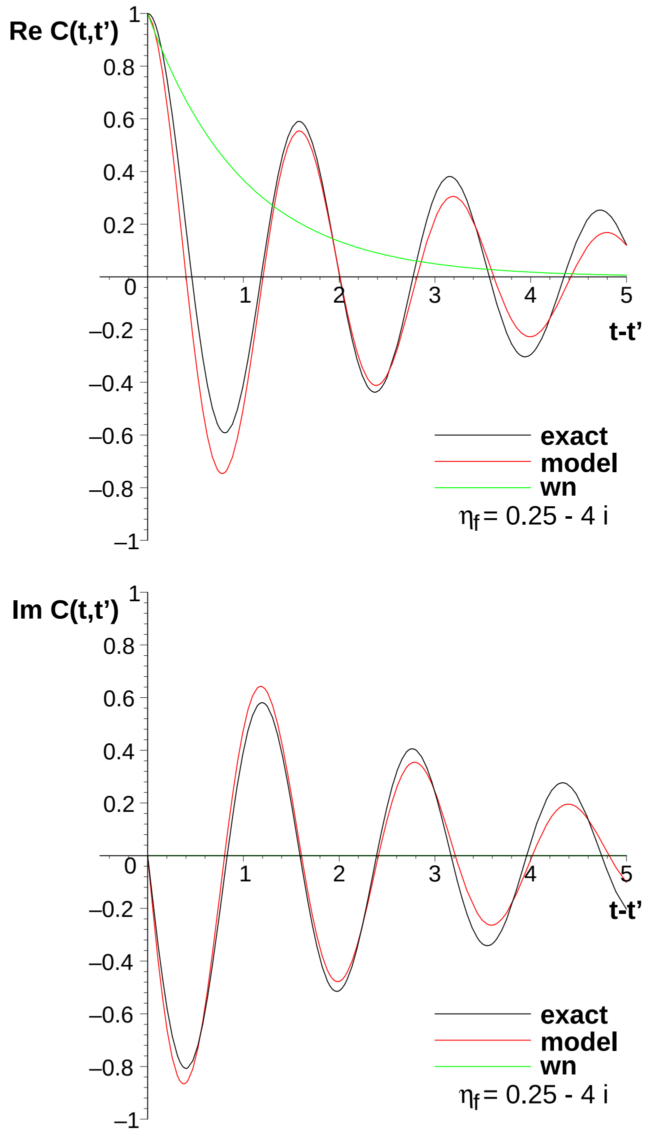

FIG. 2. Re and Im parts of $C(t,t')/C_0$ vs $t-t'$, for the same 3 functions as in Fig. 1, but for $\eta_f = 0.25 - 4i$. Note that $\operatorname{Im} C = 0$ for the white noise case in this and later figures.

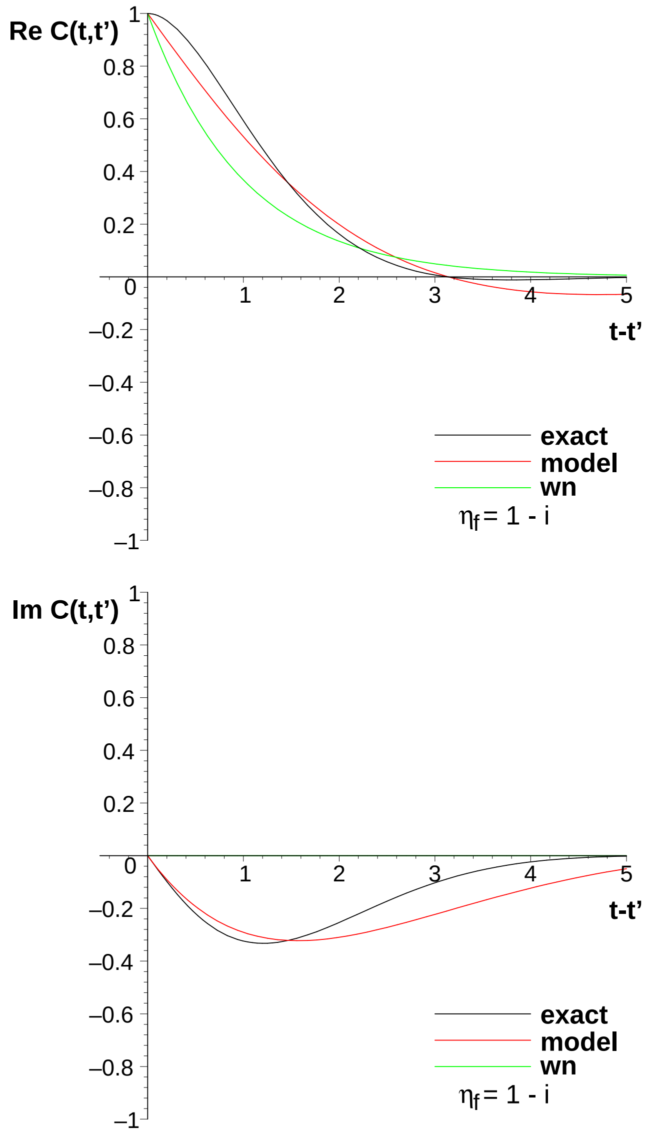

FIG. 3. Re and Im parts of $C(t,t')/C_0$ vs $t-t'$, for the same 3 functions as in Fig. 1, but for $\eta_f = 1 - i$.

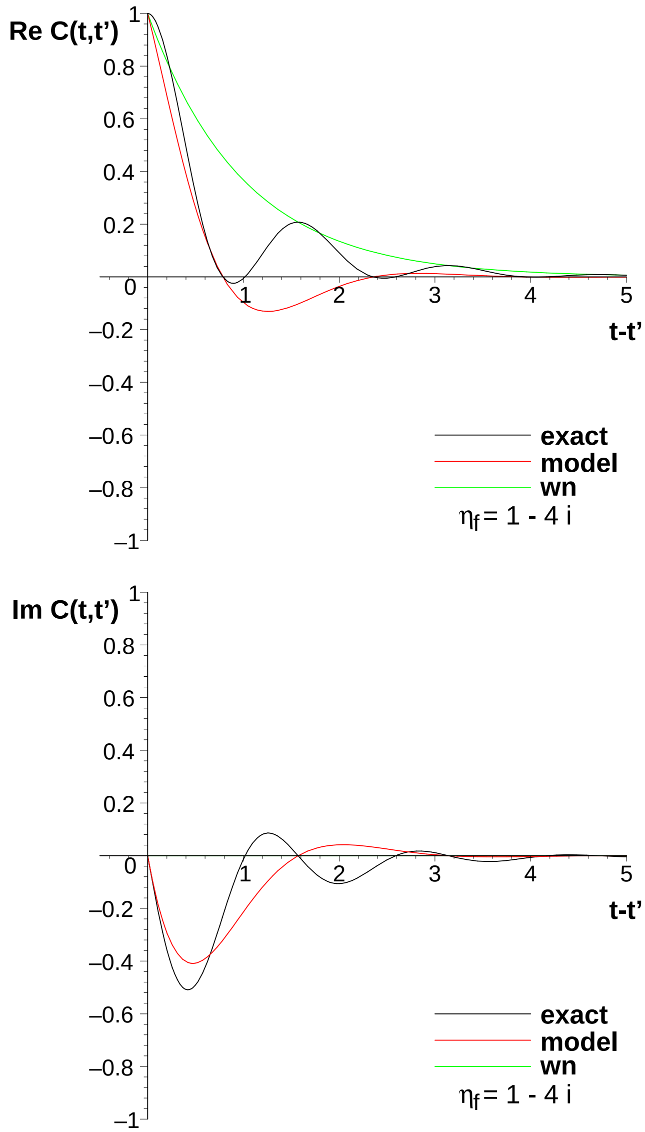

FIG. 4. Re and Im parts of $C(t,t')/C_0$ vs $t-t'$, for the same 3 functions as in Fig. 1, but for $\eta_f = 1-4i$.

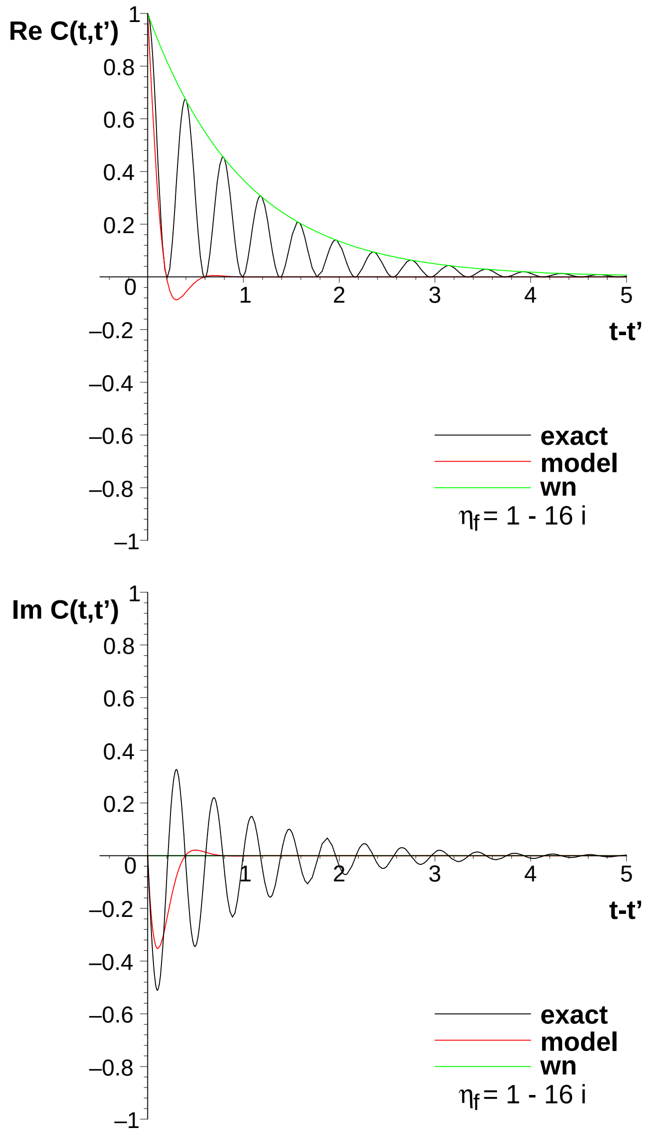

FIG. 5. Re and Im parts of $C(t,t')/C_0$ vs $t-t'$, for the same 3 functions as in Fig. 1, but for $\eta_f = 1 - 16i$.

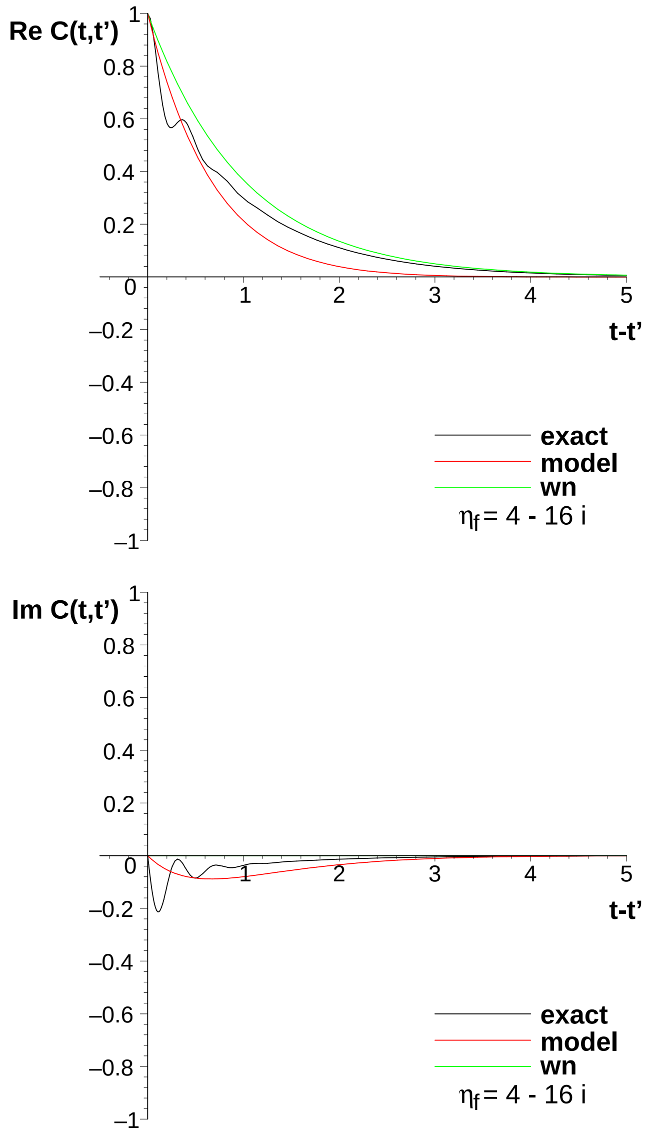

FIG. 6. Re and Im parts of $C(t,t')/C_0$ vs $t-t'$, for the same 3 functions as in Fig. 1, but for $\eta_f = 4 - 16i$.

IV. FORMULATION OF THE FULL NONLINEAR PROBLEM AND STATISTICAL CLOSURES

In this section we provide background on the general form of the nonlinear problem we are considering and on the general theory of statistical closures. In particular we will write down the Kraichnan's direct-interaction approximation, which are the starting point of our calculation. This section borrow's heavily from the BKO paper[1], but is provided for completeness to define our starting point.

A. The fundamental nonlinear stochastic process

Consider a quadratically nonlinear equation, written in Fourier space, for some stochastic variable $\psi_{\boldsymbol{k}}$ that has zero mean:

$$\left(\frac{\partial}{\partial t} + \nu_{\boldsymbol{k}}\right)\psi_{\boldsymbol{k}}(t) = \tfrac{1}{2} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}}\psi^*_{\boldsymbol{p}}(t)\psi^*_{\boldsymbol{q}}(t). \tag{19}$$

Here the *time-independent* coefficients of linear "damping" $\nu_{\boldsymbol{k}}$ and mode-coupling $M_{\boldsymbol{kpq}}$ are complex.

For each $\boldsymbol{k}$ in Eq. (19), the summation on the right-hand-side involves a sum over all possible $\boldsymbol{p}$ and $\boldsymbol{q}$ that satisfy the three-wave interaction $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q} = 0$ (this is sometimes expressed as $\boldsymbol{k} = \boldsymbol{k}_2+\boldsymbol{k}_3$, but the reality conditions $\psi_{-\boldsymbol{k}} = \psi^*_{\boldsymbol{k}}$ has been used to rearrange it). Without any loss of generality one may assume the symmetry

$$M_{\boldsymbol{kpq}} = M_{\boldsymbol{kqp}}. \tag{20}$$

Another important symmetry possessed by many such systems is

$$\sigma_{\boldsymbol{k}}M_{\boldsymbol{kpq}} + \sigma_{\boldsymbol{p}}M_{\boldsymbol{pqk}} + \sigma_{\boldsymbol{q}}M_{\boldsymbol{qkp}} = 0 \tag{21}$$

for some time-independent nonrandom *real* quantity $\sigma_{\boldsymbol{k}}$. Equation (21) is easily shown to imply that the nonlinear terms of Eq. (19) conserve the total generalized "energy," defined as

$$E \doteq \tfrac{1}{2}\sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}}|\psi_{\boldsymbol{k}}(t)|^2. \tag{22}$$

For some problems (e.g., two-dimensional turbulence), Eq. (21) may be satisfied by more than one choice of $\sigma_{\boldsymbol{k}}$; this implies the existence of more than one nonlinear invariant.

We define the *two-time correlation function* $C_{\boldsymbol{k}}(t,t') \doteq \left\langle \psi_{\boldsymbol{k}}(t)\psi^*_{\boldsymbol{k}}(t')\right\rangle$ and the *equal-time correlation function* $C_{\boldsymbol{k}}(t) \doteq C_{\boldsymbol{k}}(t,t)$ (note that the two different functions are distinguished only by the number of arguments), so that $E = \frac{1}{2}\sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} C_{\boldsymbol{k}}(t)$. In stationary turbulence, the two-time correlation function depends on only the difference of its time arguments: $C_{\boldsymbol{k}}(t,t') \doteq \mathcal{C}_{\boldsymbol{k}}(t-t')$ The renormalized *infinitesimal response function* (nonlinear Green's function) $R_{\boldsymbol{k}}(t,t')$ is the ensemble-averaged infinitesimal response to a source function $\bar{\eta}_{\boldsymbol{k}}(t) = \delta\bar{\eta}_k\delta(t-t')$ added to the right-hand side of Eq. (19) for mode $\boldsymbol{k}$ (and no source added to other modes with $\boldsymbol{p} \neq \boldsymbol{k}$). As a functional derivative,

$$R_{\boldsymbol{k}}(t,t') \doteq \left\langle \frac{\delta\psi_{\boldsymbol{k}}(t)}{\delta\bar{\eta}_{\boldsymbol{k}}(t')}\right\rangle \Big|_{\bar{\eta}_{\boldsymbol{k}}=0}. \tag{23}$$

We adopt the convention that the equal-time response function $R_{\boldsymbol{k}}(t,t)$ evaluates to $1/2$ [although $\lim_{t'\to t-} R_{\boldsymbol{k}}(t,t') = 1$].

B. Statistical closures; the direct-interaction approximation

The starting point of our derivation will be the equations of Kraichnan's direct-interaction approximation (DIA), as given in Eqs.(6-7) of BKO[1], and reproduced below as Eq. (24a), Eq. (24b), Eq. (25a), Eq. (25b).

The general form of a statistical closure in the absence of mean fields is

$$\left(\frac{\partial}{\partial t}+\nu_{\boldsymbol{k}}\right)C_{\boldsymbol{k}}(t,t')+\int_0^t d\bar{t}\,\Sigma_{\boldsymbol{k}}(t,\bar{t})C_{\boldsymbol{k}}(\bar{t},t')$$
$$=\int_0^{t'} d\bar{t}\,\mathcal{F}_{\boldsymbol{k}}(t,\bar{t})R^*_{\boldsymbol{k}}(t',\bar{t}), \tag{24a}$$

$$\left(\frac{\partial}{\partial t}+\nu_{\boldsymbol{k}}\right)R_{\boldsymbol{k}}(t,t')+\int_{t'}^t d\bar{t}\,\Sigma_{\boldsymbol{k}}(t,\bar{t})R_{\boldsymbol{k}}(\bar{t},t')$$
$$=\delta(t-t'). \tag{24b}$$

These equations specify an initial-value problem for which $t=0$ is the initial time.

The original nonlinearity in Eq. (19) is split in Eqs. (24) into two separate effects: one describing nonlinear damping ($\Sigma_{\boldsymbol{k}}$) and one modeling nonlinear noise ($\mathcal{F}_{\boldsymbol{k}}$). This structure is reminiscent of a Langevin equation. However, the nonlinear damping and noise in Eqs. (24) are determined on the basis of fully nonlinear statistics. Given appropriate forms for $\Sigma_{\boldsymbol{k}}$ and $\mathcal{F}_{\boldsymbol{k}}$, Eqs. (24) would yield an *exact* description of the second-order statistics. Unfortunately, this merely shifts the difficulty to the determination of these new functions.

The direct-interaction approximation provides specific *approximate* forms for $\Sigma_{\boldsymbol{k}}$ and $\mathcal{F}_{\boldsymbol{k}}$:

$$\Sigma_{\boldsymbol{k}}(t,\bar{t})=-\sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}}M^*_{\boldsymbol{pqk}}R^*_{\boldsymbol{p}}(t,\bar{t})C^*_{\boldsymbol{q}}(t,\bar{t}), \tag{25a}$$

$$\mathcal{F}_{\boldsymbol{k}}(t,\bar{t})=\tfrac{1}{2}\sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2 C^*_{\boldsymbol{p}}(t,\bar{t})C^*_{\boldsymbol{q}}(t,\bar{t}). \tag{25b}$$

These renormalized forms can be obtained from the formal perturbation series by retaining only selected terms. While there are infinitely many ways of obtaining a renormalized expression, Kraichnan[5] has shown that most of the resulting closed systems of equations lead to physically unacceptable solutions. For example, they might predict the physically impossible situation of a negative value for $C_{\boldsymbol{k}}(t,t)$ (i.e., a negative energy)! Such behavior cannot occur in the DIA or other realizable closures.

V. SUMMARY OF A NON-WHITE MARKOVIAN CLOSURE

Applying these techniques (demonstrated earlier for a Langevin equation) in a straightforward way to the time-dependent DIA equations leads to a Markovian Closure model to include the effects of non-white noise on the decorrelation rate. The resulting Non-White Markovian Closure equations are

$$\frac{\partial}{\partial t}C_{\boldsymbol{k}}(t) + 2\,\mathrm{Re}\,\bar{\eta}_{\boldsymbol{k}}(t)\,C_{\boldsymbol{k}}(t) = 2\,\mathrm{Re}\,F_{\boldsymbol{k}}(t), \tag{26a}$$

$$\bar{\eta}_{\boldsymbol{k}}(t) \doteq \nu_{\boldsymbol{k}} - \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} M_{\boldsymbol{kpq}}M^*_{\boldsymbol{pqk}}\Theta^*_{\boldsymbol{pqk}}(t)\,C_{\boldsymbol{q}}^{1/2}(t)C_{\boldsymbol{k}}^{-1/2}(t), \tag{26b}$$

$$F_{\boldsymbol{k}}(t) \doteq \tfrac{1}{2}\,\mathrm{Re}\sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} |M_{\boldsymbol{kpq}}|^2\Theta^*_{\boldsymbol{kpq}}(t)\,C_{\boldsymbol{p}}^{1/2}(t)\,C_{\boldsymbol{q}}^{1/2}(t), \tag{26c}$$

$$\frac{\partial}{\partial t}\Theta_{\boldsymbol{kpq}} + (\eta_{\boldsymbol{k}} + \eta_{c\boldsymbol{p}} + \eta_{c\boldsymbol{q}})\Theta_{\boldsymbol{kpq}} = C_p^{1/2}(t)C_q^{1/2}(t), \tag{26d}$$

$$\Theta_{\boldsymbol{kpq}}(0) = 0. \tag{26e}$$

This is very similar to the Bowman-Krommes-Ottaviani Realizable Markovian Closure (RMC) (as given by Eqs. (66a-e) of BKO[1]), but with the replacement of the single decay/decorrelation rate of RMC with 3 different rates in these equations. [Other Markovian models, such as Orszag's EDQNM also use a single decorrelation rate parameter.] If in Eq. (26d) we replace $\eta_{\boldsymbol{k}} = \bar{\eta}_{\boldsymbol{k}}$, $\eta_{c\boldsymbol{p}} = \mathcal{P}(\bar{\eta}_{\boldsymbol{p}})$, and $\eta_{c\boldsymbol{q}} = \mathcal{P}(\bar{\eta}_{\boldsymbol{q}})$, then these equations become identical to the RMC.

To summarize the 3 rates used here:

$\bar{\eta}_{\boldsymbol{k}}$ is the nonlinear energy damping rate for the wave energy equation for the equal time covariance $C_k(t)$ in Eq. (26a), and is defined in Eq. (26b),

$\eta_{\boldsymbol{k}}$ is the infinitesimal response function decay rate for $R_k(t,t')$, and $\eta_{c\boldsymbol{k}}$ is the decorrelation rate for $C_k(t,t')$.

In general there are time-dependent equations involving additional functions similar in form to $\Theta_{\boldsymbol{kpq}}$ that are used to define these additional rates. But in the steady-state limit, these equations simplify to $\eta_{\boldsymbol{k}} = \bar{\eta}_{\boldsymbol{k}}$ and

$$\eta_{c\boldsymbol{k}} \doteq \bar{\eta}_{\boldsymbol{k}} - (\eta_{c\boldsymbol{k}} + \eta^*_{c\boldsymbol{k}}) \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} \frac{M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} C_q}{(\eta^*_{c\boldsymbol{k}} + \eta^*_{\boldsymbol{p}} + \eta^*_{c\boldsymbol{q}})^2}$$
$$- \frac{(\eta_{c\boldsymbol{k}} + \eta^*_{c\boldsymbol{k}})}{2C_k} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} \frac{|M_{\boldsymbol{kpq}}|^2 C_p C_q}{(\eta^*_{c\boldsymbol{k}} + \eta^*_{c\boldsymbol{p}} + \eta^*_{c\boldsymbol{q}})(\eta^*_{\boldsymbol{k}} + \eta^*_{c\boldsymbol{p}} + \eta^*_{c\boldsymbol{q}})} \tag{27}$$

Thus the decorrelation rate $\eta_{c\boldsymbol{k}}$ equals the infinitesimal decay rate $\bar{\eta}_k$ plus two correction terms. For the simple fluid case with real and positive η's, the first correction term is usually positive, while the second term is usually negative.

VI. CONCLUSIONS

The NWMC extension of the RMC includes the effects of non-white noise at the expense of introducing a few new parameters, but the computational scaling of this system is still $\mathrm{O}(N_t)$, a vast improvement over the $\mathrm{O}(N_t^3)$ scaling of the DIA.

An alternate name for this Non-White Markovian Closure might be the Colored-Noise Markovian Closure, since instead of white-noise we are able keep the effects of a noise spectrum of width $\delta\omega \sim \mathrm{Re}\,\eta_f$ centered around the frequency $\omega \sim \mathrm{Im}\,\eta_f$ (i.e., this models a range of possible colored spectra).

To clarify, the Bowman-Krommes-Ottaviani Realizable Markovian Closure does include some of the effects of non-white noise and, for example, properly includes these effects when calculating the triad interaction time $\theta_{\boldsymbol{kpq}} \approx 1/(\eta_{\boldsymbol{k}} + \eta_{\boldsymbol{p}} + \eta_{\boldsymbol{q}})$. However, the assumption that the decorrelation rate is identical to the infinitesimal response decay rate, i.e., that $C_k(t,t') \propto R_k(t,t')$ (for $t > t'$), is rigorously correct only in the white noise limit, though it may not be too bad of an approximation in many regimes. $C_k(t,t') \propto R_k(t,t')$ is also a rigorous result of the Fluctuation-Dissipation Theorem in thermodynamic equilibrium. [Its interesting to note that the F-D theorem holds even if $R_k(t,t')$ is not a simple exponential function $\exp(-\eta_k(t-t'))$]. However, the Non-White Markovian Closure might still reproduce the Fluctuation-Dissipation result, because in thermodynamic equilibrium the spectrum $E_k \sim 1/k$ falls off slower than the usual Kolmogorov turbulence spectrum $E_k \sim 1/k^{5/3}$, and thus might cause the noise terms such as Eq. (26c) to be dominated by high $\boldsymbol{q}$ modes (beating with modes with high $\boldsymbol{p} = -(\boldsymbol{q} + \boldsymbol{k})$) which look like white noise on the time scale of modes with small $\boldsymbol{k}$.

REFERENCES

[1] J. C. Bowman, J. A. Krommes, and M. Ottaviani, Phys. Fluids **B5**, 3558 (1993).

[2] In particular, in footnote 185 of Sec. 7.2.1 (near p. 176) of Ref.[6], Krommes shows how to get the Galilean invariant form.

[3] Maple, a computer package for symbolic mathematics, www.maplesoft.com.

[4] Maple scripts used to obtain these results are available from the author.

[5] R. H. Kraichnan, J. Math. Phys. **2**, 124 (1961).

[6] John A. Krommes, "Fundamental statistical descriptions of plasma turbulence in magnetic fields", Manuscript in preparation, 2000.