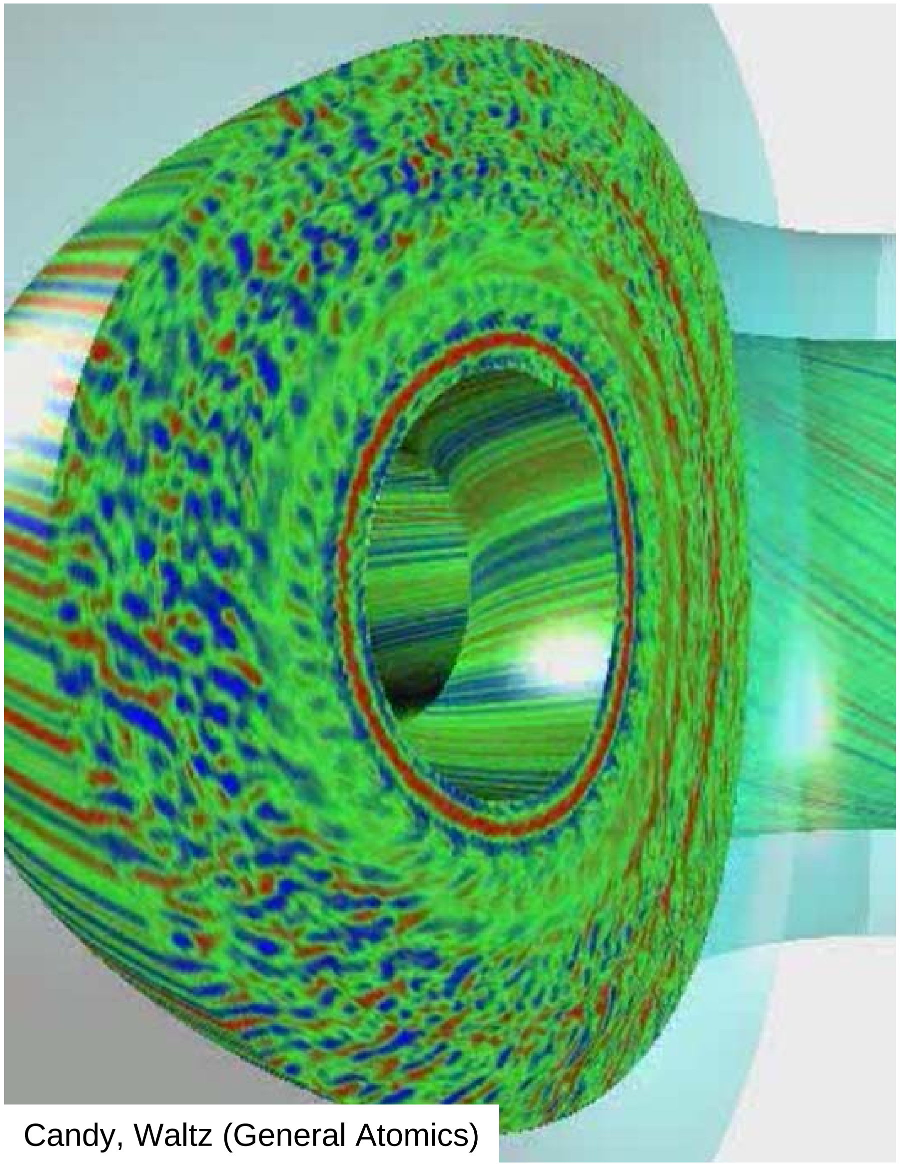
Heat Transport in Tokamaks (I): The Bad-curvature ITG Instability



Candy, Waltz (General Atomics)

Greg Hammett
Princeton Plasma Physics Lab (PPPL)
w3.pppl.gov/~hammett

CMPD/CMSO Winter School
UCLA, Jan. 2009

I. **Brief Fusion Status Report**

II. **Physical picture of effective-gravity / bad-curvature instabilities in toroidal magnetic fields, based on inverted-pendulum and Rayleigh-Taylor analogies.**

III. **Detailed linear calculation of Ion Temperature Gradient instability from drift/gyro-kinetic Eq.**

The Center For Plasma Microturbulence Studies

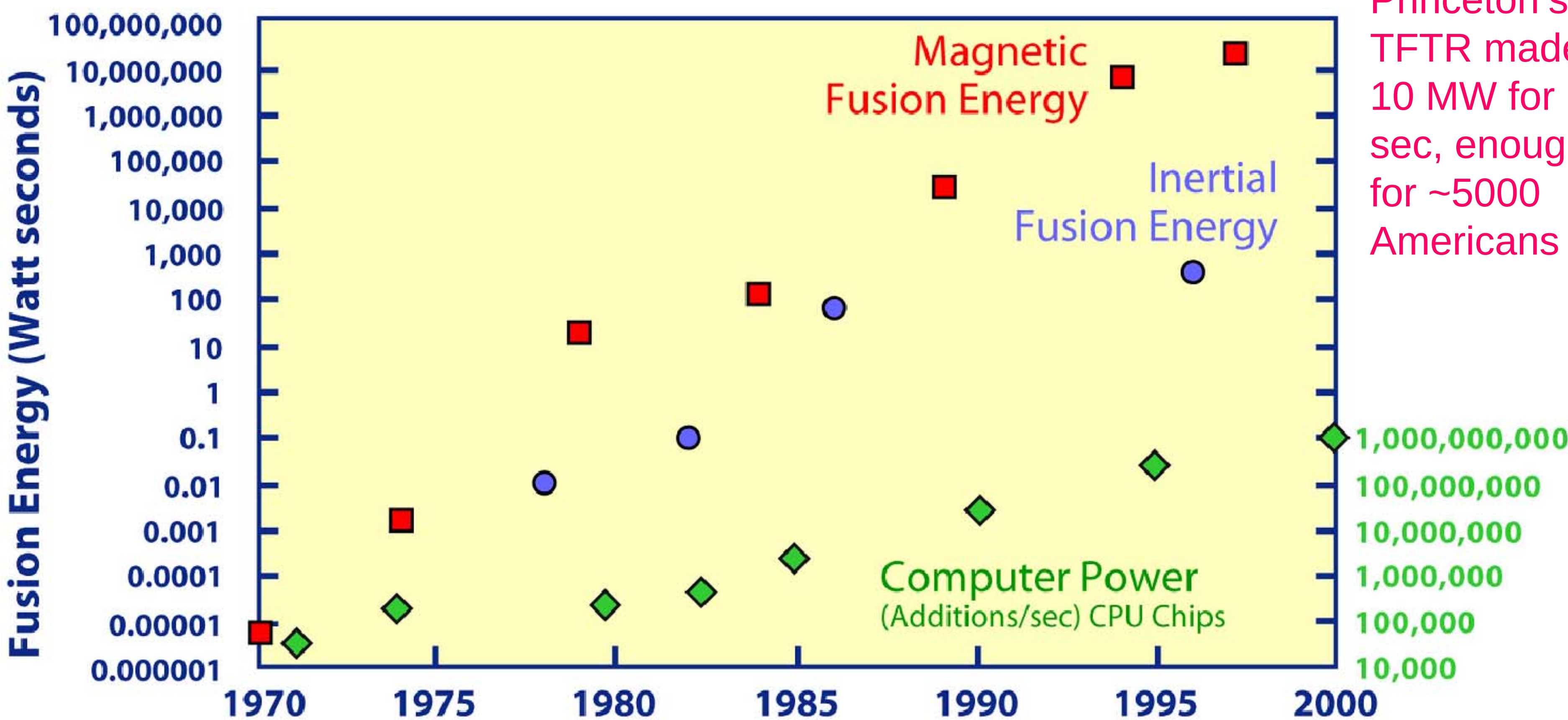
- A DOE, Office of Fusion Energy Sciences, SciDAC (Scientific Discovery Through Advanced Computing) project

- devoted to studying plasma microturbulence through direct numerical sumulation

- National Team (& 3 main codes):
  - GA (Waltz, Candy)
  - U. MD (Dorland)
  - U. CO (Parker, Chen)
  - LLNL (Nevins P.I.)
  - PPPL (Hammett, Mikkelsen, Rewoldt …)
  - MIT

- They've done all the hard work …

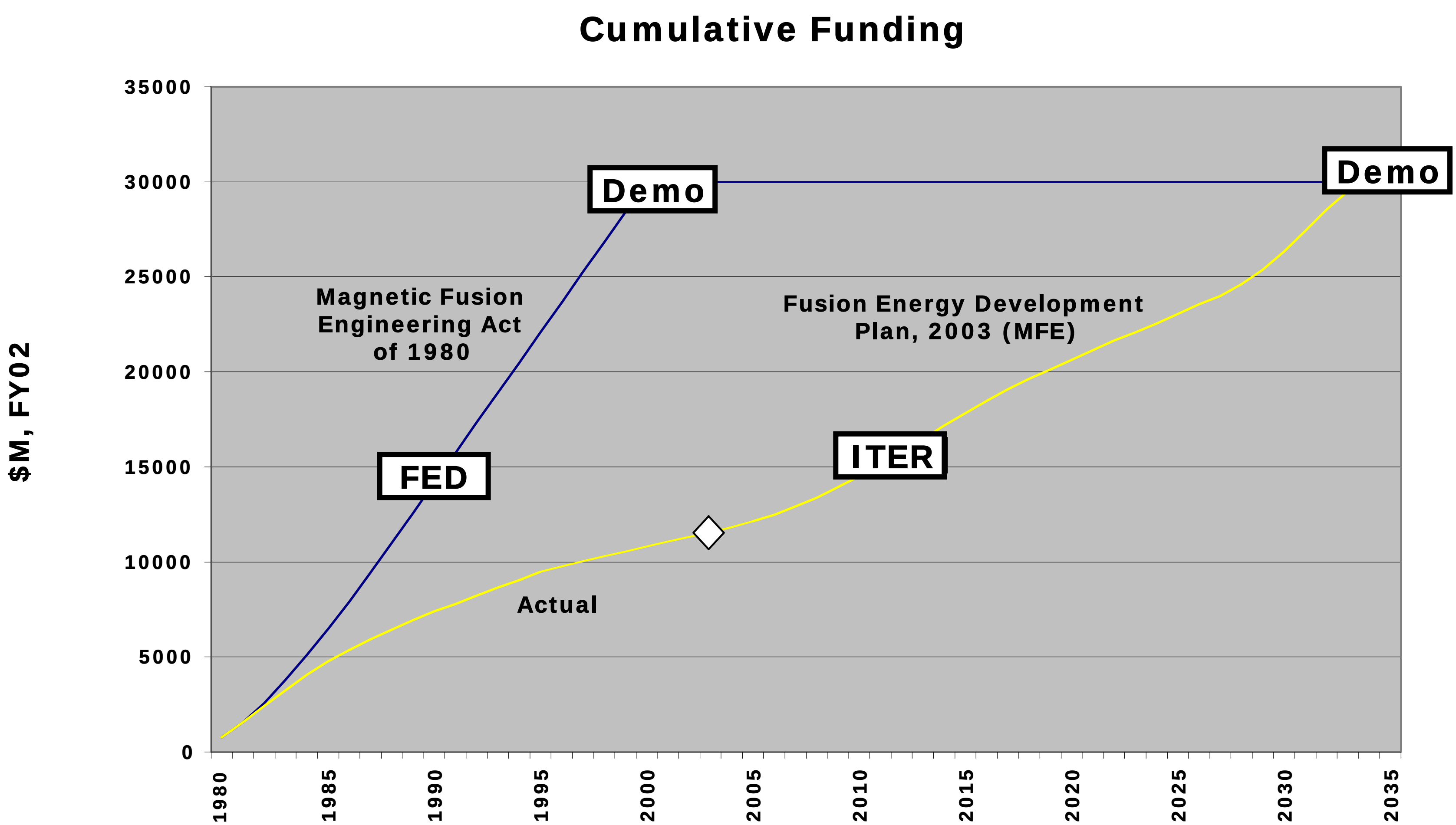
Progress in Fusion Energy has Outpaced Computer Speed



Princeton's TFTR made 10 MW for 1 sec, enough for ~5000 Americans

Some of the progress in computer speed can be attributed to plasma science.

The Estimated Development Cost for Fusion Energy is Essentially Unchanged since 1980



On budget,
if not on time.

$30B development cost tiny compared to >$100 Trillion energy needs of 21st century and potential costs of global warming. Still 40:1 payoff after discounting 50+ years.

based on slide from R.J. Goldston

Need to aggressively pursue a portfolio of alternative energy in the near term (10-30 years)

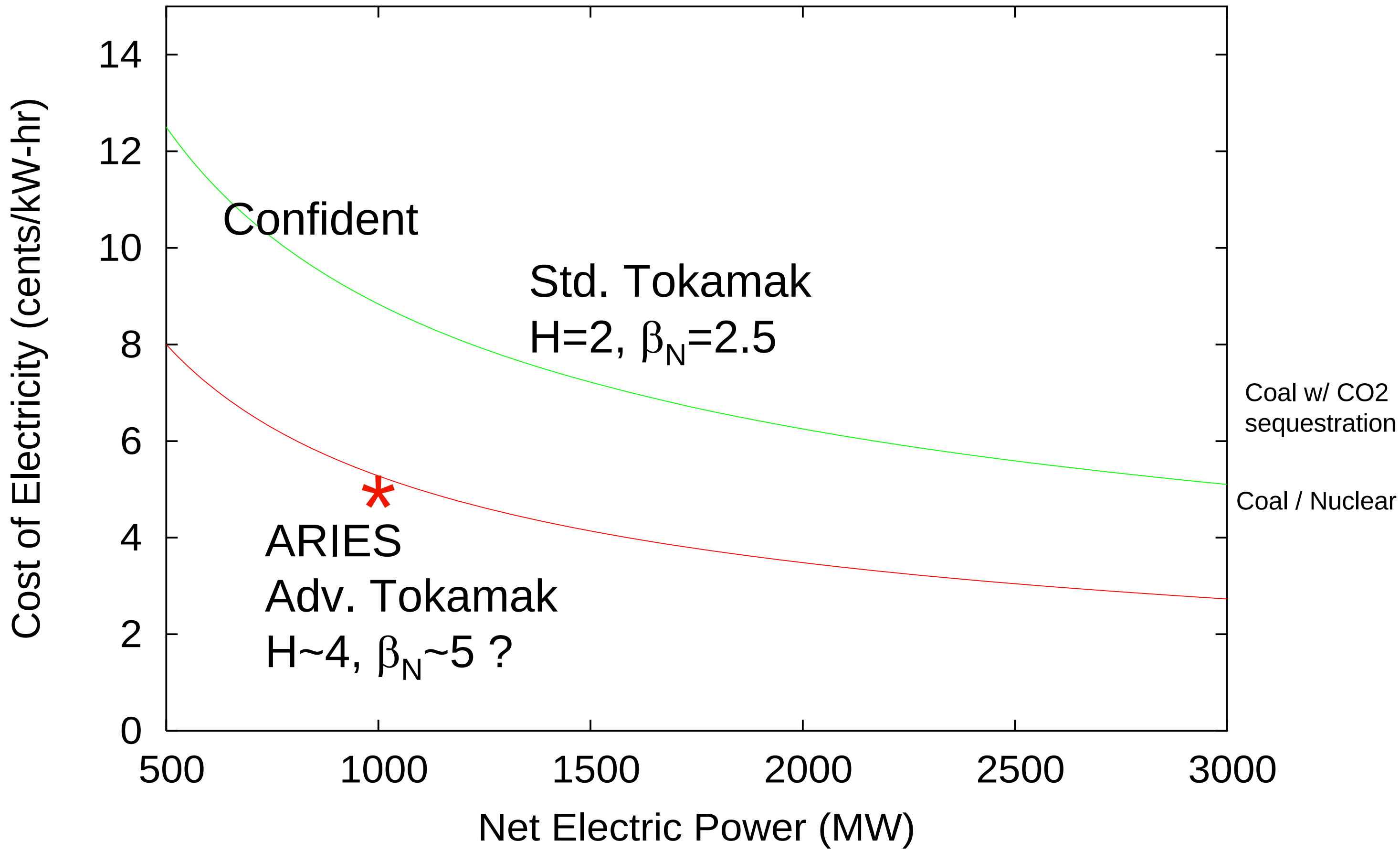
Needed to deal with global warming & energy independence economic issues

- improved building & transportation efficiency
- plug-in hybrid, CNG, vehicles
- wind power
- concentrated solar
- clean coal with CO_2 sequestration
- synfuels+biomass with CO_2 sequestration
- fission nuclear power plants
- …

However, there are uncertainties about all of these energy sources: cost, quantity, intermittency, side-effects. How much CO_2 can be stored underground long term, and at what cost? Energy demand in the developing world will continue to grow throughout this century.

Long term, still need something like fusion energy, or fission breeder reactors, or ?

↓ turbulence & ↑ β could significantly improve fusion



Can't just go to arbitrarily large reactor sizes: Heat flux to wall & Greenwald density limit get worse

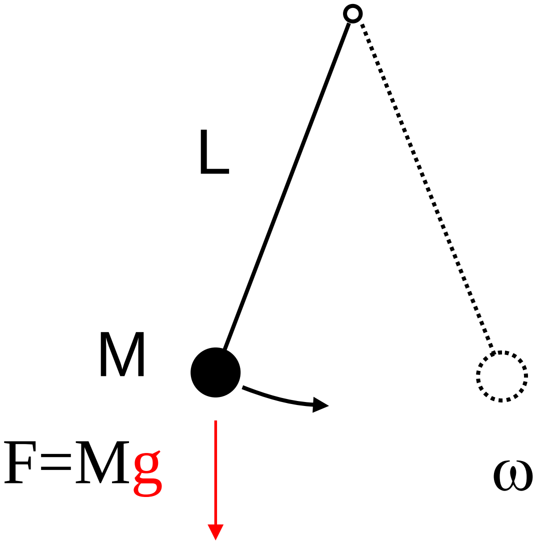
(Relative Cost of Electricity (COE) estimates in this study, see ARIES reactor studies for more detailed & lower costs estimates.)

From Galambos, Perkins, Haney, & Mandrekas 1995 Nucl.Fus. (very good), scaled to match ARIES-AT reactor design study (2001), http://aries.ucsd.edu/ARIES/

Part 1: Intuitive picture of the ITG instability

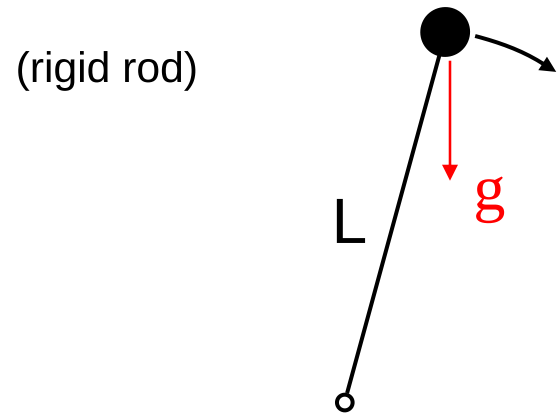
-- based on analogy with Inverted pendulum / Rayleigh-Taylor instability

Stable Pendulum



$\omega=(g/L)^{1/2}$

Unstable Inverted Pendulum

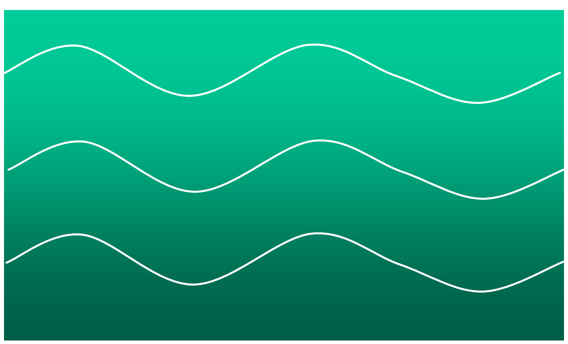


$\omega= (-g/|L|)^{1/2} = i(g/|L|)^{1/2} = i\gamma$

Instability

Density-stratified Fluid

$\rho=\exp(-y/L)$

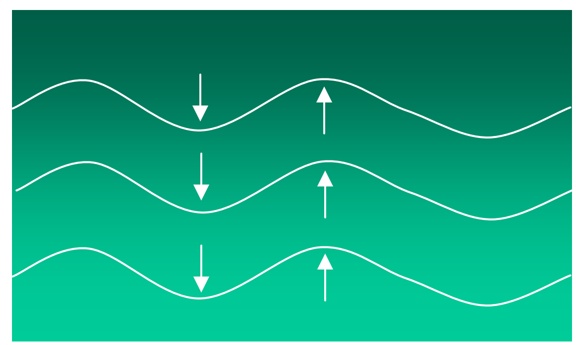


stable $\omega=(g/L)^{1/2}$

Inverted-density fluid

$\Rightarrow$Rayleigh-Taylor Instability

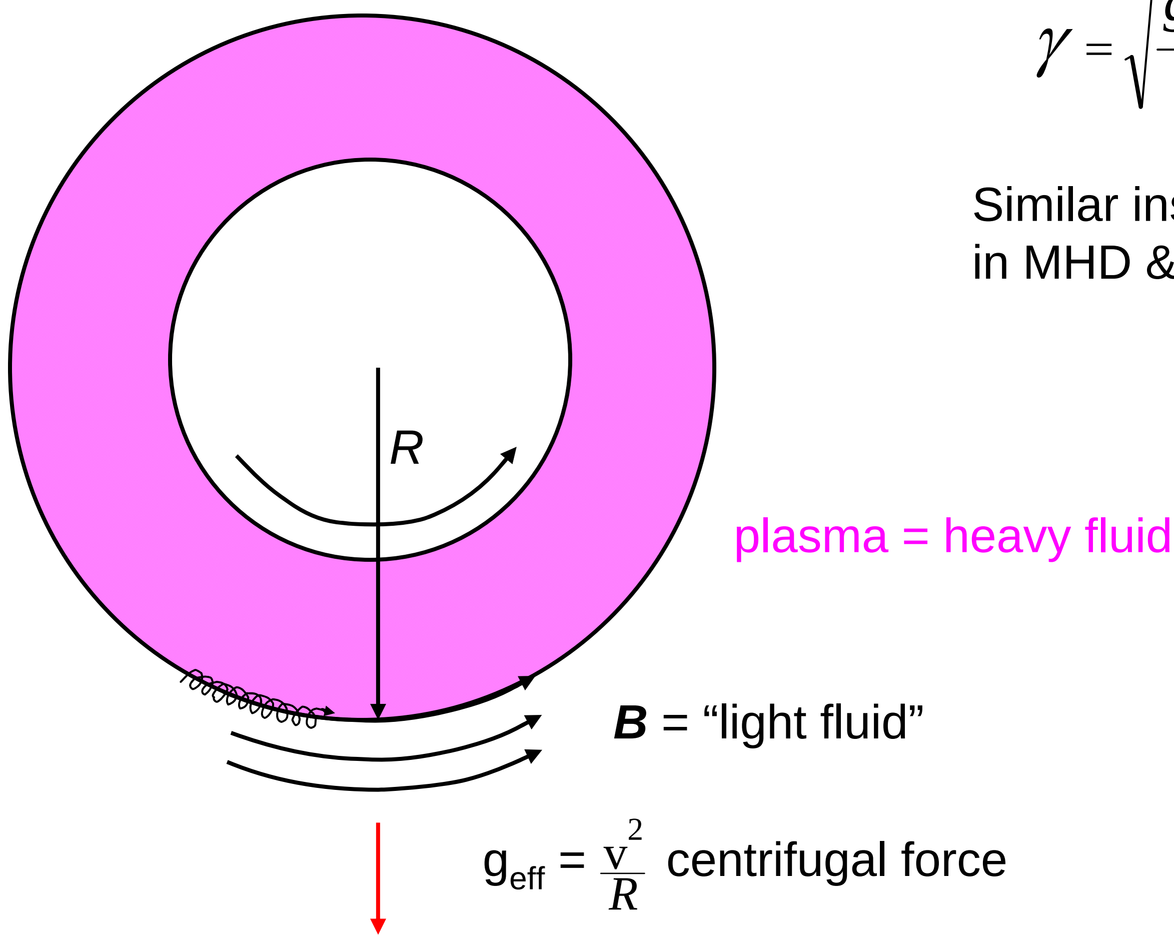
$\rho=\exp(y/L)$



Max growth rate $\gamma=(g/L)^{1/2}$

"Bad Curvature" instability in plasmas ≈ Inverted Pendulum / Rayleigh-Taylor Instability

Top view of toroidal plasma:



Growth rate:

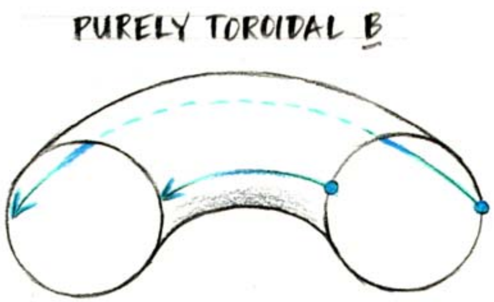
$$\gamma = \sqrt{\frac{g_{eff}}{L}} = \sqrt{\frac{\mathbf{v}_t^2}{RL}} = \frac{\mathbf{v}_t}{\sqrt{RL}}$$

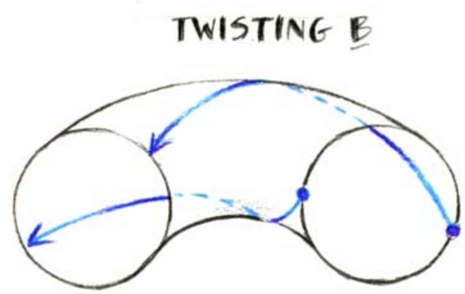
Similar instability mechanism in MHD & drift/microinstabilities

$1/L = \nabla p/p$ in MHD,
$\propto$ combination of ∇n & ∇T in microinstabilities.

The Secret for Stabilizing Bad-Curvature Instabilities

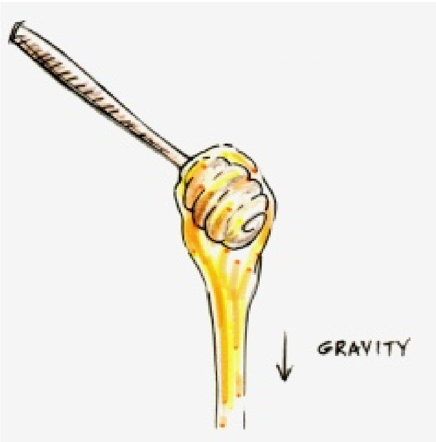
Twist in ***B*** carries plasma from bad curvature region to good curvature region:

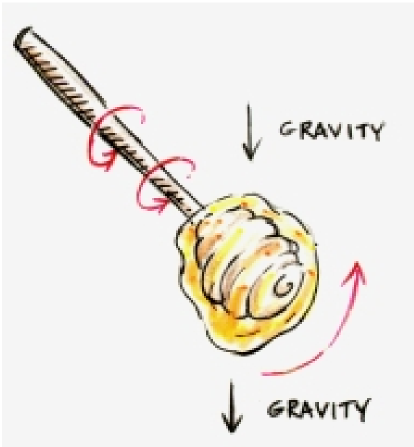




Unstable

Stable





Similar to how twirling a honey dipper can prevent honey from dripping.

Twist in B stabilizes unless

$$\boxed{\text{growth rate in bad-curvature region} > \text{propagation from bad-curvature to good curvature regions}}$$

MHD works well to lowest order in plasmas, so RHS $\Rightarrow$

$$\frac{v_t}{\sqrt{RL}} > k_\parallel v_A \sim \frac{v_A}{qR}$$

Square:
$$\frac{v_t^2 \, q^2 R^2}{v_A^2 \, RL} > 1$$

$$\text{LHS} = \frac{\beta}{2} \frac{q^2 R}{L} = \frac{1}{2} q^2 R \left| \frac{\partial \beta}{\partial r} \right| = \frac{1}{2} \alpha_{MHD}$$

While MHD works well to lowest order in plasmas, there are next-order FLR corrections that defrost the magnetic field & allow $E_{\parallel} \neq 0$ & allow the plasma to move separately from $\underline{B}$.

Still have sound waves that can connect good & bad curvature region. Unstable if:

$$\gamma > \text{connection rate}$$

$$\frac{v_t}{\sqrt{RL}} > \frac{v_t}{qR}$$

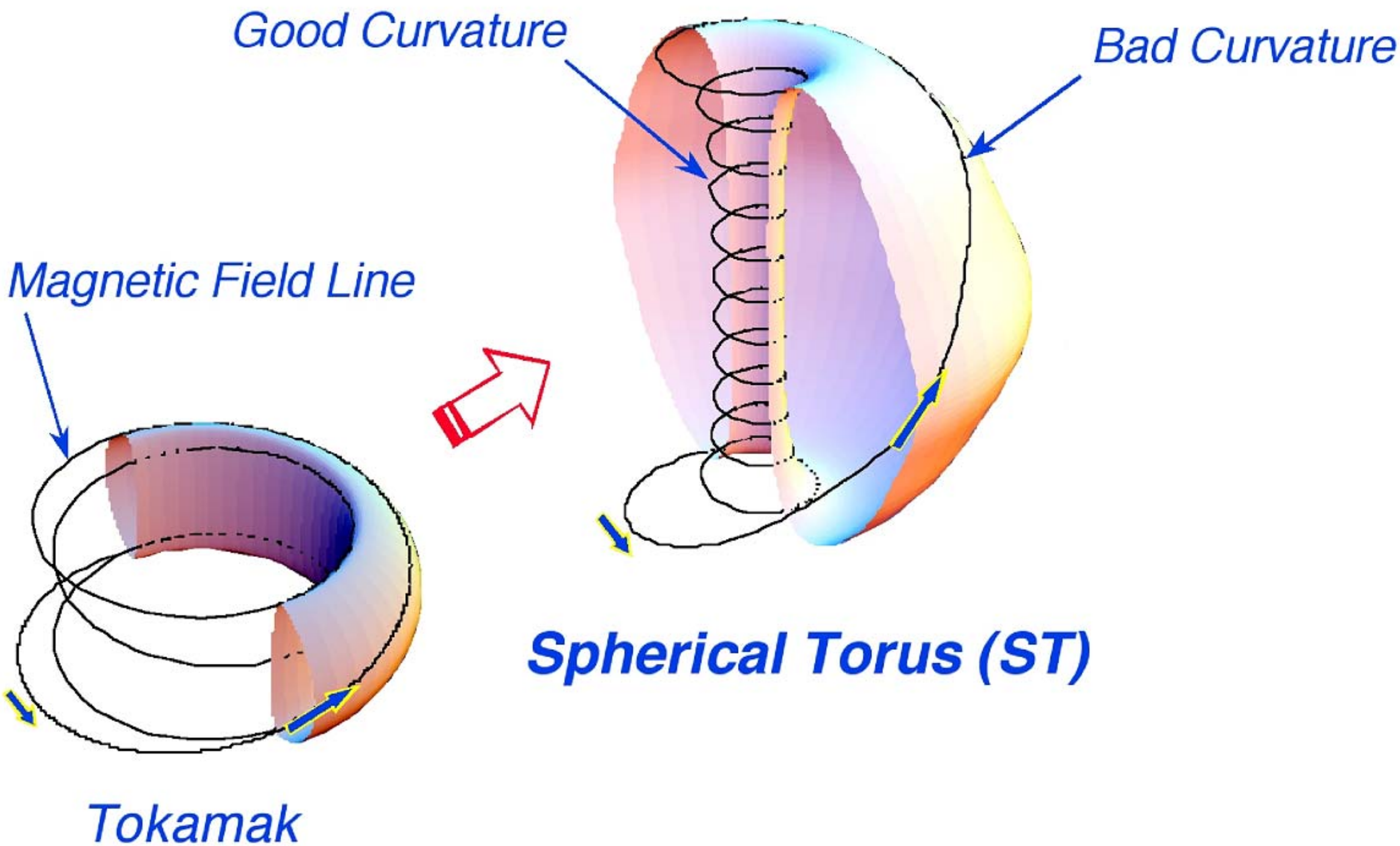
$$\boxed{\frac{R}{L} > \frac{1}{q^2}}$$

Rough, but tells us $\frac{R}{L}$ in important...

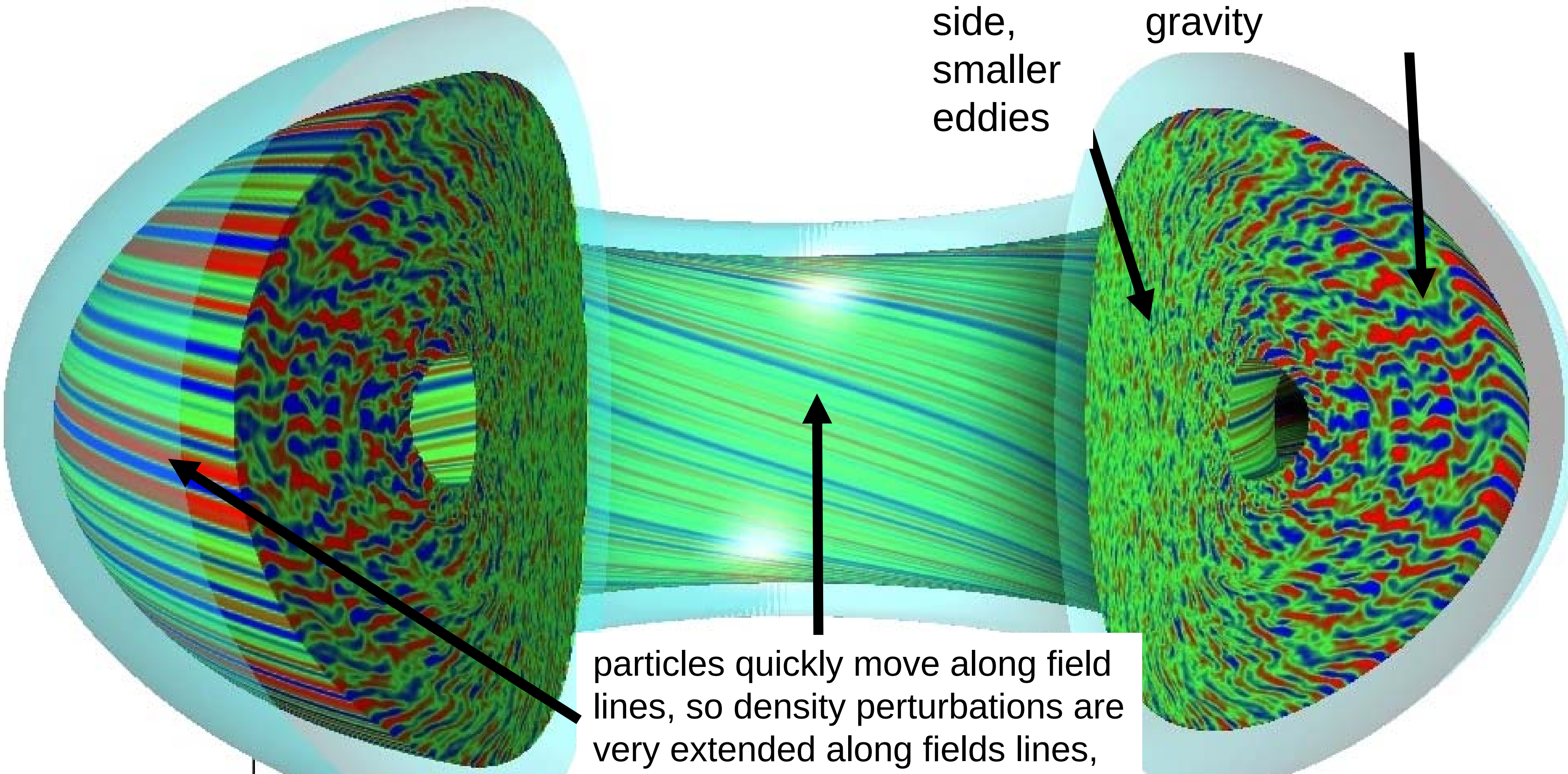
$\frac{1}{L} \sim \frac{\nabla T}{T}$

or $\sim \nabla p / p$

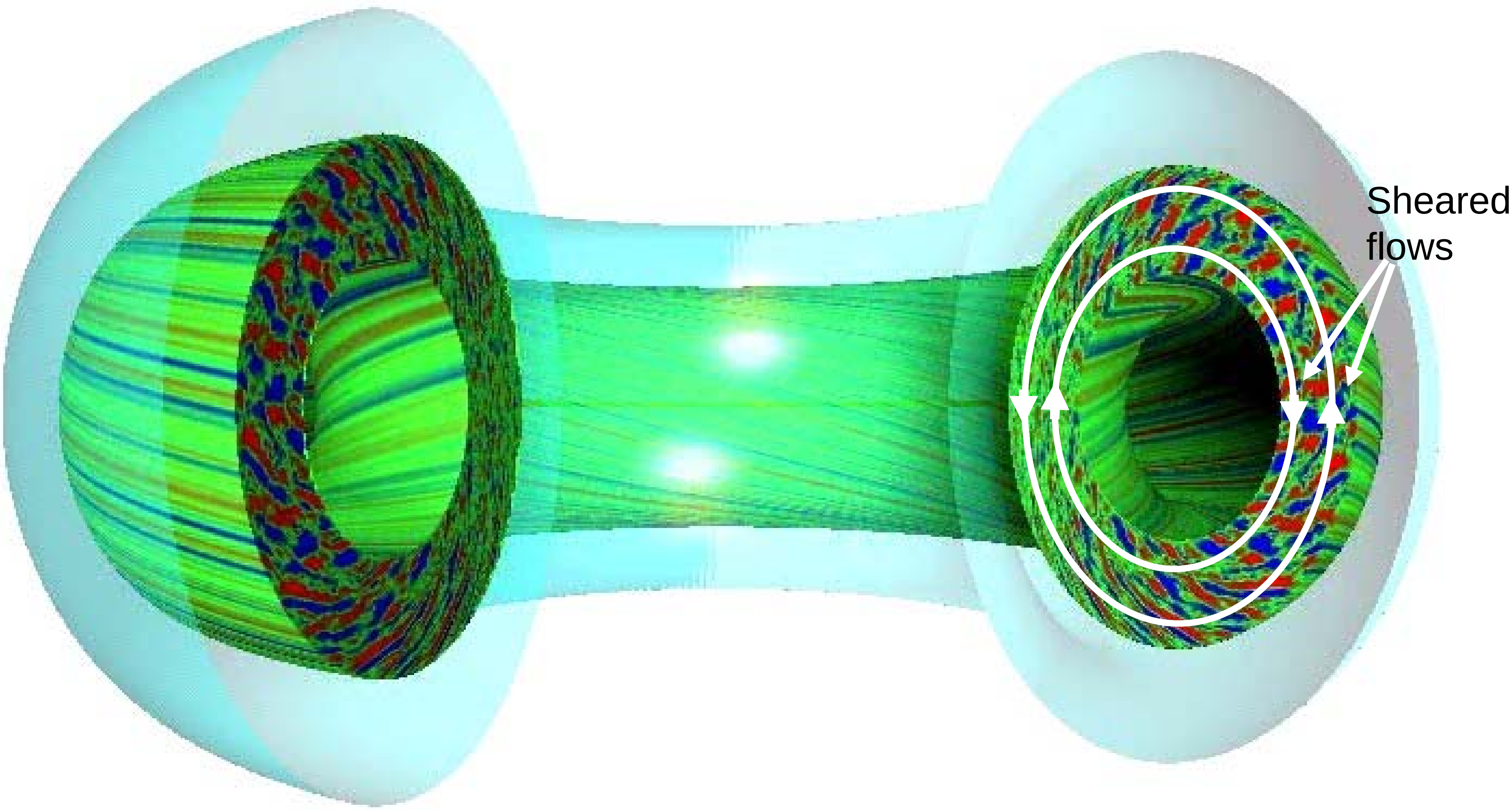
Spherical Torus has improved confinement and pressure limits (but less room in center for coils)

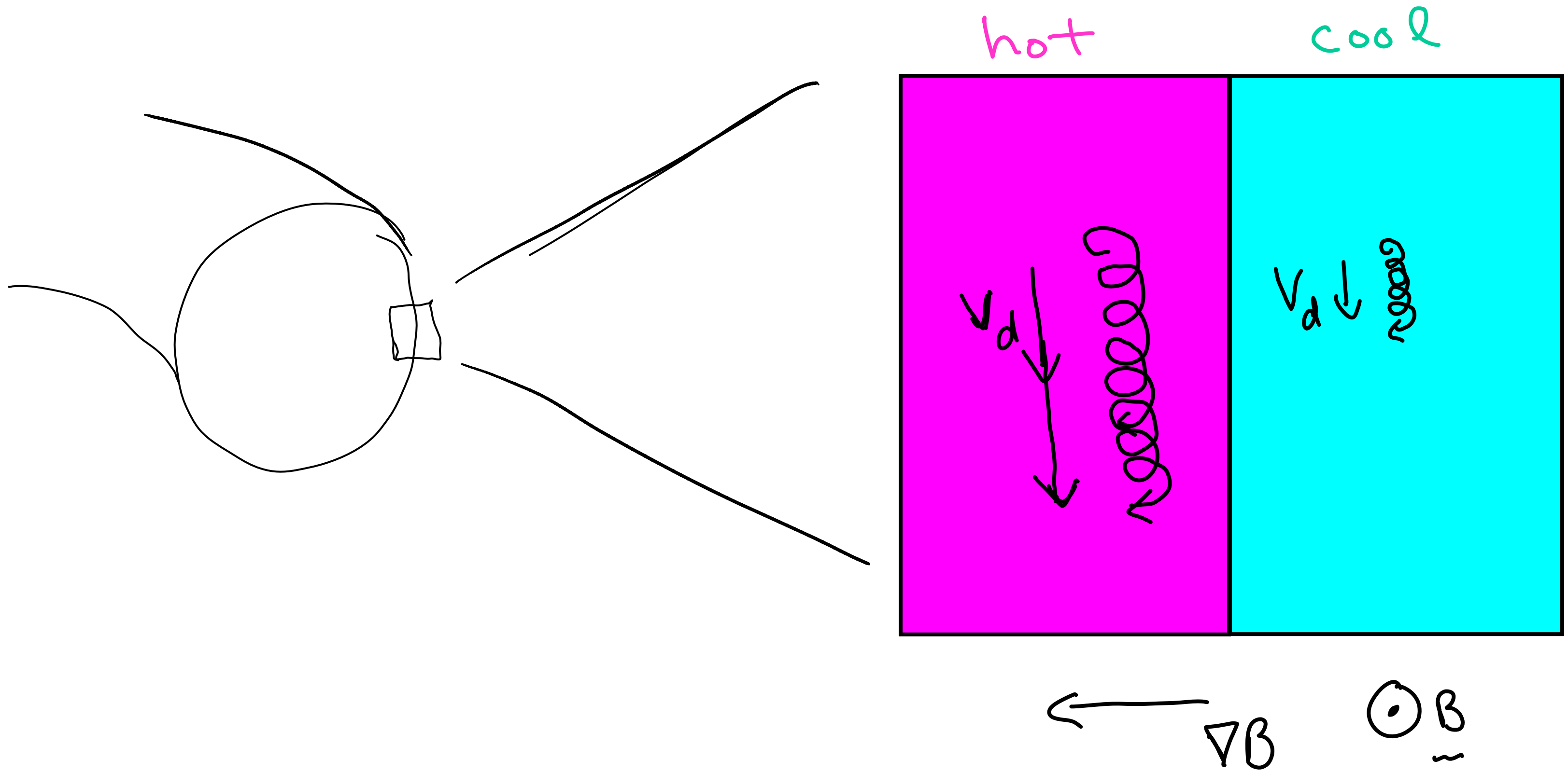


These physical mechanisms can be seen in gyrokinetic simulations and movies

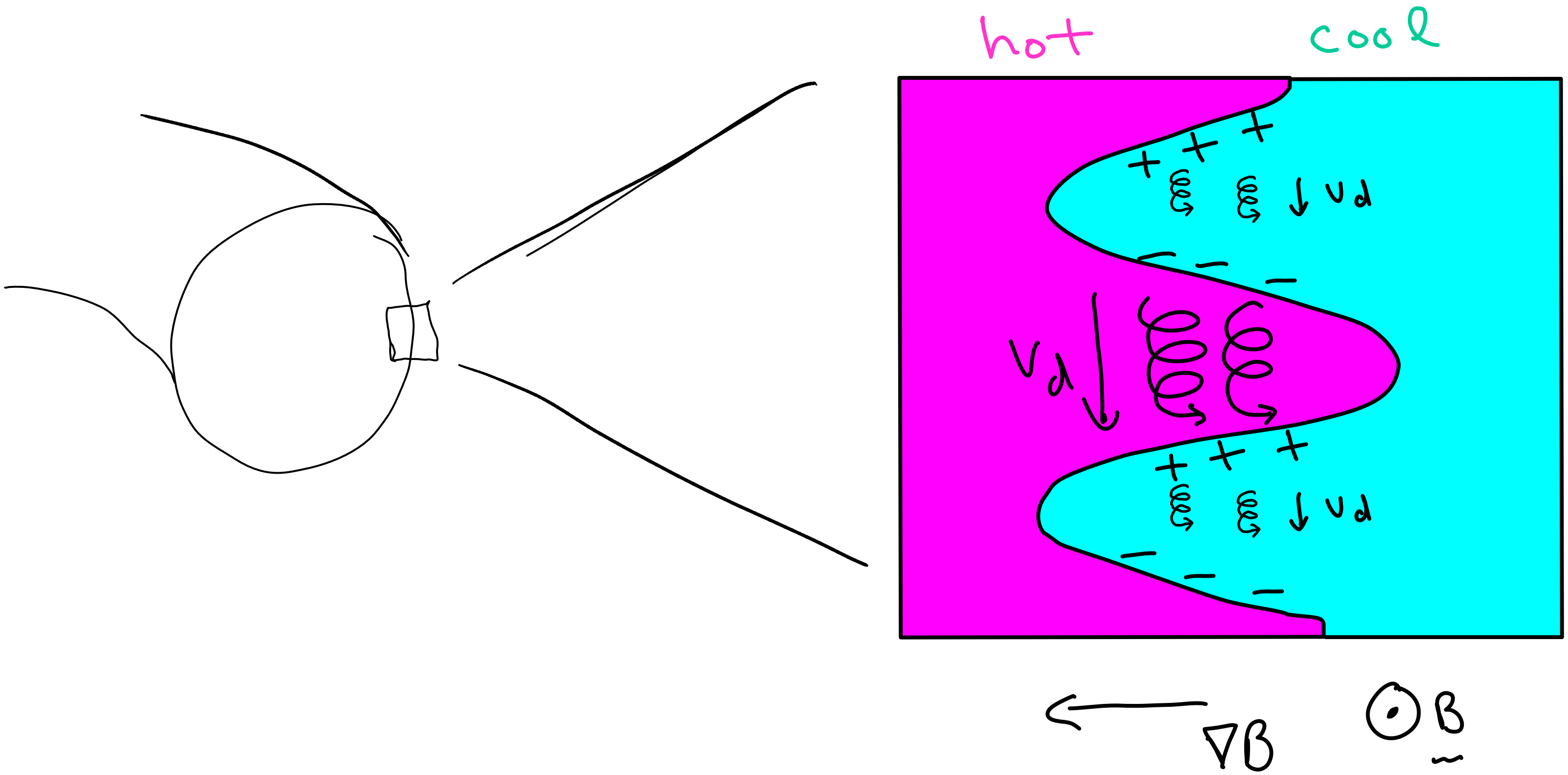


Movie http://fusion.gat.com/THEORY/images/3/35/D3d.n16.2x_0.6_fly.mpg from http://fusion.gat.com/theory/Gyromovies shows contour plots of density fluctuations in a cut-away view of a GYRO simulation (Candy & Waltz, GA). This movie illustrates the physical mechanisms described in the last few slides. It also illustrates the important effect of sheared flows in breaking up and limiting the turbulent eddies. Long-wavelength equilibrium sheared flows in this case are driven primarily by external toroidal beam injection. (The movie is made in the frame of reference rotating with the plasma in the middle of the simulation. Barber pole effect makes the dominantly-toroidal rotation appear poloidal..) Short-wavelength, turbulent-driven flows also play important role in nonlinear saturation.





hot
cool
V_d
V_d
∇B
$\odot \underset{\sim}{B}$



hot
cool
v_d
v_d
v_d
∇B
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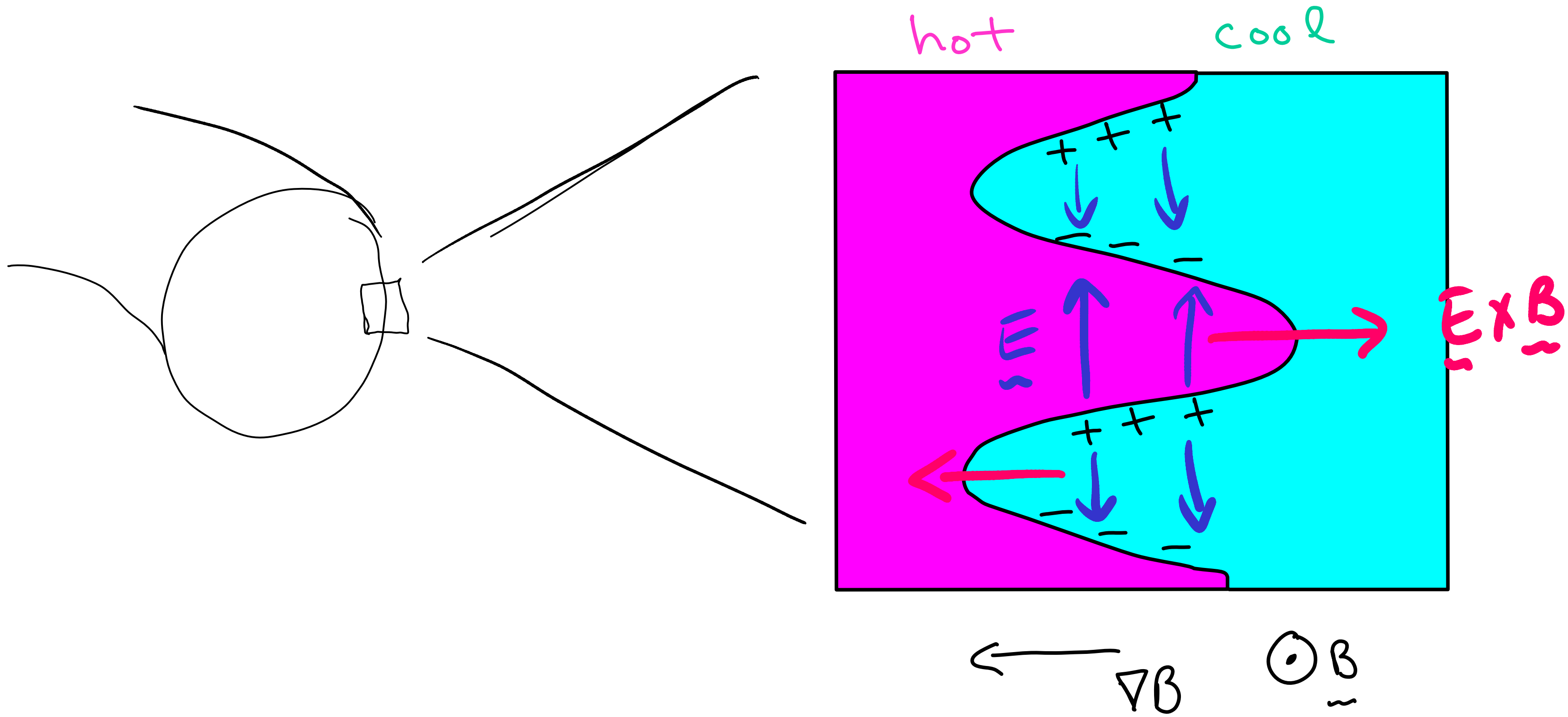
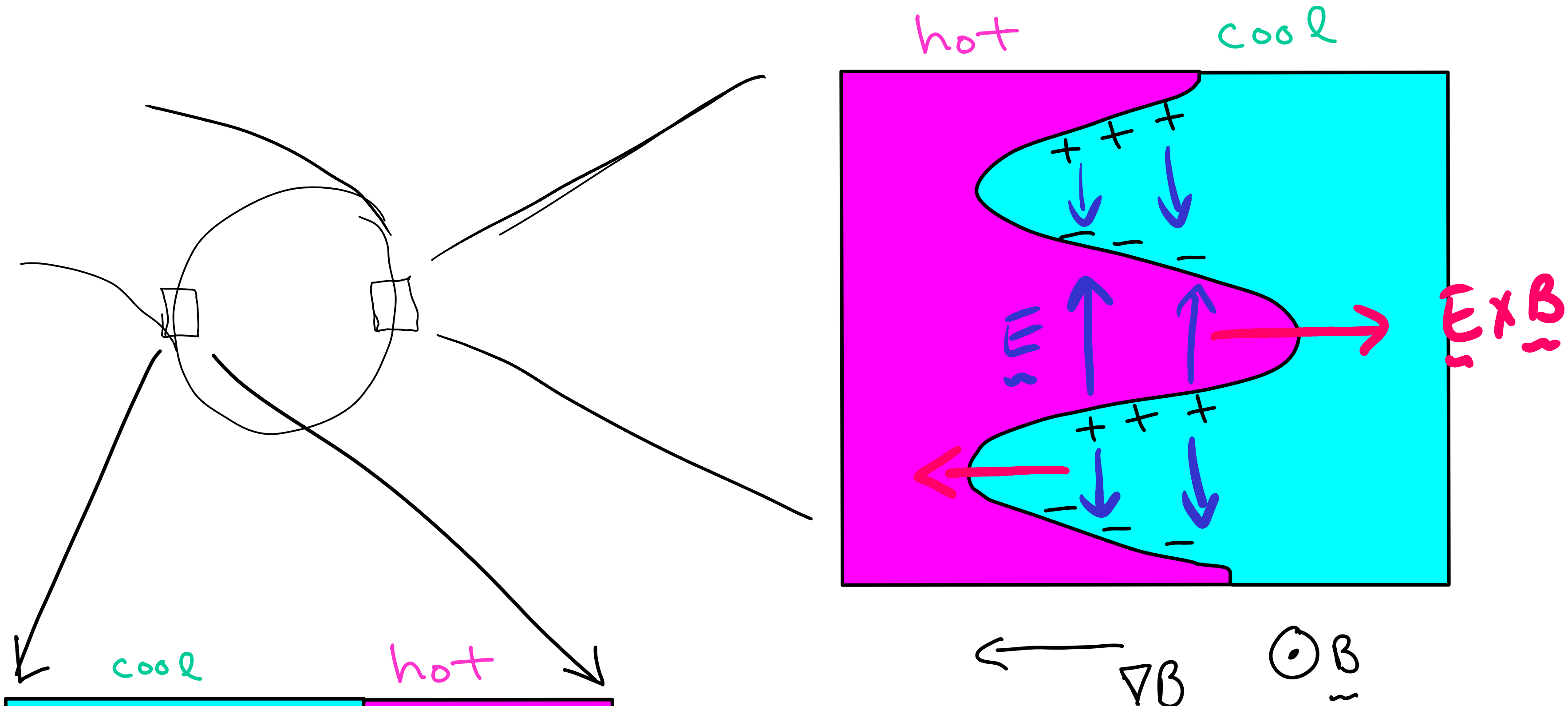
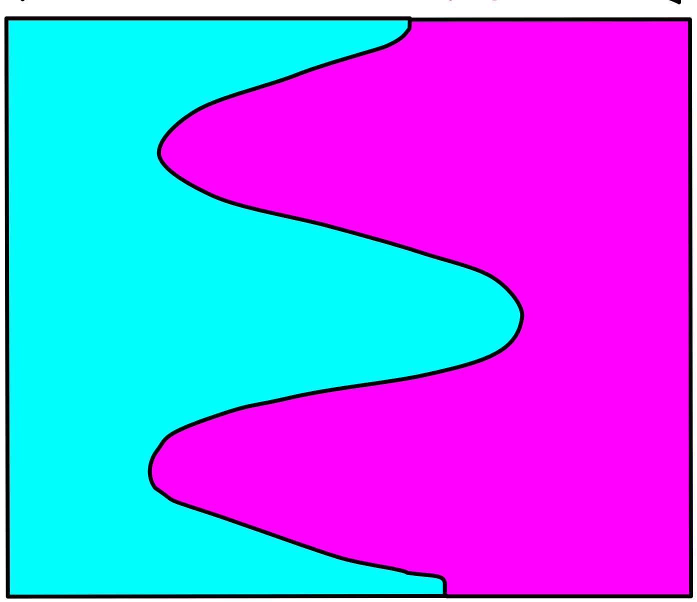


Higher energy particles ∇B drift faster, creates charge separation & thus $\underset{\sim}{E}$ field, causes $E \times B$ flow that further accentuates perturbation. Positive feedback $\Rightarrow$ instability.

Rosenbluth-Longmire picture





Can repeat this analysis on the good curvature side & find it is stable. (Leave as exercise.)

Rosenbluth-Longmire picture

Simple picture of reducing turbulence by negative magnetic shear

Particles that produce an eddy tend to follow field lines.

Reversed magnetic shear twists eddy in a short distance to point in the ``good curvature direction".

Locally reversed magnetic shear naturally produced by squeezing magnetic fields at high plasma pressure: ``Second stability" Advanced Tokamak or Spherical Torus.

Shaping the plasma (elongation and triangularity) can also change local shear

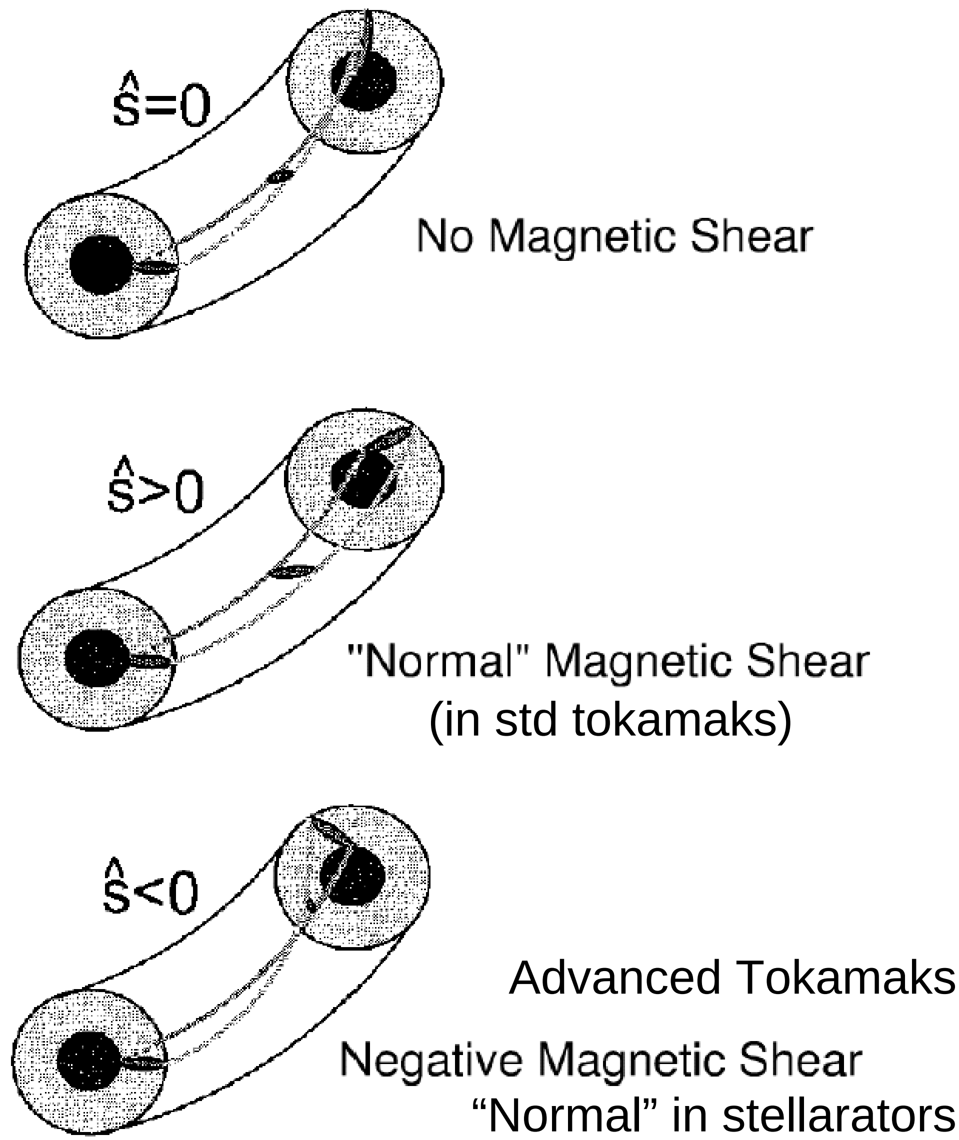
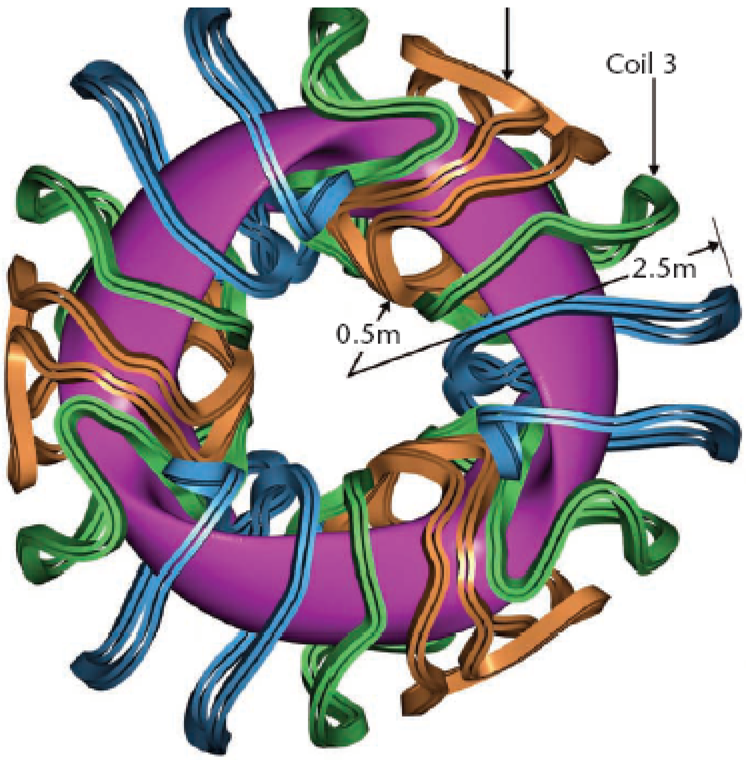


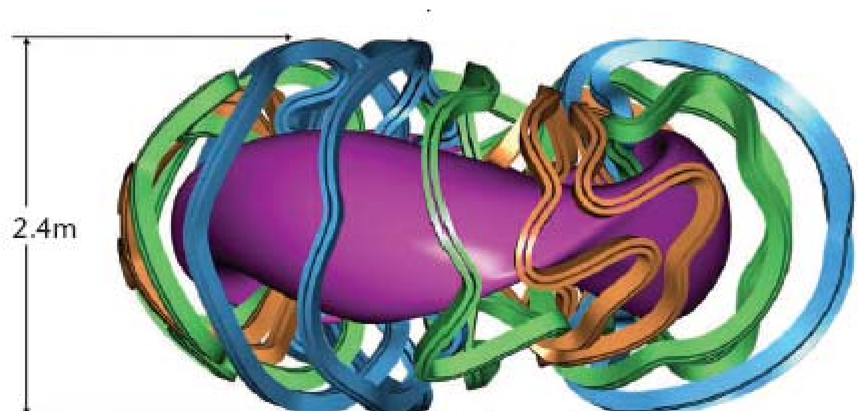
Fig. from Antonsen, Drake, Guzdar et al. Phys. Plasmas 96
Kessel, Manickam, Rewoldt, Tang Phys. Rev. Lett. 94

Improved Stellarators Being Studied

- Originally invented by Spitzer ('51), the unique idea when fusion declassified ('57)
- Mostly abandoned for tokamaks in '69. But computer optimized designs now much better than slide rules. Now studying cost reductions.
- Quasi-symmetry discovered in late 90's: don't need vector ***B*** exactly symmetric toroidally, |***B***| symmetric in field-aligned coordinates sufficient to be as good as tokamak.
- Magnetic field twist & shear provided by external coils, not plasma currents, inherently steady-state. Stellarator expts. find they don't have Greenwald density limit or hard beta limit & don't disrupt. Quasi-symmetry allows plasma spin to reduce turbulence?



Top view



Front view

Part 2: Rigorous derivation of ITG growth rate & threshold (in a simple limit) starting from the Gyrokinetic Eq.

Our starting point will be the electrostatic Gyrokinetic Eq. written in a Drift-Kinetic-like form for the full, gyro-averaged, guiding center density $\bar{f}(\underset{\sim}{R}, v_{\parallel}, \mu, t)$:

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_{\parallel}\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d) \cdot \nabla \bar{f} + \left(\frac{q}{m}E_{\parallel} - \mu \nabla_{\parallel} B + v_{\parallel}(\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \cdot \mathbf{v}_E\right) \frac{\partial \bar{f}}{\partial v_{\parallel}} = 0}$$

$$\underset{\sim}{V}_E \equiv -\frac{c}{B} \nabla \langle \Phi \rangle \times \hat{b} \qquad E_{\parallel} = -\hat{b} \cdot \nabla \langle \Phi \rangle$$

$$\mathbf{v}_d = \frac{v_{\parallel}^2}{\Omega}\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) + \frac{\mu}{\Omega}\hat{\mathbf{b}} \times \nabla B \simeq \frac{v_{\parallel}^2 + v_{\perp}^2/2}{\Omega B^2} \underset{\sim}{B} \times \nabla B$$

$$\mu = \frac{1}{2} \frac{v_{\perp}^2}{B}$$

Gyro-averaged

$$\bar{f}(\underset{\sim}{R}, v_{\parallel}, \mu, t) = \langle f(\underset{\sim}{R} + \underset{\sim}{\rho}(\theta), v_{\parallel}, \mu, \theta, t \rangle_{\theta}$$

details:

* this is not the original Drift-Kinetic Eq. of Chew, Goldberger, + Low (1956), which was for the strong E-field "MHD ordering" (see Kulsrud, Handbook of Plasma Physics, 1983)

$$V_E \sim V_t \gg V_d \sim \frac{v_\perp^2}{\Omega R} \sim V_t \frac{\rho}{R}$$

* closer to the form of the Drift-Kinetic Eq. used in neoclassical theory, where $\underset{\sim}{V}_E \sim \underset{\sim}{V}_d$ ("weak E-field")

even though $\frac{V_E}{V_t} \sim \frac{\rho}{R} \sim \varepsilon$, $\frac{V_E \cdot \nabla}{V_\parallel \hat{b} \cdot \nabla} \sim \frac{V_t \frac{\rho}{R} k_\perp}{V_t k_\parallel} \sim \frac{k_\perp \rho}{k_\parallel R}$

$$\sim 1$$

Gyrokinetic Eq. for full guiding-center density $f(\underset{\sim}{R}, v_\parallel, \mu, t)$:

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d) \cdot \nabla \bar{f} + \left(\frac{q}{m} E_\parallel - \mu \nabla_\parallel B + v_\parallel (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \cdot \mathbf{v}_E \right) \frac{\partial \bar{f}}{\partial v_\parallel} = 0}$$

In the uniform $\underset{\sim}{B}$ slab limit, this is = to Krommes GK Eq. 4
(~ p. 11-13)

Homework: show that expanding the Boltzmann factor in Cowley's Eq. 37, & gyroaveraging to get

$$\boxed{\bar{f} = F_0 - \frac{q \langle \Phi \rangle}{T_0} F_0 + h}$$

& subst. into above GK Eq. gives exactly Cowley's (Frieman-Chen) form of the GK Eq. (Cowley Eq. 40) for $\frac{\partial h}{\partial t}$ (Use uniform B slab limit for simplicity).

[& expand in consistent assumptions: $\frac{q\langle \Phi \rangle}{T} \ll 1$ but

$$F_0 \nabla_\perp \frac{q\langle \Phi \rangle}{T_0} \sim \nabla_\perp F_0 \qquad F_0 \nabla_\perp \frac{q\langle \Phi \rangle}{T_0} \sim \nabla_\perp F_0$$]

Gyrokinetic Eq. for full guiding-center density $f(\underset{\sim}{R}, v_\parallel, \mu, t)$:

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d)\cdot\nabla\bar{f} + \left(\frac{q}{m}E_\parallel - \mu\nabla_\parallel B + v_\parallel(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right)\frac{\partial \bar{f}}{\partial v_\parallel} = 0}$$

Homework: show that substituting the gyro-average of Cowley's Eq. 37:

(Straight + uniform $\underline{B}$ limit for simplicity)

$$\bar{f} = F_0 - \frac{q\langle\Phi\rangle}{T_0}F_0 + h$$

$$\frac{\partial h}{\partial t} - \frac{q}{T_0}\frac{\partial\langle\Phi\rangle}{\partial t}F_0 + v_\parallel\hat{b}\cdot\nabla h + \underset{\sim}{v}_E\cdot\nabla h + \underset{\sim}{v}_E\cdot\nabla\Big(F_0\big(1 - \underbrace{q\frac{\langle\Phi\rangle}{T_0}}_{\text{drop}}\big)\Big)$$

$$- v_\parallel\hat{b}\cdot\nabla\langle\Phi\rangle\frac{q}{T_0}F_0 \quad \underbrace{- \frac{q}{m}\hat{b}\cdot\nabla\langle\Phi\rangle\frac{\partial h}{\partial v_\parallel}}_{\text{drop}}$$

$$- \frac{q}{m}\hat{b}\cdot\nabla\langle\Phi\rangle\frac{\partial F_0}{\partial v_\parallel}\Big(1 - \underbrace{\frac{q\langle\Phi\rangle}{T_0}}_{\text{drop}}\Big) = 0$$

These 2 terms cancel

Use $\dfrac{\partial F_0}{\partial v_\parallel} = -\dfrac{m v_\parallel}{T_0}F_0$

Homework Solution outline

$$\boxed{\frac{\partial \overline{f}}{\partial t} + (v_{\|}\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d)\cdot\nabla\overline{f} + \left(\frac{q}{m}E_{\|} - \mu\nabla_{\|}B + v_{\|}(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right)\frac{\partial \overline{f}}{\partial v_{\|}} = 0}$$

Linearize: $\overline{f} = F_0 + \tilde{f}$, where F_0 satisfies Equilibrium Eq.

$$\frac{\partial}{\partial t} = 0 \qquad \underset{\sim}{E} = 0$$

$$(v_{\|}\hat{b} + \underset{\sim}{v}_d)\cdot\nabla F_0 - \mu\nabla_{\|}B\frac{\partial F_0}{\partial v_{\|}} = 0$$

Basically says $F_0 =$ const. along trajectories of banana orbits or passing orbits in a tokamak.

General Equilibrium solution could be an arbitrary function of the constants of the motion (E, μ, P_ϕ) where

$$E = \tfrac{1}{2}mv_{\|}^2 + \mu B.$$

& $P_\phi =$ canonical angular momentum

But if we neglect $\frac{|\underset{\sim}{v}_d|}{v_{\|}} \sim \frac{\rho}{R}$ get simpler Eq:

$$v_{\parallel} \hat{b}\cdot\nabla F_0 - \mu\left(\hat{b}\cdot\nabla B\right)\frac{\partial F_0}{\partial v_{\parallel}} = 0$$

Will consider Equilibrium of the form:

$$F_0(\underset{\sim}{R}, v_{\parallel}, \mu) \propto \frac{n_0(\psi)}{T_0^{3/2}(\psi)} e^{-\frac{m\left(\frac{1}{2}v_{\parallel}^2 + \mu B(\underset{\sim}{x})\right)}{T(\psi)}} \propto e^{-\frac{E}{T}}$$

Exercise: Plug this in to the previous Eq. & show it is a solution.

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_{\|}\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d)\cdot\nabla\bar{f} + \left(\frac{q}{m}E_{\|} - \mu\nabla_{\|}B + v_{\|}(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right)\frac{\partial\bar{f}}{\partial v_{\|}} = 0}$$

Linearize: $\bar{f} = F_0 + \tilde{f}$, where F_0 satisfies Equilibrium Eq.

Next order Eq:

$$\frac{\partial \tilde{f}}{\partial t} + \left(v_{\|}\hat{b} + \mathbf{v}_d\right)\cdot\nabla\tilde{f} - \mu\nabla_{\|}B\frac{\partial\tilde{f}}{\partial v_{\|}} = -\mathbf{v}_E\cdot\nabla F_0 - \left(\frac{q}{m}E_{\|} + v_{\|}(\hat{b}\cdot\nabla\hat{b})\cdot\mathbf{v}_E\right)\frac{\partial F_0}{\partial v_{\|}}$$

$$\left(-i\omega + i v_{\|}k_{\|} + i\mathbf{v}_d\cdot\mathbf{k}_{\perp}\right)\tilde{f} = -\mathbf{v}_E\cdot\nabla F_0 - \left(\frac{q}{m}E_{\|} + v_{\|}(\hat{b}\cdot\nabla\hat{b})\cdot\mathbf{v}_E\right)\frac{\partial F_0}{\partial v_{\|}}$$

Important Subtlety: $\bar{F}(\underset{\sim}{R}, v_\parallel, \mu, t)$ so

$$-\underset{\sim}{v}_E \cdot \nabla F_0 = -\underset{\sim}{v}_E \cdot \nabla\Big|_{v_\parallel, \mu, t} F_0$$

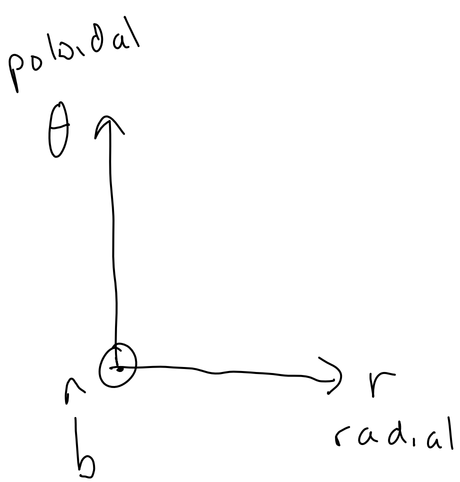
using $$F_0 \propto \frac{n_0(r)}{T_0^{3/2}(r)} e^{-\frac{\left(\frac{1}{2} m v_\parallel + m\mu B(\underset{\sim}{x})\right)}{T_0(r)}}$$

will give terms proportional to ∇n_0, ∇T_0, ~~&~~ $\mu \nabla B$

∇n_0 terms: $$-\underset{\sim}{v}_E \cdot \nabla F_0 \Rightarrow + \frac{c}{B}\left(\nabla \Phi \times \hat{b} \cdot \frac{\nabla n_0}{n_0}\right) F_0$$

$$\frac{\nabla n_0}{n_0} = -\frac{\hat{r}}{L_n}$$

$$= -\frac{c}{B} \nabla \Phi \times \hat{b} \cdot \hat{r} \frac{1}{L_n} F_0$$



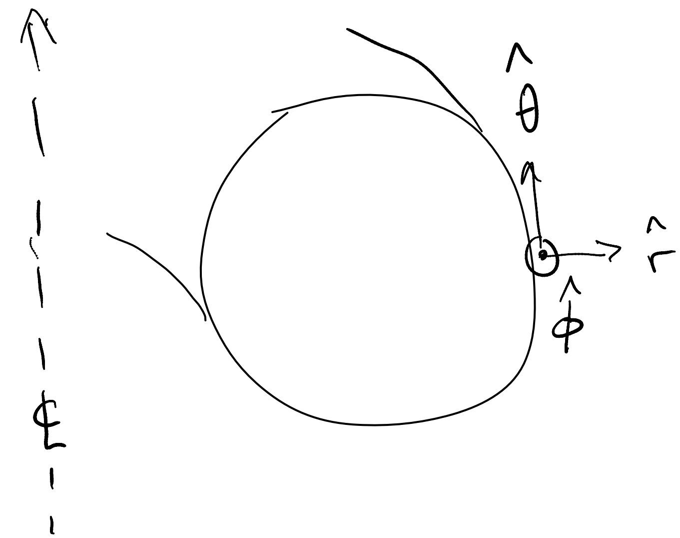
$$= -\frac{c}{B} i k_\theta \Phi \frac{1}{L_n} F_0$$

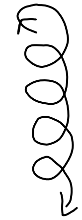
$$= \cancel{+}\, i \omega_* \frac{e\Phi}{T_0} F_0$$

$$\omega_{*i} \equiv -\frac{cT}{eB} \frac{k_\theta}{L_n}$$

$$\equiv -k_\theta \rho_s \frac{c_s}{L_n}$$

Note on sign conventions:





With $\underset{\sim}{B}$ field out of the page, the ∇B drift for ions is downward

$$\underset{\sim}{v}_d \approx -\hat{\theta}\, v_t \frac{\rho}{R} \qquad (\text{at } \theta = 0)$$

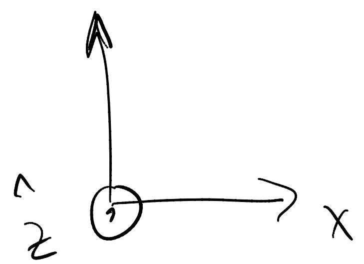
defining $\omega_{dv} = \underset{\sim}{k} \cdot \underset{\sim}{v}_d$

gives convention used in Beer's thesis!

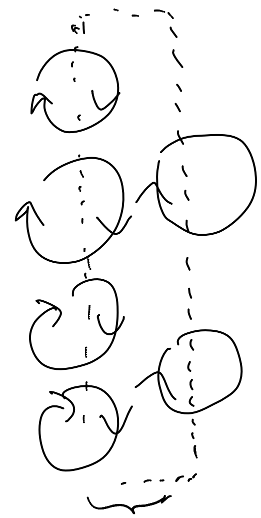
$$\omega_{dv} = \omega_d (v_\parallel^2 + \mu B)/v_t^2$$

$$\omega_d = -k_\theta \rho v_t / R$$

More on Sign Conventions



$\longleftarrow \nabla n$



with $\underset{\sim}{B}$ out of page, the diamagnetic flow V_{*i} is downward if ∇n is inward. Thus

$$\omega_{*i} \equiv \underset{\sim}{k} \cdot V_{*i} = - k_\theta v_t \frac{\rho}{L_n}$$

$$= - \frac{cT}{eB} \frac{k_\theta}{L_n}$$

(Back to RHS of linearized GK Eq., 4 slides back)

$$RHS = \underbrace{-\underset{\sim}{v}_E\cdot\nabla F_0}_{\text{part of this}} - \underbrace{\left(\frac{q}{m}E_\parallel + v_\parallel(\hat{b}\cdot\nabla\hat{b})\cdot\underset{\sim}{v}_E\right)\frac{\partial F_0}{\partial v_\parallel}}$$

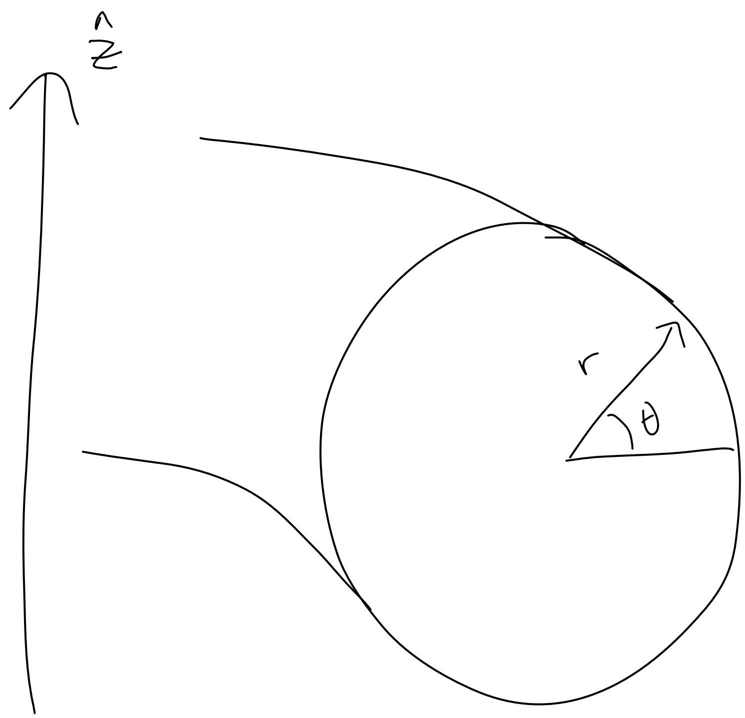
$$\propto -\frac{c}{B}\nabla\Phi\times\hat{b}\cdot\mu\nabla B \qquad\qquad \propto +v_\parallel^2(\hat{b}\cdot\nabla\hat{b})\cdot\frac{(\hat{b}\times\nabla\Phi)}{B}$$

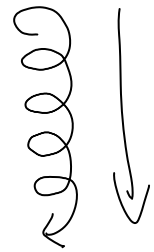
$$\propto -\nabla\Phi\cdot\Big[\underbrace{\mu\,\hat{b}\times\nabla B}_{\nabla B} + \underbrace{v_\parallel^2\,\hat{b}\times(\hat{b}\cdot\nabla\hat{b})}_{+\text{ curvature drift}}\Big]$$

$$RHS = +i\left(\omega_{*v}^T - \omega_{dv} - k_\parallel v_\parallel\right)\frac{e\Phi}{T_0}F_0$$

$$\omega_*^T = \omega_*[1+\eta(v_\parallel^2/2v_t^2 + \mu B/v_t^2 - 3/2)] \qquad\qquad \omega_{dv} = \omega_d(v_\parallel^2+\mu B)/v_t^2$$

$$\omega_* = \frac{k_\theta\rho v_t}{L_n} \qquad \eta = \frac{L_n}{L_T} \qquad \omega_d = -\frac{v_t}{R}\rho\left(k_\theta\cos\theta + k_r\sin\theta\right)$$





downward
$\underset{\sim}{v}_d$ from ∇B + curvature drift

$$\omega_d = \underset{\sim}{k} \cdot \underset{\sim}{v}_d$$

$$= -\frac{v_t \rho}{R}\left(k_\theta \cos\theta + k_r \sin\theta\right)$$

will focus on $\theta \approx 0$ here
(where bad-curvature drive is the strongest)

$$\left(-i\omega + i v_\parallel k_\parallel + i \underset{\sim}{v}_d \cdot \underset{\sim}{k}_\perp\right) \tilde{f} = -\underset{\sim}{v}_E \cdot \nabla F_0 - \left(\frac{q}{m} E_\parallel + v_\parallel \left(\hat{b}\cdot\nabla\hat{b}\right)\cdot \underset{\sim}{v}_E\right) \frac{\partial F_0}{\partial v_\parallel}$$

subst. for RHS

$$\left(-i\omega + i v_\parallel k_\parallel + i\omega_{dv}\right) \tilde{f} = -i\left(-\omega_{*v}^{T} + \omega_{dv} + k_\parallel v_\parallel\right) \frac{e\Phi}{T_0} F_0$$

$$\boxed{\tilde{f} = \frac{-\omega_{*v}^{T} + \left(k_\parallel v_\parallel + \omega_{dv}\right)}{\omega - \left(k_\parallel v_\parallel + \omega_{dv}\right)} \frac{e\Phi}{T_0} F_0}$$

Note: recover Boltzmann response when $k_\parallel v_\parallel$ &/or ω_{dv} large

$$\boxed{\tilde{f} = \frac{-\omega_{*v}^{T} + (k_\parallel v_\parallel + \omega_{dv})}{\omega - (k_\parallel v_\parallel + \omega_{dv})} \frac{e\Phi}{T_0} F_0}$$

Look for modes with

$$k_\parallel v_{ti} \ll \omega, \omega_{*v}^{T}, \omega_{dv} \ll k_\parallel v_{te}$$

(slab "η_i" version of ITG requires finite $k_\parallel v_{ti}$, but not toroidal version).

assume Boltzmann electrons

Quasineutrality: $\tilde{n}_e = \tilde{n}_i$

(additional polarization contribution to density gives $k_\perp^2 \rho_i^2$ corrections but not critical for basic ITG.)

$$\boxed{n_{e0} \frac{e\Phi}{T_e} = \int d^3v \, \frac{-\omega_{*v}^{T} + \omega_{dv}}{\omega - \omega_{dv}} F_0 \frac{e\Phi}{T_{i0}}}$$

$$n_0 \frac{e\Phi}{T_{e0}} = n_0 \frac{e\Phi}{T_{i0}} \int d^3v \, \frac{F_0}{n_0} \frac{\omega_{dv} - \omega_{*T}}{\omega - \omega_{dv}}$$

"Cold plasma" or "fast wave" approx. $\omega \gg \omega_{dv}$

$$\frac{T_{i0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\frac{T_{\wedge 0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \, \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\omega_{dv} = \omega_d(v_\parallel^2 + \mu B)/v_t^2 \qquad \omega_*^T = \omega_*[1 + \eta(v_\parallel^2/2v_t^2 + \mu B/v_t^2 - 3/2)]$$

$$\omega_d = -k_\theta \rho v_t/R \qquad \omega_* = -k_\theta \rho v_t/L_n$$

$$\int d^3v \, \frac{F_0}{n_0} \, \omega_{dv} = \int d^3v \, \frac{F_0}{n_0} \, \omega_d \left(v_\parallel^2 + \frac{1}{2} \underbrace{v_\perp^2}_{= v_x^2 + v_y^2} \right) / v_t^2$$

$$= 2\,\omega_d$$

Using useful I.D. for Maxwellian F_0 :

$$\langle v_x^{2n} \rangle = \int d^3v \, \frac{F_0}{n_0} \, v_x^{2n} = v_t^{2n} \underbrace{(2n-1)!!}_{(2n-1)(2n-3)(2n-5)\cdots 5\cdot 3\cdot 1}$$

$$\frac{T_{\wedge 0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \, \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\omega_{dv} = \omega_d (v_\parallel^2 + \mu B)/v_t^2 \qquad \omega_*^T = \omega_*[1 + \eta(v_\parallel^2/2v_t^2 + \underbrace{\mu B}_{= \frac{1}{2} v_\perp^2 = \frac{1}{2}(v_x^2 + v_y^2)}/v_t^2 - 3/2)]$$

$$\omega_d = -k_\theta \rho v_t/R \qquad \omega_* = -k_\theta \rho v_t/L_n$$

$$\int d^3v \, \frac{F_0}{n_0} \omega_*^T = \omega_* \left(1 + \eta \left(\frac{1}{2} + 1 - \frac{3}{2} \right) \right) = \omega_*$$

$$\int d^3v \, \frac{F_0}{n_0} \omega_{dv}^2 = \int d^3v \, \frac{F_0}{n_0} \omega_d^2 \left[v_\parallel^4 + 2 v_\parallel^2 \frac{1}{2} v_\perp^2 + \frac{1}{4} (v_x^2 + v_y^2)^2 \right] \frac{1}{v_t^4}$$

$$= \omega_d^2 \left[3 + 2 \cdot \frac{1}{2} (1+1) + \frac{1}{4} \left(\frac{\langle v_x^4 + 2 v_x^2 v_y^2 + v_y^4 \rangle}{v_t^4} \right) \right]$$

$$= \omega_d^2 \left[5 + \frac{1}{4} (8) \right] = 7 \omega_d^2$$

$$\frac{T_{\wedge 0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \, \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\omega_{dv} = \omega_d (v_\parallel^2 + \mu B)/v_t^2 \qquad \omega_*^T = \omega_* [1 + \eta (v_\parallel^2 / 2 v_t^2 + \underbrace{\mu B}_{= \frac{1}{2} v_\perp^2 = \frac{1}{2}(v_x^2 + v_y^2)} / v_t^2 - 3/2)]$$

$$\omega_d = -k_\theta \rho v_t / R \qquad \omega_* = -k_\theta \rho v_t / L_n$$

$$\int d^3v \, \frac{F_0}{n_0} \, \omega_{dv} \, \omega_*^T = \omega_d \omega_* \Bigg\{ 2 + \eta \int d^3v \, \frac{F_0}{n_0} \, \frac{\left(v_\parallel^2 + \frac{1}{2} v_\perp^2 \right)}{v_t^2} \, \frac{\left(\frac{1}{2} v_\parallel^2 + \frac{1}{2} v_\perp^2 - \frac{3}{2} v_t^2 \right)}{v_t^2} \Bigg\}$$

$$= \omega_d \omega_* \left\{ 2 + \eta \left[\frac{1}{2} 3 + \frac{1}{2} 2 - \frac{3}{2} + \frac{1}{2} \cdot 2 \cdot \frac{1}{2} + \frac{1}{4} 8 - \frac{1}{2} \cdot 2 \cdot \frac{3}{2} \right] \right\}$$

$$\int d^3v \, \frac{F_0}{n_0} \omega_{dv} \omega_*^T$$

$$= \omega_d \omega_* \left\{ 2 + \eta \left[\cancel{\tfrac{1}{2} 3} + \cancel{\tfrac{1}{2} 2} - \cancel{\tfrac{3}{2}} + \cancel{\tfrac{1}{2} \cdot 2 \cdot \tfrac{1}{2}} + \tfrac{1}{4} 8 - \cancel{\tfrac{1}{2} \cdot 2} \cdot \cancel{\tfrac{3}{2}} \right] \right\}$$

$$= \omega_d \omega_* 2 (1 + \eta)$$

Combine results from last 2 pages:

$$\boxed{\frac{T_{i0}}{T_{e0}} = 2 \frac{\omega_d}{\omega} - \frac{\omega_*}{\omega} + 7 \frac{\omega_d^2}{\omega^2} - 2 \frac{\omega_d \omega_*}{\omega^2} (1 + \eta)}$$

This defines a dispersion relation ω vs. $\underset{\sim}{k}$

$$\boxed{\frac{T_{i0}}{T_{e0}} = 2\frac{\omega_d}{\omega} - \frac{\omega_*}{\omega} + 7\frac{\omega_d^2}{\omega^2} - 2\frac{\omega_d\,\omega_*}{\omega^2}(1+\eta)}$$

Consider the flat density limit: $\nabla n \to 0$, but $\nabla T \neq 0$

$$\omega_* = -k_\theta \rho \frac{v_t}{L_n} \longrightarrow 0 \qquad\qquad \eta = \frac{\frac{1}{T}\nabla T}{\frac{1}{n}\nabla n} = \frac{L_n}{L_T} \to \infty$$

$$\omega_*\eta = -k_\theta \rho \frac{v_t}{L_n}\frac{L_n}{L_T} \equiv \bar{\omega}_{*T}$$

$$\omega^2 \frac{T_{i0}}{T_{e0}} - 2\omega_d\omega + 2\omega_d\bar{\omega}_{*T} - 7\omega_d^2 = 0$$

$$\boxed{\omega = \frac{2\omega_d \pm \sqrt{4\omega_d^2 - 4\frac{T_{i0}}{T_{e0}}\left(2\omega_d\bar{\omega}_{*T} - 7\omega_d^2\right)}}{2\left(T_{i0}/T_{e0}\right)}}$$

From last page:

$$\omega = \frac{2\omega_d \pm \sqrt{4\omega_d^2 - 4\frac{T_{i0}}{T_{e0}}\left(2\omega_d \overline{\omega}_{*T} - 7\omega_d^2\right)}}{2\left(T_{i0}/T_{e0}\right)}$$

Consider large temperature gradient limit: $\omega_{*T} \propto \nabla T$ ↑

Growth rate:

$$\gamma = \frac{\sqrt{2\omega_d \overline{\omega}_{*T}}}{\sqrt{T_{i0}/T_{e0}}} = \frac{\sqrt{2}\, h_\theta \rho_i}{\sqrt{T_{i0}/T_{e0}}} \frac{v_{ti}}{\sqrt{R L_T}}$$

Fundamental scaling of bad-curvature driven instabilities.

Go back to general D.R.:

$$\omega = \frac{2\omega_d \pm \sqrt{4\omega_d^2 - 4\frac{T_{i0}}{T_{e0}}\left(2\omega_d \overline{\omega}_{*T} - 7\omega_d^2\right)}}{2(T_{i0}/T_{e0})}$$

$$= \frac{2\omega_d \pm \sqrt{\left(4 + 28\frac{T_{i0}}{T_{e0}}\right)\omega_d^2 - 8\frac{T_{i0}}{T_{e0}}\omega_d \overline{\omega}_{*T}}}{2(T_{i0}/T_{e0})}$$

Instability exists if

$$8\frac{T_{i0}}{T_{e0}}\omega_d \overline{\omega}_{*T} > \omega_d^2\left(4 + 28\frac{T_{i0}}{T_{e0}}\right)$$

$$\frac{1}{R}\frac{1}{L_T} > \frac{1}{R^2}\left(\frac{1}{2}\frac{T_{e0}}{T_{i0}} + \frac{1}{2}7\right)$$

$$\bigstar \quad \boxed{\frac{R}{L_T} > \frac{1}{2}\left(7 + \frac{T_{e0}}{T_{i0}}\right)}$$

Compare w/ Romanelli 1990 (Eq. 12):

$$\eta_i = (\tfrac{5}{3} + \tau/4) 2\epsilon_n$$

or

$$\frac{L_n}{L_T} = \left(\frac{5}{3} + \frac{1}{4} \frac{T_e}{T_i} \right) 2 \frac{L_n}{R}$$

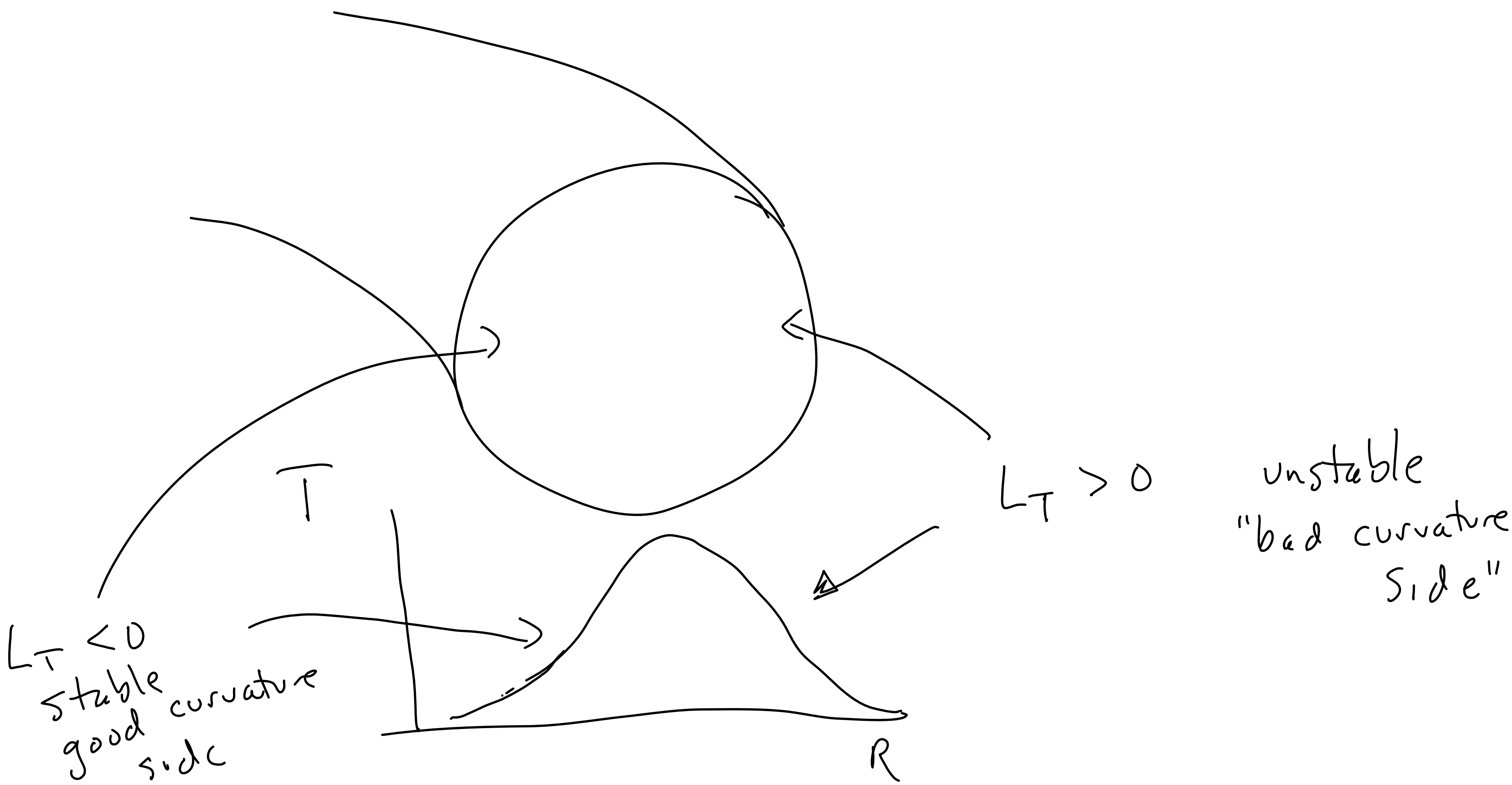
$$\boxed{\frac{R}{L_{Tcrit}} = \frac{10}{3} + \frac{1}{2} \frac{T_{eo}}{T_{io}}}$$

$$= 3.33 + 0.5 \frac{T_{eo}}{T_{io}}$$

vs. my $\displaystyle \frac{R}{L_{Tcrit}} = 3.5 + 0.5 \frac{T_{eo}}{T_{io}}$ } Very close. Diff. is presumably because Romanelli simplifies ω_D (See after his Eq. 6)

Note there is an instability only if $\omega_d \overline{\omega}_{*T} > 0$

$$\omega_d \overline{\omega}_{*T} = (k_\theta \rho)^2 \frac{v_t^2}{R L_T} \qquad \frac{1}{L_T} \equiv -\frac{1}{T}\frac{\partial T}{\partial R}$$



Why does this get the $\frac{T_{i0}}{T_{e0}}$ dependence of

$\frac{R}{L_{T\,crit}}$ wrong?   More accurate: $\boxed{\frac{R}{L_T} > \frac{R}{L_{T\,crit}} = \frac{4}{3}\left(1 + \frac{T_{i0}}{T_{e0}}\right)}$

Because near marginal stability, the expansion of the resonant denominator

$$\frac{1}{\omega - \omega_{dv}} \simeq \frac{1}{\omega}\left(1 + \frac{\omega_{dv}}{\omega} + \cdots\right)$$

breaks down, since $\omega \sim \omega_d$ near marginal stability...

More general result for threshold for instability:

$$\frac{R_0}{L_{T\,crit}} = \mathrm{Max}\left[\left(1+\frac{T_i}{T_e}\right)\left(1.33+1.91\,\frac{\hat{s}}{q}\right)\left(1-1.5\,\frac{r}{R_0}\right)\left(1+0.3\,\frac{r\,d\kappa}{dr}\right),\; 0.8\,\frac{R_0}{L_n}\right]$$

Found by fits to lots of GS2 Gyrokinetic stability calculations (Jenko, Dorland Hammett, PoP 2001), guided by previous analytic results (Romanelli, Hahm + Tang) in some limits.

ITG References

- Mike Beer's Thesis 1995
  http://w3.pppl.gov/~hammett/collaborators/mbeer/afs/thesis.html
- Romanelli & Briguglio, Phys. Fluids B 1990
- Biglari, Diamond, Rosenbluth, Phys. Fluids B 1989
- Jenko, Dorland, Hammett, PoP 2001
- Candy & Waltz, PRL …
- Kotschenreuther et al.
- Dorland et al, PRL …
- Dimits et al….
- …
- Earlier history:
  - slab eta_i mode: Rudakov and Sagdeev, 1961
  - Sheared-slab eta_i mode: Coppi, Rosenbluth, and Sagdeev, Phys. Fluids 1967
  - Toroidal ITG mode: Coppi and Pegoraro 1977, Horton, Choi, Tang 1981, Terry et al. 1982, Guzdar et al. 1983… (See Beer's thesis)