Part 2: Rigorous derivation of ITG growth rate & threshold (in a simple limit) starting from the Gyrokinetic Eq.

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Our starting point will be the electrostatic Gyrokinetic Eq. written in a Drift-Kinetic-like form for the full, gyro-averaged, guiding center density $\bar{f}(\underset{\sim}{R}, v_{\parallel}, \mu, t)$:

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_{\parallel}\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d)\cdot\nabla\bar{f} + \left(\frac{q}{m}E_{\parallel} - \mu\nabla_{\parallel}B + v_{\parallel}(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right)\frac{\partial \bar{f}}{\partial v_{\parallel}} = 0}$$

$$\underset{\sim}{V}_E \equiv -\frac{c}{B}\nabla\langle\Phi\rangle \times \hat{b} \qquad E_{\parallel} = -\hat{b}\cdot\nabla\langle\Phi\rangle$$

$$\mathbf{v}_d = \frac{v_{\parallel}^2}{\Omega}\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}) + \frac{\mu}{\Omega}\hat{\mathbf{b}}\times\nabla B \simeq \frac{v_{\parallel}^2 + v_{\perp}^2/2}{\Omega B^2}\underset{\sim}{B}\times\nabla B$$

$$\mu = \frac{1}{2}\frac{v_{\perp}^2}{B}$$

Gyro-averaged

$$\bar{f}(\underset{\sim}{R}, v_{\parallel}, \mu, t) = \left\langle f(\underset{\sim}{R} + \underset{\sim}{\rho}(\theta), v_{\parallel}, \mu, \theta, t\right\rangle_{\theta}$$

details:

* this is not the original Drift-Kinetic Eq. of Chew, Goldberger, + Low (1956), which was for the strong E-field "MHD ordering" (see Kulsrud, Handbook of Plasma Physics, 1983)

$$V_E \sim V_t \gg V_d \sim \frac{v_\perp^2}{\Omega R} \sim v_t \frac{\rho}{R}$$

* closer to the form of the Drift-Kinetic Eq. used in neoclassical theory, where $\underset{\sim}{V}_E \sim \underset{\sim}{V}_d$ ("weak E-field")

even though $\frac{V_E}{V_t} \sim \frac{\rho}{R} \sim \varepsilon$, $\frac{V_E \cdot \nabla}{V_\parallel \hat{b} \cdot \nabla} \sim \frac{V_t \frac{\rho}{R} k_\perp}{V_t k_\parallel} \sim \frac{k_\perp \rho}{k_\parallel R}$

$$\sim 1$$

Gyrokinetic Eq. for full guiding-center density $f(\underset{\sim}{R}, v_{\|}, \mu, t)$:

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_{\|}\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d) \cdot \nabla \bar{f} + \left(\frac{q}{m} E_{\|} - \mu \nabla_{\|} B + v_{\|} (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \cdot \mathbf{v}_E\right) \frac{\partial \bar{f}}{\partial v_{\|}} = 0}$$

In the uniform $\underset{\sim}{B}$ slab limit, this is = to Krommes GK Eq. 4
(~ p. 11-13)

Homework: show that expanding the Boltzmann factor in Cowley's Eq. 37, & gyroaveraging to get

$$\boxed{\bar{f} = F_0 - \frac{q\langle\Phi\rangle}{T_0} F_0 + h}$$

& subst. into above GK Eq.
gives exactly Cowley's (Frieman-Chen) form of the GK Eq.
(Cowley Eq. 40) for $\frac{\partial h}{\partial t}$ (Use uniform B slab limit for simplicity).

[& expand in consistent assumptions: $\frac{q\langle\Phi\rangle}{T} \ll 1$ but

$$F_0 \nabla_\perp \frac{q\langle\Phi\rangle}{T_0} \sim \nabla_\perp F_0 \qquad F_0 \nabla_\perp \frac{q\langle\Phi\rangle}{T_0} \sim \nabla_\perp F_0$$
]

Gyrokinetic Eq. for full guiding-center density $f(\underset{\sim}{R}, v_{\|}, \mu, t)$:

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_{\|}\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d) \cdot \nabla \bar{f} + \left(\frac{q}{m}E_{\|} - \mu\nabla_{\|}B + v_{\|}(\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}) \cdot \mathbf{v}_E\right)\frac{\partial \bar{f}}{\partial v_{\|}} = 0}$$

Homework: show that substituting the gyro-average of Cowley's Eq. 37:

(Straight + uniform $\underline{B}$ limit for simplicity)

$$\bar{f} = F_0 - \frac{q\langle\Phi\rangle}{T_0}F_0 + h$$

$$\frac{\partial h}{\partial t} - \frac{q}{T_0}\frac{\partial\langle\Phi\rangle}{\partial t}F_0 + v_{\|}\hat{b}\cdot\nabla h + \underset{\sim}{v}_E\cdot\nabla h + \underset{\sim}{v}_E\cdot\nabla\left(F_0\left(1 - \underbrace{\frac{q\langle\Phi\rangle}{T_0}}_{\text{drop}}\right)\right)$$

$$- v_{\|}\hat{b}\cdot\nabla\langle\Phi\rangle\frac{q}{T_0}F_0 \quad \underbrace{- \frac{q}{m}\hat{b}\cdot\nabla\langle\Phi\rangle\frac{\partial h}{\partial v_{\|}}}_{\text{drop}}$$

$$- \frac{q}{m}\hat{b}\cdot\nabla\langle\Phi\rangle\frac{\partial F_0}{\partial v_{\|}}\left(1 - \underbrace{\frac{q\langle\Phi\rangle}{T_0}}_{\text{drop}}\right) = 0$$

These 2 terms cancel

Use $\frac{\partial F_0}{\partial v_{\|}} = -\frac{m v_{\|}}{T_0}F_0$

Homework Solution outline

$$\boxed{\frac{\partial \overline{f}}{\partial t} + (v_{\|}\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d)\cdot\nabla\overline{f} + \left(\frac{q}{m}E_{\|} - \mu\nabla_{\|}B + v_{\|}(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right)\frac{\partial \overline{f}}{\partial v_{\|}} = 0}$$

Linearize: $\overline{f} = F_0 + \tilde{f}$, where F_0 satisfies Equilibrium Eq.

$$\frac{\partial}{\partial t} = 0 \qquad \underset{\sim}{E} = 0$$

$$(v_{\|}\hat{b} + \underset{\sim}{v}_d)\cdot\nabla F_0 - \mu\nabla_{\|}B\frac{\partial F_0}{\partial v_{\|}} = 0$$

Basically says $F_0 = $ const. along trajectories of banana orbits or passing orbits in a tokamak.

General Equilibrium solution could be an arbitrary function of the constants of the motion (E, μ, P_ϕ) where

$$E = \tfrac{1}{2}mv_{\|}^2 + \mu B.$$

& P_ϕ = canonical angular momentum

But if we neglect $\frac{|\underset{\sim}{v}_d|}{v_{\|}} \sim \frac{\rho}{R}$ get simpler Eq:

$$v_{\parallel}\,\hat{b}\cdot\nabla F_0 - \mu\left(\hat{b}\cdot\nabla B\right)\frac{\partial F_0}{\partial v_{\parallel}} = 0$$

Will consider Equilibrium of the form:

$$F_0(\underset{\sim}{R}, v_{\parallel}, \mu) \propto \frac{n_0(\psi)}{T_0^{3/2}(\psi)}\, e^{-\frac{m\left(\frac{1}{2}v_{\parallel}^2 + \mu B(\underset{\sim}{x})\right)}{T(\psi)}} \propto e^{-\frac{E}{T}}$$

Exercise: Plug this in to the previous Eq. & show it is a solution.

$$\frac{\partial \bar{f}}{\partial t} + (v_{\|}\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d)\cdot\nabla\bar{f} + \left(\frac{q}{m}E_{\|} - \mu\nabla_{\|}B + v_{\|}(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right)\frac{\partial \bar{f}}{\partial v_{\|}} = 0$$

Linearize: $\bar{f} = F_0 + \tilde{f}$, where F_0 satisfies Equilibrium Eq.

Next order Eq:

$$\frac{\partial \tilde{f}}{\partial t} + \left(v_{\|}\hat{b} + \underset{\sim}{v}_d\right)\cdot\nabla\tilde{f} - \mu\nabla_{\|}B\frac{\partial \tilde{f}}{\partial v_{\|}} = -\underset{\sim}{V}_E\cdot\nabla F_0 - \left(\frac{q}{m}E_{\|} + v_{\|}(\hat{b}\cdot\nabla\hat{b})\cdot\underset{\sim}{V}_E\right)\frac{\partial F_0}{\partial v_{\|}}$$

$$\left(-i\omega + i v_{\|}h_{\|} + i\underset{\sim}{v}_d\cdot\underset{\sim}{h}_{\perp}\right)\tilde{f} = -\underset{\sim}{V}_E\cdot\nabla F_0 - \left(\frac{q}{m}E_{\|} + v_{\|}(\hat{b}\cdot\nabla\hat{b})\cdot\underset{\sim}{V}_E\right)\frac{\partial F_0}{\partial v_{\|}}$$

Important Subtlety: $\bar{F}(\underset{\sim}{R}, v_\parallel, \mu, t)$ so

$$-\underset{\sim}{v}_E\cdot\nabla F_0 = -\underset{\sim}{v}_E\cdot\nabla\Big|_{v_\parallel,\mu,t} F_0$$

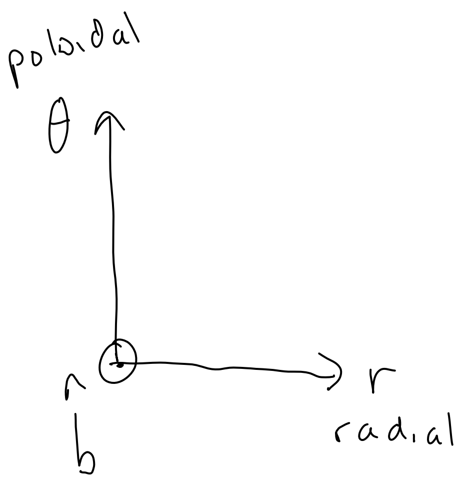
using $$F_0 \propto \frac{n_0(r)}{T_0^{3/2}(r)}\, e^{-\frac{\left(\frac{1}{2} m v_\parallel + m\mu B(\underset{\sim}{x})\right)}{T_0(r)}}$$

will give terms proportional to ∇n_0, ∇T_0, $\cancel{\&}$ $\mu \nabla B$

∇n_0 terms: $$-\underset{\sim}{v}_E\cdot\nabla F_0 \Rightarrow +\frac{c}{B}\left(\nabla\Phi\times\hat{b}\cdot\frac{\nabla n_0}{n_0}\right)F_0$$

$$\frac{\nabla n_0}{n_0} = -\frac{\hat{r}}{L_n}$$

$$= -\frac{c}{B}\nabla\Phi\times\hat{b}\cdot\hat{r}\,\frac{1}{L_n}F_0$$

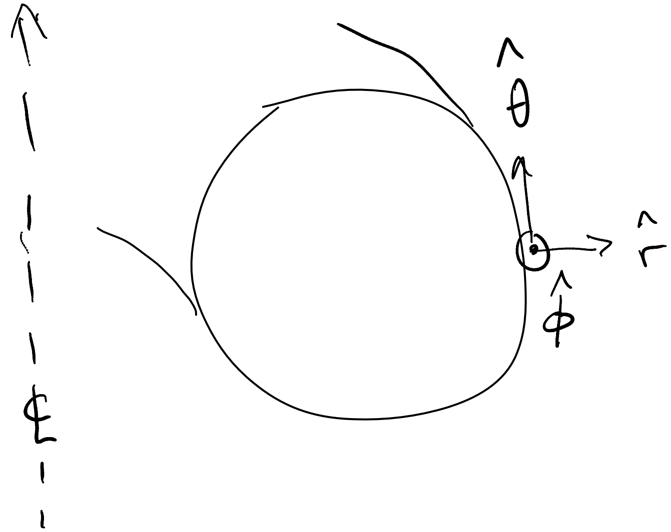


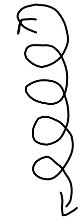
$$= -\frac{c}{B}\, i k_\theta \Phi\, \frac{1}{L_n} F_0$$

$$= + i\,\omega_* \frac{e\Phi}{T_0} F_0$$

$$\boxed{\omega_{*i} \equiv -\frac{cT}{eB}\frac{k_\theta}{L_n} \equiv -k_\theta \rho_s \frac{c_s}{L_n}}$$

Note on sign conventions:





With $\underset{\sim}{B}$ field out of the page, the ∇B drift for ions is downward

$$\underset{\sim}{v}_d \approx -\hat{\theta}\, v_t \frac{\rho}{R} \quad (\text{at } \theta = 0)$$

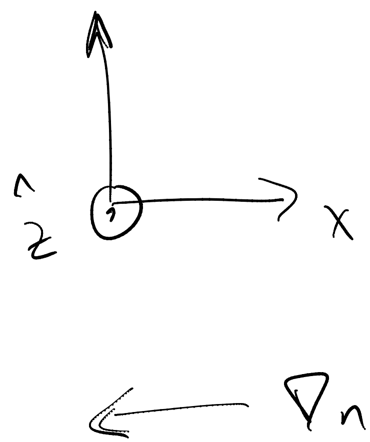
defining $\omega_{dv} = \underset{\sim}{k} \cdot \underset{\sim}{v}_d$

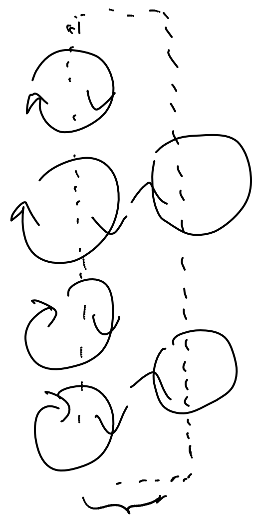
gives convention used in Beer's thesis!

$$\omega_{dv} = \omega_d (v_\parallel^2 + \mu B)/v_t^2$$

$$\omega_d = -k_\theta \rho v_t / R$$

More on Sign Conventions





with $\underset{\sim}{B}$ out of page, the diamagnetic flow v_{*i} is downward if ∇n is inward. Thus

$$\omega_{*i} \equiv \underset{\sim}{k} \cdot v_{*i} = - k_\theta v_t \frac{\rho}{L_n}$$

$$= - \frac{cT}{eB} \frac{k_\theta}{L_n}$$

(Back to RHS of linearized GK Eq., 4 slides back)

$$RHS = \underbrace{-\underset{\sim}{v}_E \cdot \nabla F_0}_{\text{part of this}} - \underbrace{\left(\frac{q}{m} E_\parallel + v_\parallel (\hat{b}\cdot\nabla\hat{b})\cdot \underset{\sim}{v}_E\right)\frac{\partial F_0}{\partial v_\parallel}}$$

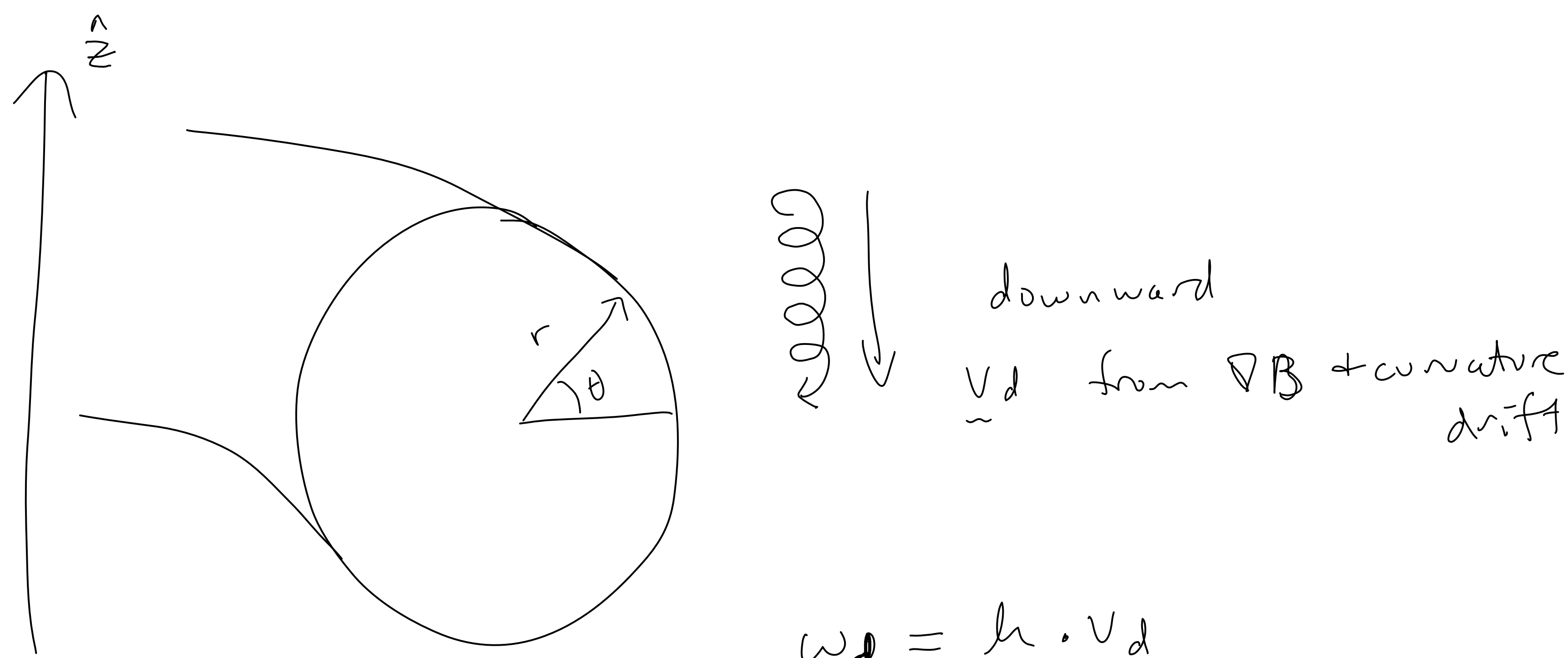
$$\propto -\frac{c}{B}\nabla\Phi\times\hat{b}\cdot\mu\nabla B \qquad \propto + v_\parallel^2 (\hat{b}\cdot\nabla\hat{b})\cdot\left(\frac{\hat{b}\times\nabla\Phi}{B}\right)$$

$$\propto -\nabla\Phi\cdot\left[\underbrace{\mu\,\hat{b}\times\nabla B}_{\nabla B} + \underbrace{v_\parallel^2\,\hat{b}\times(\hat{b}\cdot\nabla\hat{b})}_{\text{+ curvature drift}}\right]$$

$$RHS = +i\left(\omega_{*v}^T - \omega_{dv} - k_\parallel v_\parallel\right)\frac{e\Phi}{T_0}F_0$$

$$\omega_*^T = \omega_*[1 + \eta(v_\parallel^2/2v_t^2 + \mu B/v_t^2 - 3/2)] \qquad \omega_{dv} = \omega_d(v_\parallel^2 + \mu B)/v_t^2$$

$$\omega_* = \frac{k_\theta \rho v_t}{L_n} \qquad \eta = \frac{L_n}{L_T} \qquad \omega_d = -\frac{v_t}{R}\rho\left(k_\theta\cos\theta + k_r\sin\theta\right)$$



$$\omega_d = \underset{\sim}{k} \cdot \underset{\sim}{v}_d$$

$$= -\frac{v_t \rho}{R}\left(k_\theta \cos\theta + k_r \sin\theta\right)$$

will focus on $\theta \simeq 0$ here

(where bad-curvature drive is the strongest)

$$\left(-i\omega + i v_\parallel h_\parallel + i \underset{\sim}{v_d} \cdot \underset{\sim}{h_\perp}\right) \tilde{f} = -\underset{\sim}{v_E} \cdot \nabla F_0$$

$$- \left(\frac{q}{m} E_\parallel + v_\parallel \left(\hat{b} \cdot \nabla \hat{b}\right) \cdot \underset{\sim}{v_E} \right) \frac{\partial F_0}{\partial v_\parallel}$$

subst. for RHS

$$\left(-i\omega + i v_\parallel h_\parallel + i\omega_{dv}\right) \tilde{f} = -i \left(-\omega_{*v}^T + \omega_{dv} + h_\parallel v_\parallel\right) \frac{e\Phi}{T_0} F_0$$

$$\boxed{\tilde{f} = \frac{-\omega_{*v}^T + (h_\parallel v_\parallel + \omega_{dv})}{\omega - (h_\parallel v_\parallel + \omega_{dv})} \frac{e\Phi}{T_0} F_0}$$

Note: recover Boltzmann response when $h_\parallel v_\parallel$ & or ω_{dv} large

$$\boxed{\tilde{f} = \frac{-\omega_{*v}^{T} + (k_\parallel v_\parallel + \omega_{dv})}{\omega - (k_\parallel v_\parallel + \omega_{dv})} \frac{e\Phi}{T_0} F_0}$$

Look for modes with

$$k_\parallel v_{ti} \ll \omega, \omega_{*v}^{T}, \omega_{dv} \ll k_\parallel v_{te}$$

(slab "η_i" version of ITG requires finite $k_\parallel v_{ti}$, but not toroidal version).

assume Boltzmann electrons

Quasineutrality: $\tilde{n}_e = \tilde{n}_i$

(additional polarization contribution to density gives $k_\perp^2 \rho_i^2$ corrections but not critical for basic ITG.)

$$\boxed{n_{e0} \frac{e\Phi}{T_e} = \int d^3v \, \frac{-\omega_{*v}^{T} + \omega_{dv}}{\omega - \omega_{dv}} F_0 \frac{e\Phi}{T_{i0}}}$$

$$n_0 \frac{e\Phi}{T_{e0}} = n_0 \frac{e\Phi}{T_{i0}} \int d^3v \, \frac{F_0}{n_0} \frac{\omega_{dv} - \omega_{*T}}{\omega - \omega_{dv}}$$

"Cold plasma" or "fast wave" approx. $\omega >> \omega_{dv}$

$$\frac{T_{i0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\frac{T_{i0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \, \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\omega_{dv} = \omega_d (v_\parallel^2 + \mu B)/v_t^2 \qquad \omega_*^T = \omega_*[1 + \eta(v_\parallel^2/2v_t^2 + \mu B/v_t^2 - 3/2)]$$

$$\omega_d = -k_\theta \rho v_t/R \qquad \omega_* = -k_\theta \rho v_t/L_n$$

$$\int d^3v \, \frac{F_0}{n_0} \omega_{dv} = \int d^3v \, \frac{F_0}{n_0} \omega_d \left(v_\parallel^2 + \frac{1}{2} \overset{= v_x^2 + v_y^2}{v_\perp^2} \right) / v_t^2$$

$$= 2\,\omega_d$$

Using useful I.D. for Maxwellian F_0 :

$$\langle v_x^{2n} \rangle = \int d^3v \, \frac{F_0}{n_0} v_x^{2n} = v_t^{2n} \underbrace{(2n-1)!!}_{(2n-1)(2n-3)(2n-5)\cdots 5\cdot 3\cdot 1}$$

$$\frac{T_{\wedge 0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \, \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\omega_{dv} = \omega_d (v_\parallel^2 + \mu B)/v_t^2 \qquad \omega_*^T = \omega_* [1 + \eta(v_\parallel^2/2v_t^2 + \underbrace{\mu B}_{= \frac{1}{2} v_\perp^2 = \frac{1}{2}(v_x^2 + v_y^2)}/v_t^2 - 3/2)]$$

$$\omega_d = -k_\theta \rho v_t / R \qquad \omega_* = -k_\theta \rho v_t / L_n$$

$$\int d^3v \, \frac{F_0}{n_0} \omega_*^T = \omega_* \left(1 + \eta \left(\frac{1}{2} + 1 - \frac{3}{2} \right) \right) = \omega_*$$

$$\int d^3v \, \frac{F_0}{n_0} \omega_{dv}^2 = \int d^3v \, \frac{F_0}{n_0} \omega_d^2 \left[v_\parallel^4 + 2 v_\parallel^2 \frac{1}{2} v_\perp^2 + \frac{1}{4} (v_x^2 + v_y^2)^2 \right] \frac{1}{v_t^4}$$

$$= \omega_d^2 \left[3 + 2 \cdot \frac{1}{2} (1+1) + \frac{1}{4} \left(\frac{\langle v_x^4 + 2 v_x^2 v_y^2 + v_y^4 \rangle}{v_t^4} \right) \right]$$

$$= \omega_d^2 \left[5 + \frac{1}{4} (8) \right] = 7 \omega_d^2$$

$$\frac{T_{i0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \, \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\omega_{dv} = \omega_d (v_\parallel^2 + \mu B)/v_t^2 \qquad \omega_*^T = \omega_* [1 + \eta (v_\parallel^2/2v_t^2 + \underbrace{\mu B}_{= \frac{1}{2} v_\perp^2 = \frac{1}{2}(v_x^2 + v_y^2)}/v_t^2 - 3/2)]$$

$$\omega_d = -k_\theta \rho v_t / R \qquad \omega_* = -k_\theta \rho v_t / L_n$$

$$\int d^3v \, \frac{F_0}{n_0} \, \omega_{dv} \, \omega_*^T = \omega_d \omega_* \Bigg\{ 2 + \eta \int d^3v \, \frac{F_0}{n_0} \, \frac{\left(v_\parallel^2 + \frac{1}{2} v_\perp^2\right)}{v_t^2} \, \frac{\left(\frac{1}{2} v_\parallel^2 + \frac{1}{2} v_\perp^2 - \frac{3}{2} v_t^2 \right)}{v_t^2} \Bigg\}$$

$$= \omega_d \omega_* \left\{ 2 + \eta \left[\frac{1}{2} 3 + \frac{1}{2} 2 - \frac{3}{2} + \frac{1}{2} \cdot 2 \cdot \frac{1}{2} + \frac{1}{4} 8 - \frac{1}{2} \cdot 2 \cdot \frac{3}{2} \right] \right\}$$

$$\int d^3v \, \frac{F_0}{n_0} \, \omega_{dv} \, \omega_*^T$$

$$= \omega_d \omega_* \left\{ 2 + \eta \left[\cancel{\tfrac{1}{2} 3} + \cancel{\tfrac{1}{2} 2} - \cancel{\tfrac{3}{2}} + \cancel{\tfrac{1}{2} \cdot 2 \cdot \tfrac{1}{2}} + \tfrac{1}{4} 8 - \cancel{\tfrac{1}{2} \cdot 2 \cdot \tfrac{3}{2}} \right] \right\}$$

$$= \omega_d \omega_* 2 (1 + \eta)$$

Combine results from last 2 pages:

$$\boxed{\frac{T_{i0}}{T_{e0}} = 2 \frac{\omega_d}{\omega} - \frac{\omega_*}{\omega} + 7 \frac{\omega_d^2}{\omega^2} - 2 \frac{\omega_d \omega_*}{\omega^2} (1 + \eta)}$$

This defines a dispersion relation ω vs. $\underset{\sim}{k}$

$$\boxed{\frac{T_{i0}}{T_{e0}} = 2\frac{\omega_d}{\omega} - \frac{\omega_*}{\omega} + 7\frac{\omega_d^2}{\omega^2} - 2\frac{\omega_d\,\omega_*}{\omega^2}(1+\eta)}$$

Consider the flat density limit: $\nabla n \to 0$, but $\nabla T \neq 0$

$$\omega_* = -k_\theta \rho \frac{v_t}{L_n} \longrightarrow 0 \qquad \eta = \frac{\frac{1}{T}\nabla T}{\frac{1}{n}\nabla n} = \frac{L_n}{L_T} \to \infty$$

$$\omega_* \eta = -k_\theta \rho \frac{v_t}{L_n}\frac{L_n}{L_T} \equiv \bar{\omega}_{*T}$$

$$\omega^2 \frac{T_{i0}}{T_{e0}} - 2\omega_d \omega + 2\omega_d \bar{\omega}_{*T} - 7\omega_d^2 = 0$$

$$\boxed{\omega = \frac{2\omega_d \pm \sqrt{4\omega_d^2 - 4\frac{T_{i0}}{T_{e0}}\left(2\omega_d\bar{\omega}_{*T} - 7\omega_d^2\right)}}{2(T_{i0}/T_{e0})}}$$

From last page:

$$\omega = \frac{2\omega_d \pm \sqrt{4\omega_d^2 - 4\frac{T_{i0}}{T_{e0}}\left(2\omega_d \bar{\omega}_{*T} - 7\omega_d^2\right)}}{2\left(T_{i0}/T_{e0}\right)}$$

Consider large temperature gradient limit: $\omega_{*T} \propto \nabla T$

Growth rate:

$$\gamma = \frac{\sqrt{2\omega_d \bar{\omega}_{*T}}}{\sqrt{T_{i0}/T_{e0}}} = \frac{\sqrt{2}\, k_\theta \rho_i}{\sqrt{T_{i0}/T_{e0}}} \frac{v_{ti}}{\sqrt{R L_T}}$$

Fundamental scaling of bad-curvature driven instabilities.

Go back to general D.R.:

$$\omega = \frac{2\omega_d \pm \sqrt{4\omega_d^2 - 4\frac{T_{i0}}{T_{e0}}\left(2\omega_d \overline{\omega}_{*T} - 7\omega_d^2\right)}}{2(T_{i0}/T_{e0})}$$

$$= \frac{2\omega_d \pm \sqrt{\left(4 + 28\frac{T_{i0}}{T_{e0}}\right)\omega_d^2 - 8\frac{T_{i0}}{T_{e0}}\omega_d \overline{\omega}_{*T}}}{2(T_{i0}/T_{e0})}$$

Instability exists if

$$8\frac{T_{i0}}{T_{e0}}\omega_d \overline{\omega}_{*T} > \omega_d^2\left(4 + 28\frac{T_{i0}}{T_{e0}}\right)$$

$$\frac{1}{R}\frac{1}{L_T} > \frac{1}{R^2}\left(\frac{1}{2}\frac{T_{e0}}{T_{i0}} + \frac{1}{2}7\right)$$

$$\boxed{\frac{R}{L_T} > \frac{1}{2}\left(7 + \frac{T_{e0}}{T_{i0}}\right)}$$

Note: (1) To reduce growth rate far above marginal stability, want to reduce omega_d ~ 1/R, but (2) to raise the instability threshhold, want to raise omega_d ~ 1/R

Compare w/ Romanelli 1990 (Eq. 12):

$$\eta_i = (\tfrac{5}{3} + \tau/4) 2\epsilon_n$$

or

$$\frac{L_n}{L_T} = \left(\frac{5}{3} + \frac{1}{4}\frac{T_e}{T_i}\right) 2 \frac{L_n}{R}$$

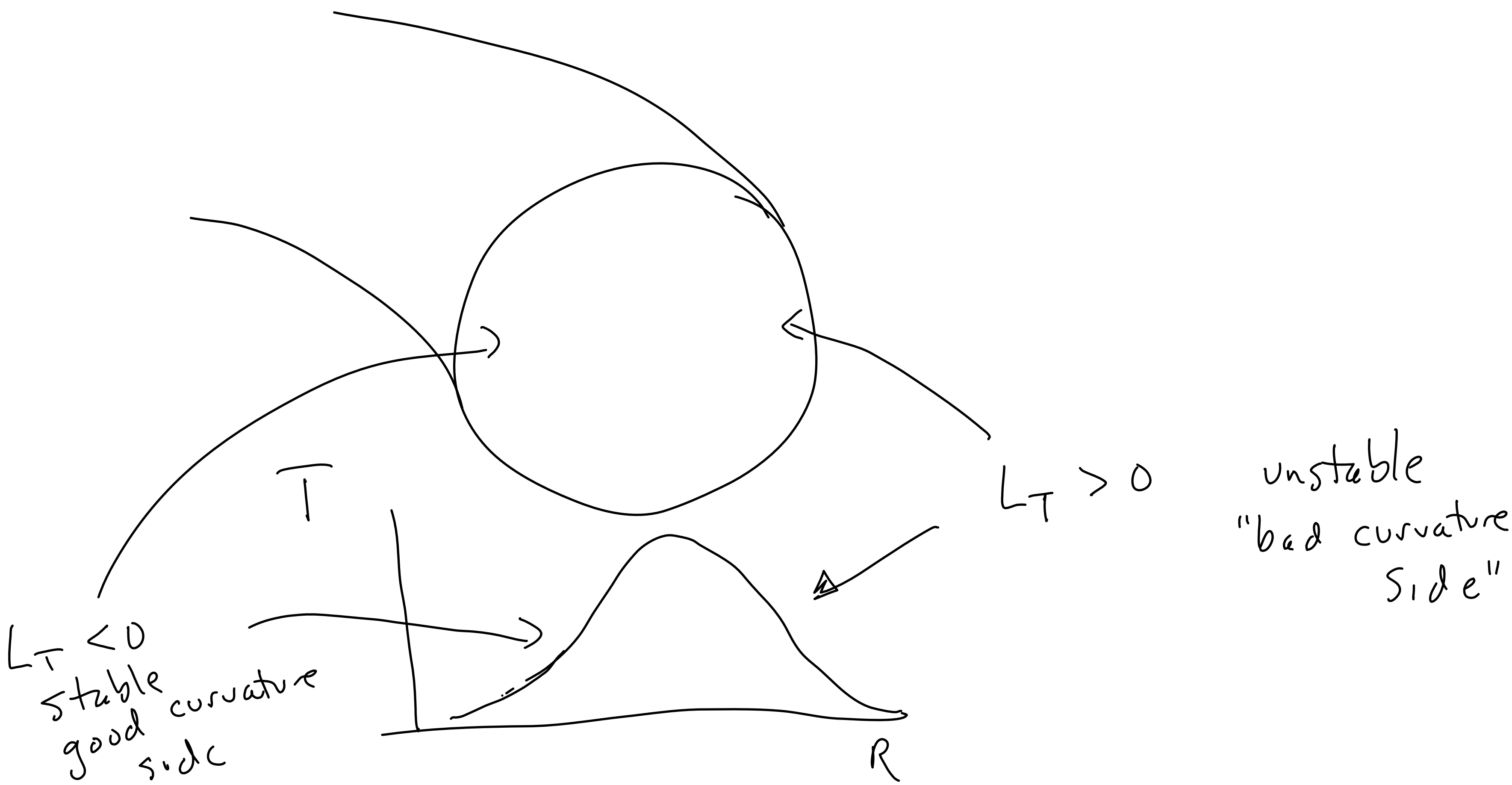
$$\boxed{\frac{R}{L_{Tcrit}} = \frac{10}{3} + \frac{1}{2}\frac{T_{eo}}{T_{io}}}$$

$$= 3.33 + 0.5 \frac{T_{eo}}{T_{io}}$$

vs. my $\dfrac{R}{L_{Tcrit}} = 3.5 + 0.5 \dfrac{T_{eo}}{T_{io}}$ } Very close. Diff. is presumably because Romanelli simplifies wov (See after his Eq. 6)

Note there is an instability only if $\omega_d \overline{\omega}_{*T} > 0$

$$\omega_d \overline{\omega}_{*T} = (h_\theta \rho)^2 \frac{v_t^2}{R L_T} \qquad \frac{1}{L_T} \equiv -\frac{1}{T}\frac{\partial T}{\partial R}$$



Why does this get the $\frac{T_{i0}}{T_{e0}}$ dependence of

$\frac{R}{L_{Tcrit}}$ wrong? More accurate: $\boxed{\frac{R}{L_T} > \frac{R}{L_{Tcrit}} = \frac{4}{3}\left(1 + \frac{T_{i0}}{T_{e0}}\right)}$

Because near marginal stability, the expansion of the resonant denominator

$$\frac{1}{\omega - \omega_{dv}} \simeq \frac{1}{\omega}\left(1 + \frac{\omega_{dv}}{\omega} + \dots\right)$$

breaks down, since $\omega \sim \omega_d$ near marginal stability...

To get this more accurately, need to include resonance effects. Can write the exact plasma response in terms of the Z function, without expanding 1/(omega-omega_dv), see Beer and Hammett 1996, "Toroidal gyrofluid equations for simulations of tokamak turbulence", Phys. Plasmas 3, 4046, and references therein. This introduces stabilizing effects from Landau damping from the spread in drift velocities in omega_dv, which increase with T_i, causing the critical R/L_Tcrit to increase at higher T_i.

More general result for threshold for instability:

$$\frac{R_0}{L_{T\,crit}} = \mathrm{Max}\left[\left(1+\frac{T_i}{T_e}\right)\left(1.33+1.91\,\frac{\hat{s}}{q}\right)\left(1-1.5\,\frac{r}{R_0}\right)\left(1+0.3\,\frac{r\,dk}{dr}\right),\ 0.8\,\frac{R_0}{L_n}\right]$$

Found by fits to lots of GS2 Gyrokinetic stability calculations (Jenko, Dorland Hammett, PoP 2001), guided by previous analytic results (Romanelli, Hahm + Tang) in some limits.

ITG References

- Mike Beer's Thesis 1995
  http://w3.pppl.gov/~hammett/collaborators/mbeer/afs/thesis.html
- Romanelli & Briguglio, Phys. Fluids B 1990
- Biglari, Diamond, Rosenbluth, Phys. Fluids B 1989
- Jenko, Dorland, Hammett, PoP 2001
- Candy & Waltz, PRL …
- Kotschenreuther et al.
- Dorland et al, PRL …
- Dimits et al….
- …
- Earlier history:
  - slab eta_i mode: Rudakov and Sagdeev, 1961
  - Sheared-slab eta_i mode: Coppi, Rosenbluth, and Sagdeev, Phys. Fluids 1967
  - Toroidal ITG mode: Coppi and Pegoraro 1977, Horton, Choi, Tang 1981, Terry et al. 1982, Guzdar et al. 1983… (See Beer's thesis)