Non-Local Fluid Closure Approximations to Model Long Mean-Free-Path Dynamics.

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Outline

* Motivation: long λ_{mfp} important in fusion & many other cases.

  Chapman-Enskog breaks down

* Phase-mixing paradigm problem

  fluid closure fit to $\nu \sim |k| v_t$ damping

  Fourier transform $\Rightarrow$ non-local heat flux

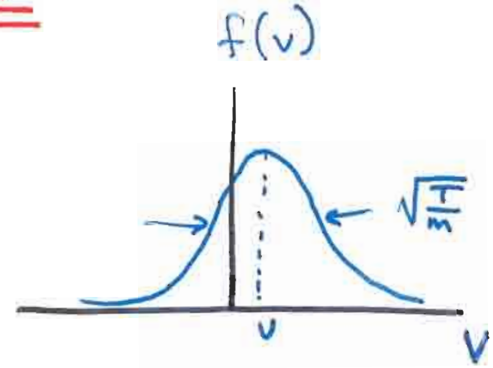
* Alternate derivation: renormalization

* Caveats

Classical Fluids are Highly Collisional

$\lambda_{mfp} \sim 10^{-4}$ cm in air

Particles move together in LTE
Local Maxwellian specified by n, v, T

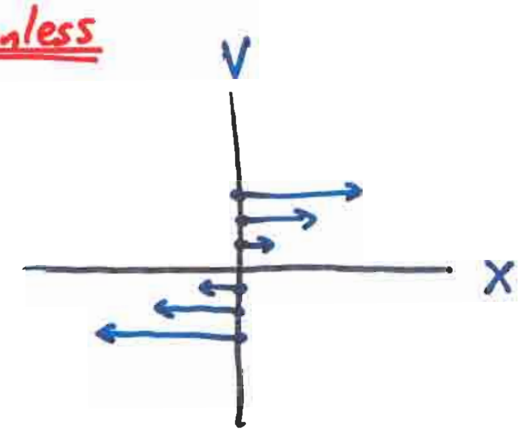


Particles slowly diffuse apart.

Fusion Plasmas are often Highly Collisionless

$\lambda_{mfp} \sim$ kilometers $\gg$ tokamak size

Particles quickly separate + "phase-mix"
$\Rightarrow$ collisionless Landau-damping



Other Low-Collisionality Systems:

* Often occur in space + astrophysical plasmas.
* The "gas" of stars evolving as a galaxy.

* Computer chips getting so small that feature sizes are approaching λ_{mfp}.

Widely thought that fluid simulations of plasma turbulence were inherently unable to model important kinetic effects.

Classical Chapman-Enskog closure procedure used by Braginskii fails for collisionless plasmas and misses kinetic effects such as Landau damping. (Improved by Chang & Callen)

"When collisions are infrequent, ... (the) heat flow depends on the detailed nature of the velocity distribution function, and cannot be determined in any simple way from the macroscopic (fluid) equations."
Spitzer, Physics of Fully Ionized Gases, 1962, p. 25 (see also p. 159).

"A property of Langmuir waves that is predicted by the Vlasov theory but which is completely outside the scope of fluid theory is the collisionless damping of electrostatic potentials..."
Krall and Trivelpiece, Principles of Plasma Physics, 1973, p. 386.

"What has happened to the Landau damping? One cannot expect the Landau damping to manifest itself in such a procedure, a power series expansion in $(k\, v_{th} / \omega)$, for in the Landau problem, in this limit, the damping goes as

$$\mathrm{Im}\,(Z) \sim \exp\left(-\frac{1}{(k\, v_{th} / \omega)^2}\right) = e^{-\frac{1}{\varepsilon^2}}$$

i.e., the damping goes to zero faster than any power of $(k\, v_{th} / \omega)$."
Oberman, Matt-57, Project Matterhorn, Princeton, 1960, p.8.

Paradigm Problem: Approximating e^{-1/ϵ^2}
(the Universal Language of Mathematics)

Taylor-series expansion:

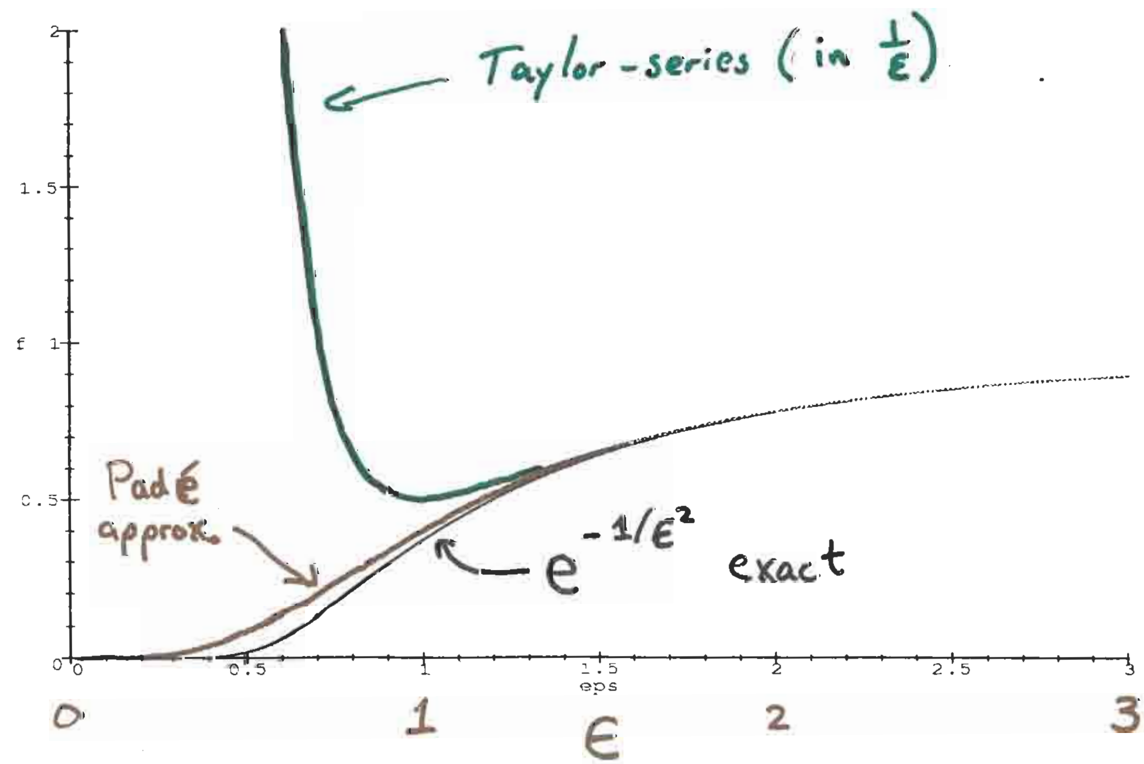
$$
\begin{aligned}
f(x) &\approx f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \ldots \\
e^{-1/\epsilon^2} &\approx 0 \quad + \epsilon\, 0 \quad + \frac{1}{2}\epsilon^2\, 0 \quad + \ldots
\end{aligned}
$$

or substitute $x = 1/\epsilon^2$, and try expanding around $x = 0$:

$$
e^{-x} \approx 1 - x + \frac{1}{2}x^2 = 1 - \frac{1}{\epsilon^2} + \frac{1}{2}\frac{1}{\epsilon^4} + \ldots
$$

Diverges as $\epsilon \to 0$!!!

(Reminiscent of "renormalization" problems in quantum mechanics, etc., where perturbation expansions must be summed to all orders...)



Relevant to plasmas:

$$\mathrm{Im}(Z) \propto e^{-\left(\frac{\omega}{k v_t}\right)^2} = e^{-\frac{1}{\varepsilon^2}}$$

$\varepsilon \to 0$ in cold plasma fluid limit

Padé Approximations of e^{-1/ϵ^2}

Padé Approximations are Ratio of Polynomials.
(Taylor series are a simple polynomial)

Example:

$$e^{-1/\epsilon^2} = \frac{1}{e^{+1/\epsilon^2}} \approx \frac{1}{1 + \frac{1}{\epsilon^2} + \frac{1}{2}\frac{1}{\epsilon^4} \ldots} = \frac{2\epsilon^4}{1 + 2\epsilon^2 + 2\epsilon^4}$$

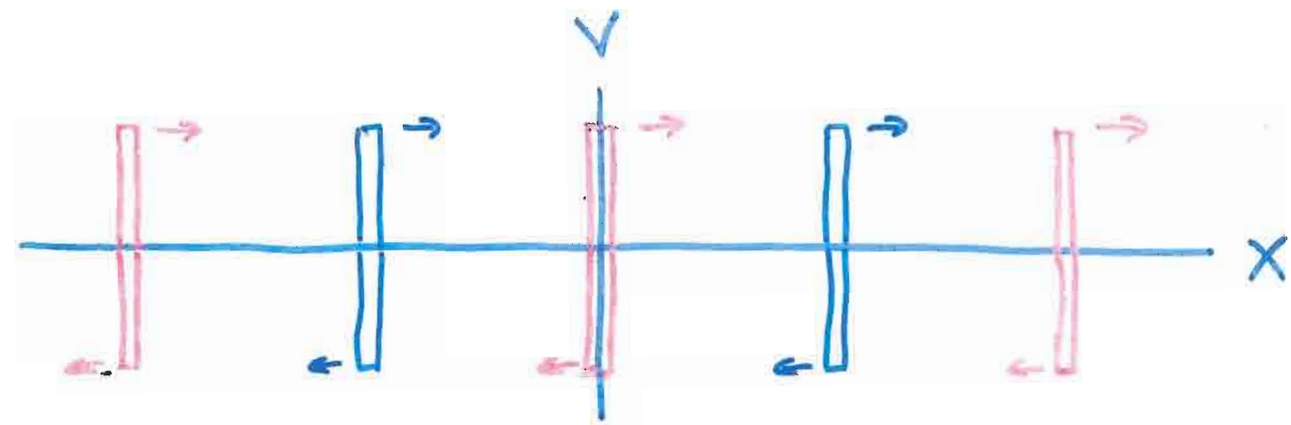
"4-pole approximation"

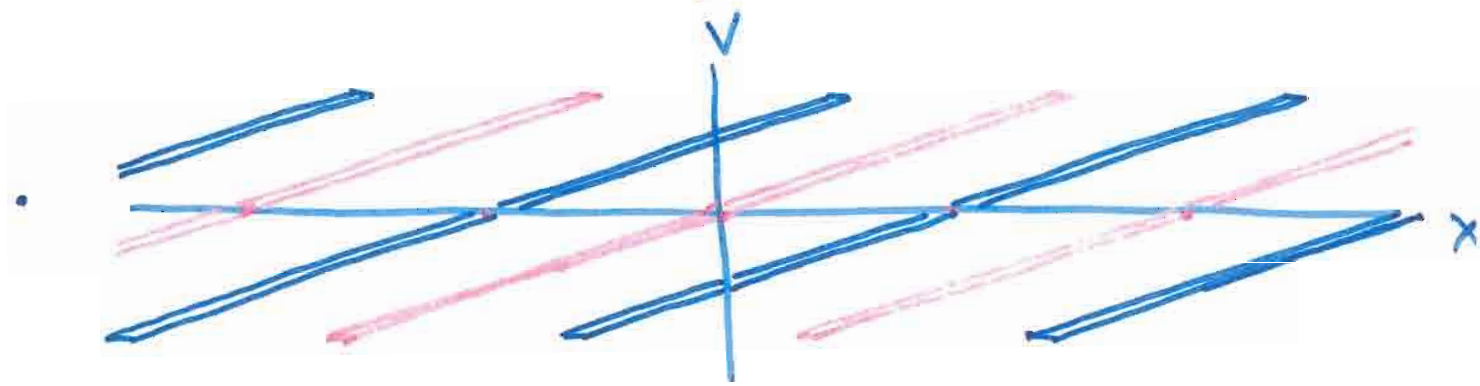
Padé approximations are often much more robust (they have bounded errors) than Taylor series.

The trick for more complicated equations is finding the equivalent of a Padé approximation.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0$$

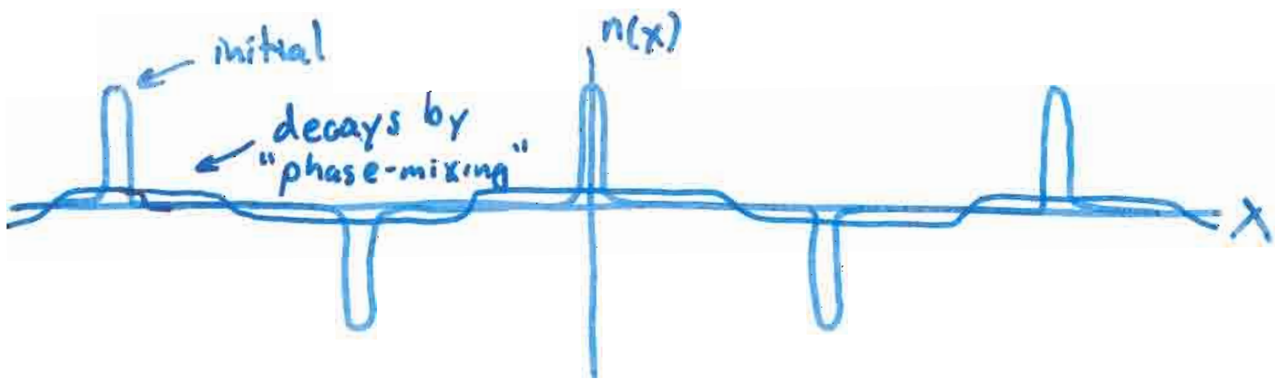
Contours of $f(x,v)$:





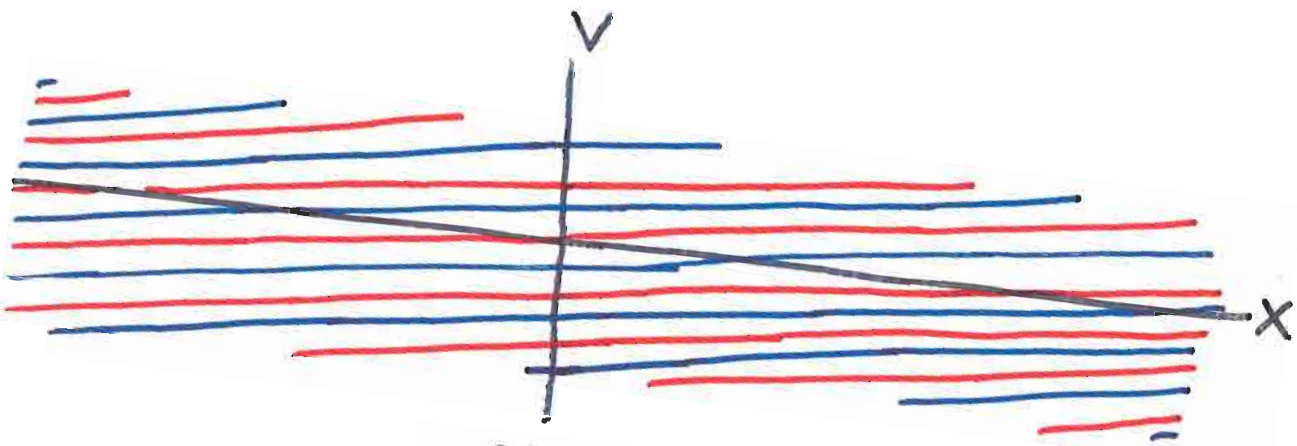
As time ↑, goes to finer velocity scales.

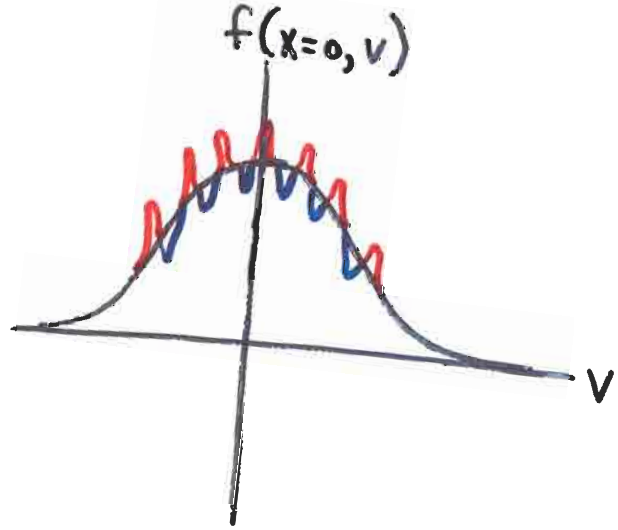
$$n(x) = \int dv\, f$$



Phase-mixing damping rate $\sim |k|\, v_t$, roughly analagous to shearing rate $|\nabla u|$ or eddy-turnover time $|k| u$ in Navier-Stok

Long time limit:





Even weak collisions will wipe out such small scale velocity features

$$C(f) = \nu_{coll} V_t^2 \frac{\partial^2 f}{\partial v^2}$$

$$f \propto e^{-ikvt}$$

$$\frac{\partial^2 f}{\partial v^2} \approx -k^2 t^2 f$$

Collisions dominate at $\tau \propto \nu_{coll}^{1/3}$

A Simple Phase-Mixing Paradigm

(Carl Oberman reminded me of this view of Landau damping.)

Consider a 1-D kinetic Eq. for $f(z,v,t)$, with no $\underset{\sim}{E}$ field:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} = 0$$

The exact solution is $f(z,v,t) = f_0(z-vt, v)$

Consider a single Fourier mode in z with a Maxwellian distribution in v:

$$f_0 = n_0 e^{ikz} f_M(v)$$

$$f = n_0 e^{ik(z-vt)} \frac{1}{\sqrt{2\pi v_t^2}} e^{-v^2/(2v_t^2)}$$

At any fixed v, f oscillates in time with $\omega = kv$ & no damping.

However, any v-moment of f will exponentially decay in time:

$$n(z,t) = \int dv\, f = n_0 \frac{e^{ikz}}{\sqrt{2\pi v_t^2}} \underbrace{\int dv}_{\text{mixing}} \underbrace{e^{-ikvt}}_{\text{phases}} e^{-v^2/(2v_t^2)}$$

$$\boxed{n(z,t) = n_0 e^{ikz} e^{-k^2 v_t^2 t^2/2}}$$

The Closure Problem in the Fluid Moment Hierarchy

$\int dv\, v^\ell *$ $$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} + \frac{e}{m}E_\parallel \frac{\partial f}{\partial v} = 0$$

$$\frac{\partial}{\partial t}\int dv\, v^\ell f + \frac{\partial}{\partial z}\int dv\, v^{\ell+1} f - \frac{e}{m}E_\parallel \int dv\, \ell v^{\ell-1} f = 0$$

$\ell=0 \Rightarrow$ $$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nu) = 0$$

$\ell=1 \Rightarrow$ $$\frac{\partial}{\partial t}(mnu) + \frac{\partial}{\partial z}(u\,mnu) = -\frac{\partial p}{\partial z} + enE_\parallel$$

$\ell=2 \Rightarrow$ $$\frac{\partial p}{\partial t} + \frac{\partial}{\partial z}(up) = -2p\frac{\partial u}{\partial z} - \frac{\partial q}{\partial z}$$

etc.

Exact, nonlinear, conservation laws (particles, momentum, energy, ...)

But each $\langle v^\ell \rangle$ Eq. requires knowledge of $\langle v^{\ell+1} \rangle$

= Infinite Hierarchy $\Rightarrow$ Closure problem

Usual approximations: neglect a higher moment

$p=0$ cold plasma approximation. No phase-mixing.

$q=0$, or $q=\infty$ (isothermal $\nabla T=0$) still no phase-mixing.

Seemingly "better" approximation! $\langle v^{10} \rangle = \alpha \langle v^8 \rangle \dfrac{\langle v^8 \rangle}{\langle v^6 \rangle}$

still fails to reproduce phase-mixing.

Traditional fluid equs. fail to reproduce phase-mixing
(even if higher-order moments are kept.)

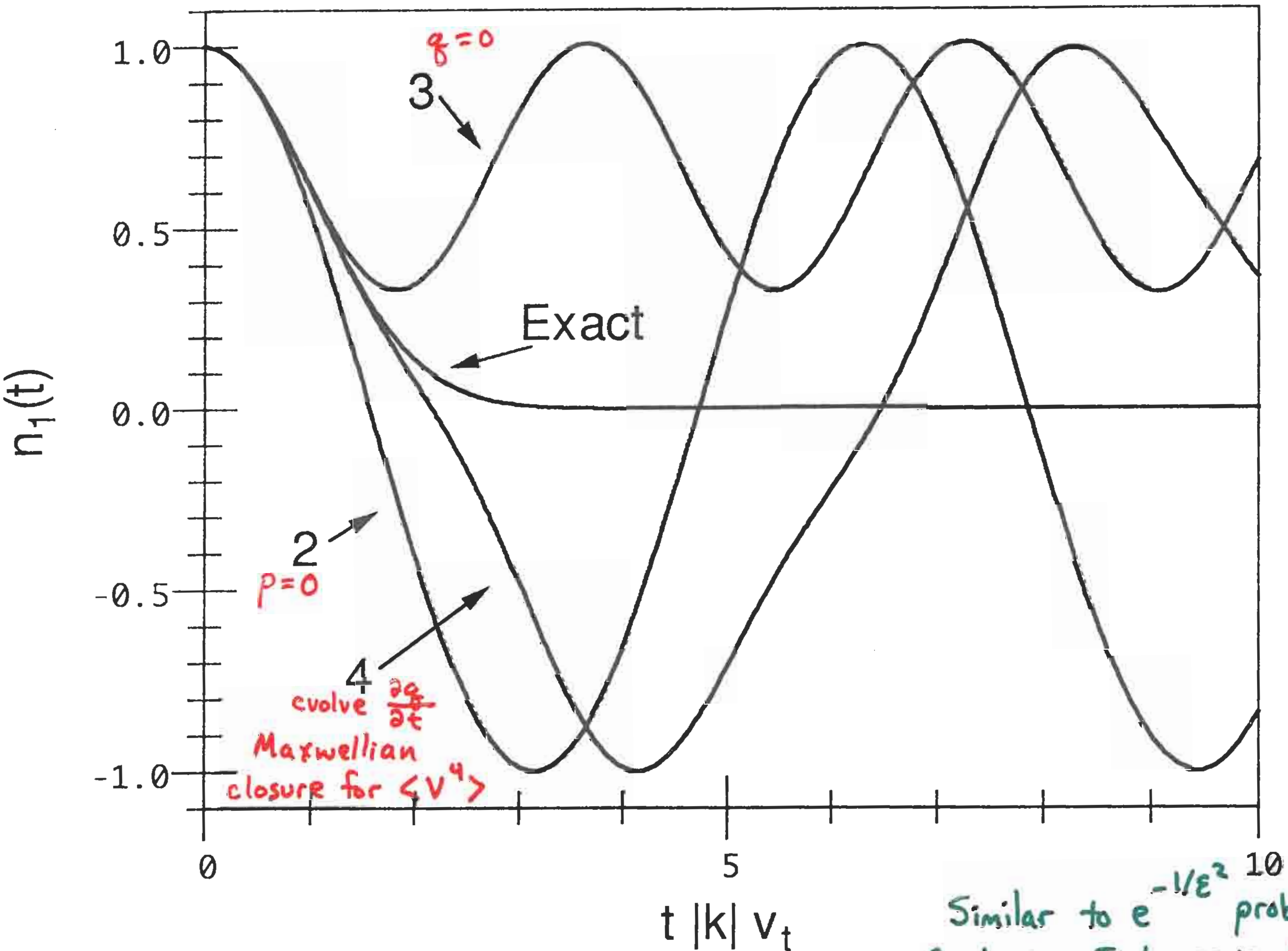


Fig. 1

Similar to e^{-1/ε^2} problem:
Good as a Taylor-series for short time,
breaks down for long time...

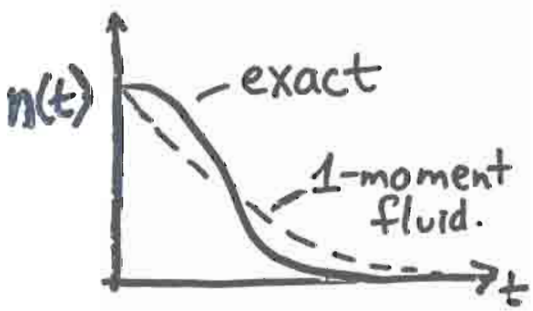
Diffusive-type Closure needed to model Collisionless Phase-Mixing.

Simplest possible 1-moment fluid model (too simple for most purposes). Start with exact density conservation Eq.:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nu) = 0$$

Approximate higher moment (nu) in terms of lower moments (n), with a diffusive-type term to model phase-mixing damping:

$$nu \approx -D\frac{\partial n}{\partial z} \quad \Rightarrow \quad \frac{\partial n_k}{\partial t} + Dk^2 n_k = 0$$



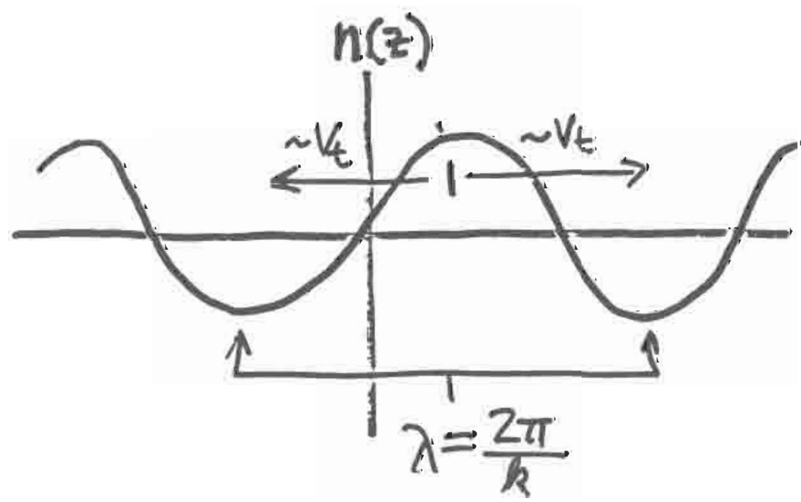
Fluid Model: $n_k(t) \propto e^{-Dk^2 t}$

Exact Kinetic: $\propto e^{-k^2 v_t^2 t^2/2}$

$$\boxed{D = \sqrt{\frac{2}{\pi}}\frac{v_t}{|k|}}$$

Damping rate $\nu \sim |k| v_t \sim Dk^2$

(D is an integral operator in real space.)



New closure approximation for fluid moments successfully models long-time-scale phase-mixing.

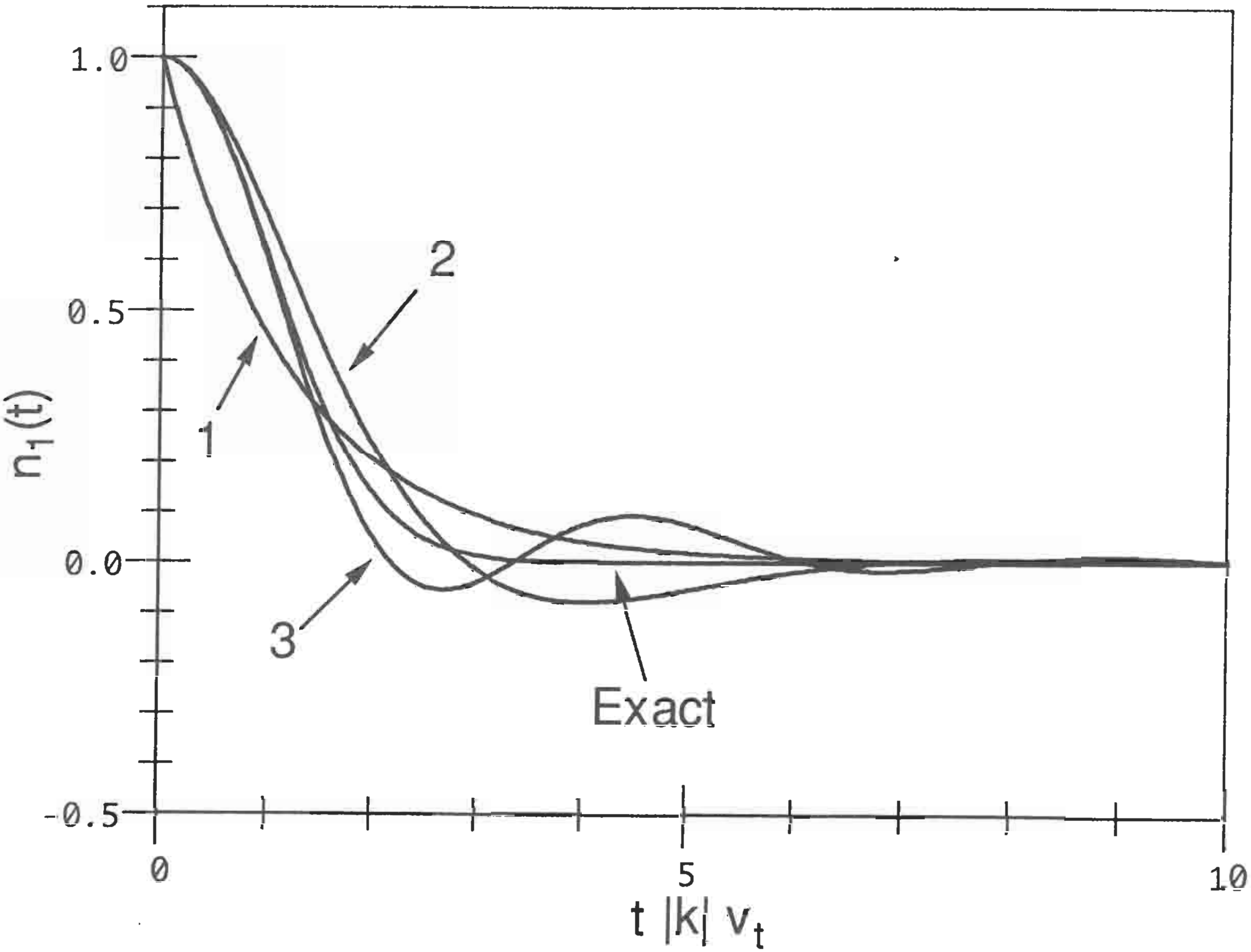


Fig. 2

New Landau-fluid closure approximation converges rapidly as more moments are kept.

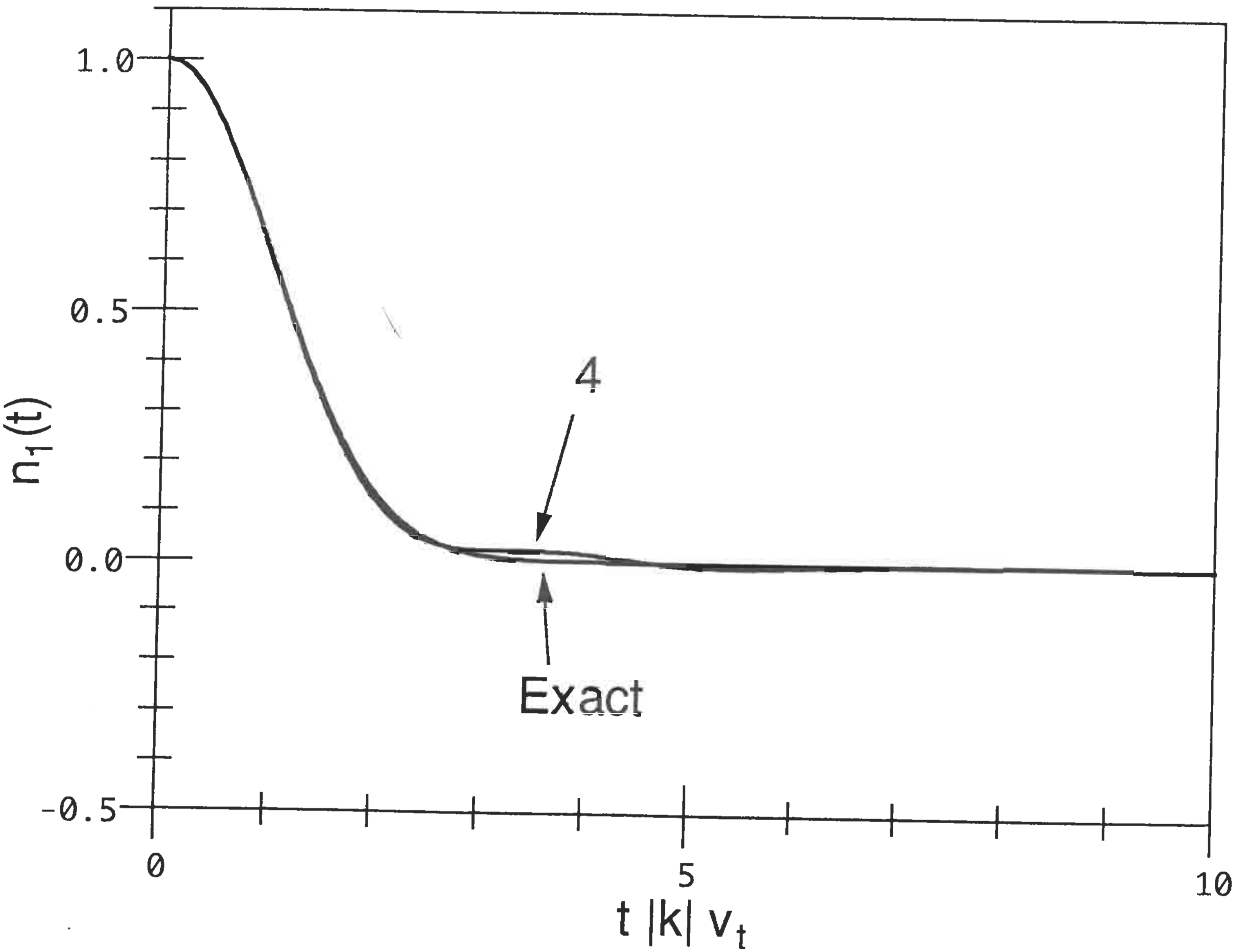


Fig. 3

$$q = -n\chi\nabla T$$

Thermal Diffusivity
(Conductivity)

$$\chi \cong \frac{v_t^2}{\nu_{coll} + |k|v_t}$$

Collisional limit
Braginskii
Chapman-Enskog

Landau-damping
Phase-mixing limit
on heat flux.

related to "flux limiters"
in laser-plasma problems.

$\chi \approx \frac{v_t}{|k|} \Rightarrow$ non-local operator in real space, not simple diffusion.

We developed these Landau-closure approximations for tokamak drift-wave turbulence. Other applications:

* α-particle driven TAE instability (Spong, Hedrick, et. al.)
* Resistive-wall MHD (Bondeson & Ward)
* Laser-plasma filamentation (Kaiser et. al.)
* Extend Zakharov Eqs. of Langmuir turbulence, ionospheric applications (Goldman & Newman, H. Rose et. al.)

Probably other useful applications.

Tokamak core microturbulence is small scale & small amplitude

$n = n_0 + \tilde{n}(z,t)$

$T = T_0 + \tilde{T}(z,t)$

$\uparrow$

Small

How to generalize non-local heat flux to $n(z) \neq n_0$?

Constraints? Have to satisfy entropy $\uparrow$

Test problems?

Luciani, Mora, & Virmont 1983

nonlocal heat flux semi-empirical

orig. for Laser-plasma interactions

compared w/ expts. & codes with large variations in n & T

still used: S. Nayakshin, "Feeding the black hole w/ condensing accretion flows..." astro-ph/0402469

Small fluctuation limit:

$$q_{LMV}(z) = \int \frac{dz'}{2\lambda} \underbrace{q_{cl}(z')}_{-n \frac{V_t^2}{\nu_{coll}} \frac{\partial T}{\partial z'}} e^{-\frac{|z-z'|}{\lambda}}$$

Integrate by parts:

$$q_{LMV} \propto -\int dz' \, \frac{T(z+z')\, \mathrm{Sgn}(z')}{\lambda} \, e^{-\frac{|z'|}{\lambda}}$$

differs from our

$$q_{LF} \propto -\int dz' \, \frac{T(z+z')}{z'} \cdot g\left(\frac{z'}{\lambda}\right)$$

Fourier transform to find damping rate γ for mode k:

$$q = -n\chi \nabla T$$

$$q_k = -n \frac{v_t}{|k|} ik T_k$$

Fourier-Transform $q_k \rightarrow q(z)$

$$q(z) = -n_0 v_{t0} \underbrace{\text{P.V.}\int_{-\infty}^{\infty} dz' \frac{T(z+z')}{z'}}_{\text{Hilbert Transform}}$$

incl. collisions (Snyder, Hammett, Dorland 97)

$$q(z) = -n_0 v_{t0} \underbrace{\int_0^{\infty} dz' \frac{T(z+z') - T(z-z')}{z'}}_{\text{Non-local derivative}} \underbrace{\frac{1}{1+\frac{z'^2}{\lambda^2}}}_{\substack{\text{cuttoff} \\ \sim \lambda_{mfp}}}$$

Renormalized Dissipation Approach

Parker & Carati, PRL (1995)

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \frac{\partial}{\partial v}\left(\nu_0 v f + \nu_0 \underset{\substack{\uparrow \\ =1 \text{ norm.}}}{V_t^2} \frac{\partial f}{\partial v}\right)$$

Chapman - Enskog : $\nu_0 \to \infty$
Here : $\nu_0 \to 0$

$$f(x,v,t) = \int dk \int d\omega\, e^{ikx - i\omega t}\, \hat{f}(k,\omega,v)$$

Hermite polynomial basis useful
[Hammett et.al 93, Armstrong et.al 70,
Grant & Feix 67, Sadowski 67]

$$\hat{f}(k,\omega,v) = e^{-\frac{v^2}{2}} \sum_{\ell} \alpha_\ell(k,\omega)\, h_\ell(v)$$

Orthonormal Hermite Polynomials

$$\alpha_\ell = \int dv\, h_\ell(v)\, \hat{f}(k,\omega,v)$$ same info as $\int dv\, v^\ell f$

$$(-i\omega + l\nu_0)\alpha_l + ik\left\{\sqrt{l}\,\alpha_{l-1} + \sqrt{l+1}\,\alpha_{l+1}\right\} = 0$$

dominates for sufficiently large l

$\alpha_{l+1} \simeq 0$

$\omega \to 0$ "long time behavior in renormalization group theories"
(Forster, Nelson, Stephen 77)

$$l\nu_0\alpha_l + ik\sqrt{l}\,\alpha_{l-1} = 0$$

Solve for α_l & insert into next lower Eq:

$$\underbrace{\left[(l-1)\nu_0 + \frac{k^2}{\nu_0}\right]}_{\equiv (l-1)\,\tilde{\nu}[l-1]}\alpha_{l-1} + ik\sqrt{l-1}\,\alpha_{l-1} = 0$$

renormalized collision rate

$$\tilde{\nu}[l-1] = \nu_0 + \frac{k^2}{(l-1)\,\tilde{\nu}[l]}$$

recursion formula

Fixed point for $\ell \ll \frac{h^2}{v_0^2}$

$$\tilde{v}[\ell] \approx \frac{|h|}{\sqrt{\ell}}$$

putting in dimensional factors gives same scaling as original approach

$$\tilde{v}[\ell] \approx \frac{|h| v_t}{\sqrt{\ell}}$$

various extensions possible...

Caveats

Landau-fluid closures are <u>approximations</u>

won't always work

Conservation of trouble

2 main challenges:

① If small scale features in $f(v)$ become important

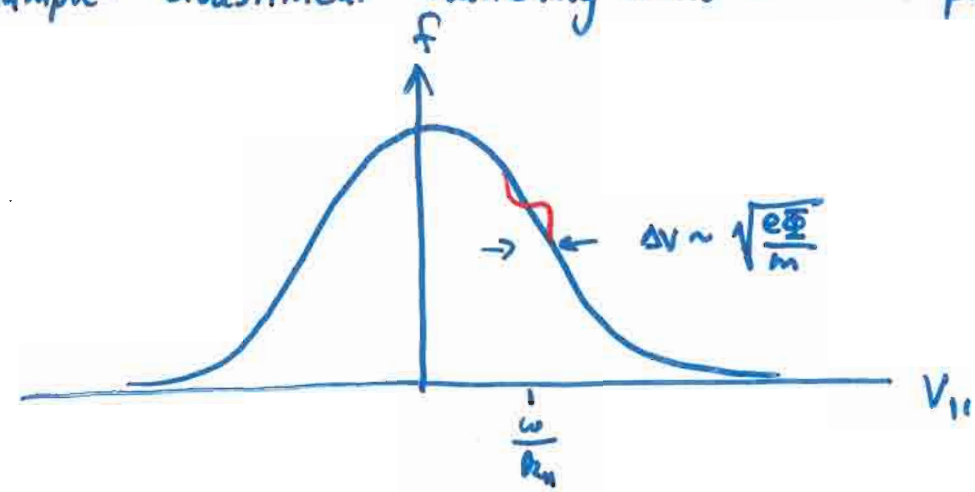
- weak turbulence
- particle trapping in wave. f. in $v_\parallel$ directn

Mattor et al.

② Multidimensional extensions in the presence of recurring orbits...

More complete discussion in paper....

Example: Quasilinear flattening turns off damping:



To resolve Δv scale requires $N \sim \left(\frac{V_t}{\Delta v}\right)^2$ Hermite polynomials or fluid moments.

For some problems (plasma echoes, ITG weak-turbulence very near marginal stability - Mattor) the Landau-Fluid approach would require a huge # of moments $\Rightarrow$ inefficient.

But for tokamak turbulence in most parameter regimes:

*) Many modes at various $\frac{\omega}{k_{\parallel}}$, Can't turn off one mode without driving another... Probably gives rise to turbulent diffusion in $v_{\parallel}$? + Landau-Fluid approx. okay...

*) Even weak collisions can be important at small v scales:

$$\frac{\partial f}{\partial t} = \nu V_t^2 \frac{\partial^2 f}{\partial v^2} \sim \nu \left(\frac{V_t}{\Delta v}\right)^2 \tilde{f}$$

*) Tokamak turbulence driven primarily by spatial gradients of density + temperature, not sensitive to velocity space details. Basic picture of Nonlinear ExB coupling between unstable modes + Landau-damped modes....

Caveats: (Hammett et.al., Plasma Physics Contr. Fus. 35, 973 (1993))

There are some cases where the Landau-fluid approximations don't work well (or require many moments to converge), where it is necessary to follow full details of full $f(x, v, t)$. (plasma echoes, quasilinear flattening near narrow resonances). (Mattor, Phys. Fluids B4, 3952 (1992)).

But works well as a model of rate at which f phase-mixes to small velocity scales which can then be ignored.

(Philisophically similar to subgrid turbulence models, or to the EDQNM simplified vs. of the DIA)

Usually there are many processes which quickly wipe out small velocity scales (collisions) or make them incoherent (turbulence).

Should be well-suited to typical strong-turbulence tokamak regimes.

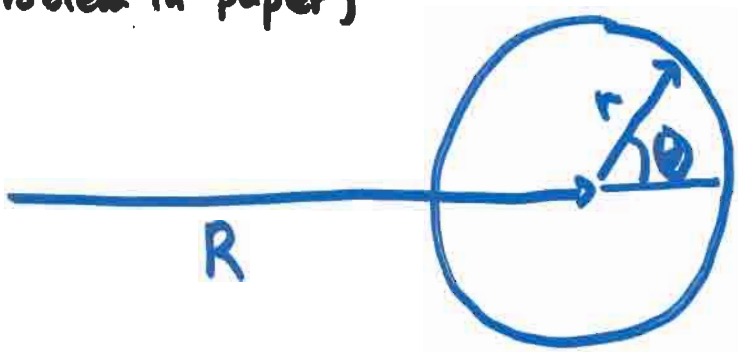
Provide accurate linear growth rates for instabilities which drive the turbulence, & for damped modes.

ExB nonlinearities in fluid Eqs. couple these together.

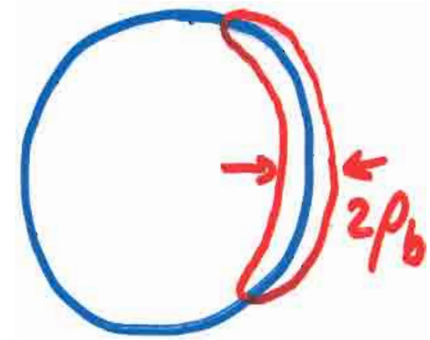
Fluid moment Eqs. express important conservation laws (particles, momentum, magnetic moment, parallel energy, ...) which constrain the nonlinear dynamics of the turbulence.

Multi-dimensional Challenge

(for illustration only, more precise problem in paper)



banana orbits...



$$\rho_b \sim \frac{\rho}{\sqrt{\varepsilon}} \ll r$$

$$\frac{\partial f}{\partial t} + \frac{v_{\parallel}}{R}\frac{\partial f}{\partial \theta} + \frac{E\rho}{R}\sin\theta\,\frac{\partial f}{\partial r} = 0$$

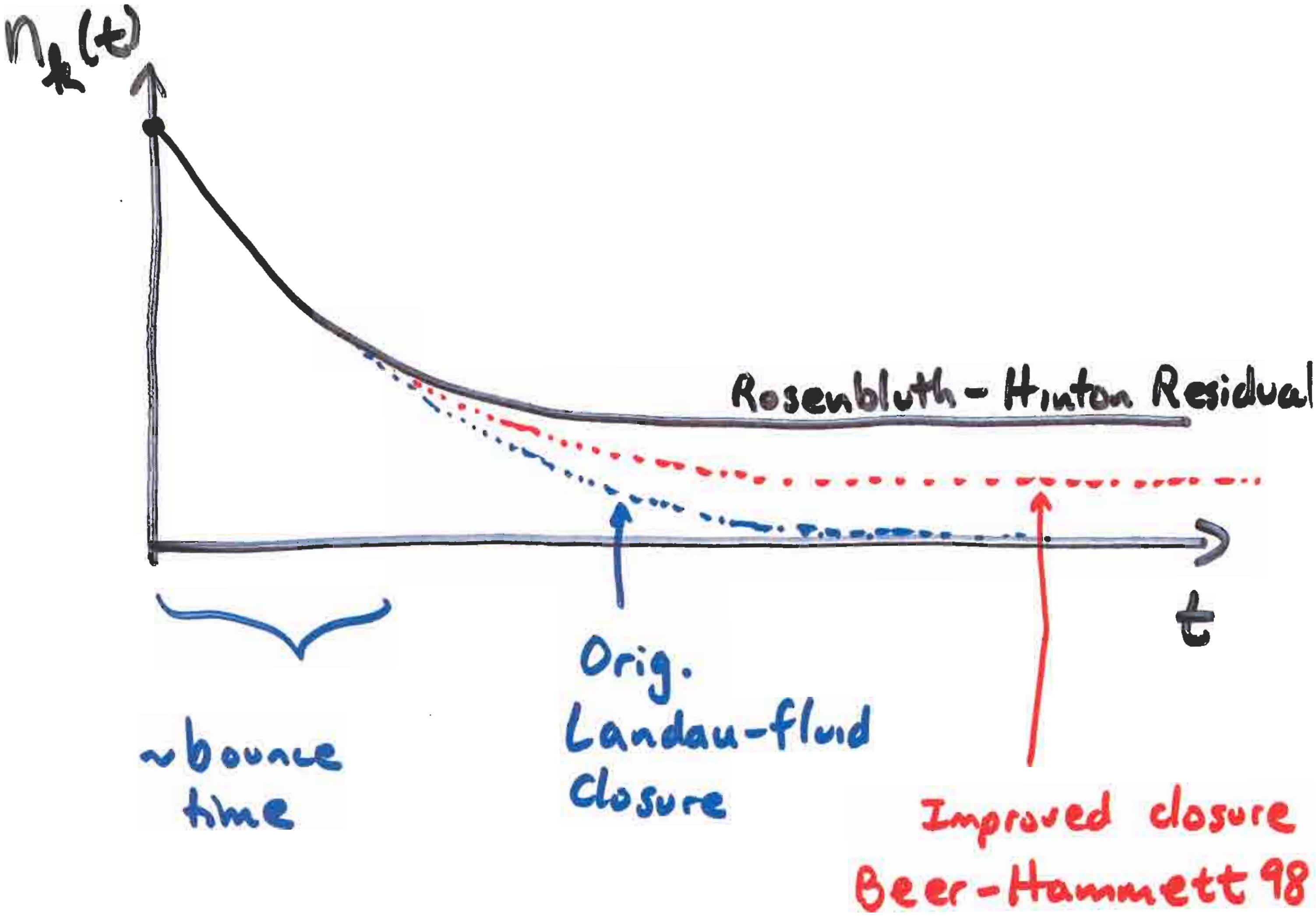
I.C. $f(r, \theta, E, \lambda, t=0) \propto e^{-E}\, e^{ikr}$

$$v_{\parallel} = \pm\sqrt{2E}\,\sqrt{1-\lambda B(\theta)}$$

$$B(\theta) = B_0(1 - \varepsilon\cos\theta)$$

$$n_k(t) = \int d^3v\, f$$

$$\propto B(\theta)\int_0^{\infty} dE\,\sqrt{E}\int_{-\lambda_{max}}^{\lambda_{max}} \frac{d\lambda}{\sqrt{1-\lambda B(\theta)}}\, f$$



Would like to do better.

Direct Continuum/Eulerian codes
for solving $f(\vec{x}, E, \lambda, t)$ (5D + time)
very successful in recent years.

- advanced algorithms: pseudo-spectral
  implicit linear dynamics,
  high order Gauss-Legendre integration

- parallelize well

- Less resolution in velocity space
  needed than in $\vec{x}$

  ~ 10 E
  ~ 30 λ
  $\sim 64 \times 96 \times 32$ $\vec{x}$

Class of interesting problems
where resolution needed is modest

Not as extreme as high-Reynolds #
Navier-Stokes. But some important problems
need more resolution.

References

Hammett & Perkins Phys. Rev. Lett. 64, 3019 (1990)

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Mattor (92) Phys. Fluids B 4 3952

Armstrong, Harding, Knorr, Montgomery (70)
Methods in Comput. Physics 9 29

Grant & Feix 67 Phys. Fluids 10 1356

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