The Gyrokinetic Regime
Geometry
Velocity Space
Linear How-To

Coordinates

Start from Vlasov equation with a collision operator:

$$\frac{\partial \mathcal{F}_s}{\partial t} + \mathbf{v} \cdot \nabla \mathcal{F}_s + \mathbf{a} \cdot \frac{\partial \mathcal{F}_s}{\partial \mathbf{v}} = C(\mathcal{F}_s)$$

Acknowledge parallel and perpendicular dynamics are different:

$$\mathbf{v} = \mathbf{v}_\perp + \hat{\mathbf{b}} v_\parallel.$$

Go to energy and magnetic moment coordinates

$$E = \tfrac{1}{2} m v^2 \qquad \mu = \tfrac{1}{2} m v_\perp^2 / B$$

with inverse transformation

$$v_\parallel^2 = 2(E - \mu B)/m \qquad \mathbf{v}_\perp = v_\perp(\mathbf{e}_1 \cos\xi + \mathbf{e}_2 \sin\xi)$$

where $\mathbf{e}_1, \mathbf{e}_2, \hat{\mathbf{b}}$ form an orthogonal coordinate system.

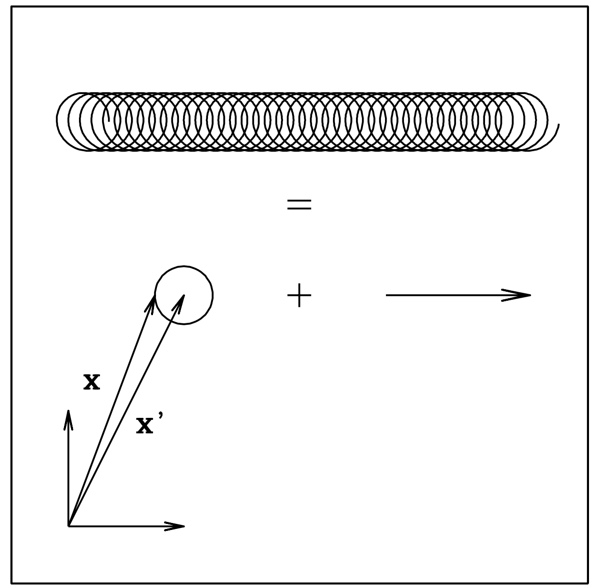
Guiding Center Transformation

- First key step is coordinate transformation, from

$$(\mathbf{x}, v_{\|}, v_{\perp}, \xi) \quad \rightarrow \quad (\mathbf{x}', E, \mu, \xi)$$

- Second key step is average over ξ

- The difference between $\mathbf{x}$ and $\mathbf{x}'$ is the difference between the position of the particle and its guiding center.



(Perpendicular velocity is subtle!)

Ordering Assumptions

- Define:

$$\left|\frac{\rho}{F_0}\frac{\partial F_0}{\partial \mathbf{x}}\right| \sim \frac{\rho}{L} \equiv \rho_*$$

- Require **slow** evolution of equilibrium:

$$\left|\frac{1}{\Omega F_0}\frac{\partial F_0}{\partial t}\right| \sim \rho_*^3$$

(transport time scale, τ)

- For fluctuations, require

$$\frac{\omega}{\Omega} \sim \rho \hat{\mathbf{b}} \cdot \nabla' \sim \frac{\delta f}{f} \sim \frac{\delta B}{B} \sim \frac{v_E}{v_t} \sim \rho_*$$

but allow

$$\rho \hat{\mathbf{b}} \times \nabla' = k_\perp \rho \sim 1$$

- Note there are *three* time scales:

$$\Omega^{-1}, \qquad \omega^{-1}, \qquad \tau$$

Dynamical Equation

- Expand F and fields in small parameter ($\sim \rho_*$)
- Find equilibrium is independent of gyrophase ξ, $F_0 = F_0(E, \mu, \mathbf{x})$. Solubility condition yields $\hat{\mathbf{b}} \cdot \nabla F_0 = 0$
- Assume equilibrium has isotropic pressure: $F_0 = F_0(E, \mathbf{x}_\perp)$
- Perturbed distribution function still has ξ dependence. GK equation describes evolution of h, the non-adiabatic, ξ-independent part:
$$\left(\frac{d}{dt} + v_\parallel \hat{\mathbf{b}} \cdot \nabla + i\omega_d + C\right) h = i\omega_*^T \chi - q \frac{\partial F_0}{\partial \epsilon} \frac{\partial \chi}{\partial t}.$$

Notation Defined

- Time derivative includes nonlinear terms:

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{c}{B}\{\chi, h\}.$$

- Generalized potential is

$$\chi = J_0(\gamma)\left(\Phi - \frac{v_\parallel}{c}A_\parallel\right) + \frac{J_1(\gamma)}{\gamma}\frac{mv_\perp^2}{q}\frac{\delta B_\parallel}{B}.$$

- Argument of Bessel functions is $\gamma = k_\perp v_\perp/\Omega$

- Curvature and ∇B drifts from ω_d:

$$\omega_d = \mathbf{k}_\perp \cdot \mathbf{B}_0 \times \left(mv_\parallel^2 \hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}} + \mu\nabla B_0\right)/(mB_0\Omega),$$

Integrals over Velocity

- To find fields, need Maxwell's equations. Sources (charge, current) are integrals over velocity and sums over species

- Integrals evaluated at position $\mathbf{x}$; requires coordinate transformation for $h = h(E, \mu, \mathbf{x}')$. Example:

$$\int d^3v\, h = \frac{B}{m^2} \int \frac{d\epsilon\, d\mu\, d\xi}{|v_\parallel|} h \exp{(iL)} \equiv \frac{1}{2\pi} \int d^2v\, d\xi\, h \exp{(iL)}$$

  where $L = (\mathbf{v} \times \hat{\mathbf{b}} \cdot k_\perp)/\Omega$ accounts for the gyrophase dependence.

- Integral over ξ results in Bessel functions:

$$\frac{1}{2\pi} \int d\xi\, h \exp{(iL)} = h J_0(\gamma)$$

Maxwell's Equations

- Poisson's equation:

$$\nabla_\perp^2 \Phi = 4\pi \sum_s \int d^2v\, q \left[q\Phi \frac{\partial F_0}{\partial E} + J_0(\gamma) h \right]$$

- Ampere's law:

$$\nabla_\perp^2 A_\parallel = -\frac{4\pi}{c} \sum_s \int d^2v\, q v_\parallel J_0(\gamma) h$$

- Perturbed force balance:

$$\frac{\delta B_\parallel}{B} = -\frac{4\pi}{B^2} \sum_s \int d^2v\, m v_\perp^2 \frac{J_1(\gamma)}{\gamma} h$$

Review of Key Points

- **Existence of multiple space and time scales:**
  - Dynamics slow compared to Ω
  - Equilibrium frozen on dynamical time scale
  - Weak variation of equilibrium scale lengths
- **Equilibrium quantities constant on flux surface**
- **Small amplitude fluctuations**

Review of Key Points

- Velocity-space coordinates are (E, μ)
  - Can trade magnetic moment μ for pitch angle $\lambda = \mu/E$

- $k_\parallel \ll k_\perp$ implies high toroidal mode numbers.

- No restriction on any of

$$\beta, \quad k_\perp \rho, \quad \frac{\omega}{k_\parallel v_t}, \quad \frac{\omega}{\omega_d}, \quad \frac{\omega}{\omega_b}, \quad \frac{\omega}{\nu}, \quad \frac{\omega}{\omega_{NL}}$$

Additional Nonlinearities

- "Parallel" nonlinearity ordered small:

$$\hat{\mathbf{b}} \cdot \nabla\Phi \left(\frac{\partial \delta f}{\partial v_\parallel}\right) \sim \hat{\mathbf{b}} \cdot \nabla\Phi \left(\frac{\delta f}{v_t}\right) \ll \hat{\mathbf{b}} \cdot \nabla\Phi \left(\frac{\partial F_0}{\partial v_\parallel}\right) \sim \hat{\mathbf{b}} \cdot \nabla\Phi \left(\frac{F_0}{v_t}\right)$$

  Likely a good assumption in fully developed turbulence

- Nonlinearities in Maxwell equations:

$$\delta(n\Phi) \sim n_0(\delta\Phi) + (\delta n)\Phi_0 + (\delta n)(\delta\Phi)$$

  Dropped because fluctuation amplitudes are ordered small (no perp gradient here)

- Time evolution of equilibrium strictly forbidden

References

- P. Catto, Plasma Physics, 20:719, 1978
- T. Antonsen and B. Lane, Phys. Fluids, 23:1205, 1980
- E. A. Frieman and L. Chen, Phys. Fluids, 24:502, 1982